The Minimal Time-Varying Realization of a Nonlinear Time-Invariant System [⋆]

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Abstract: The state realization is called minimal if it is either accessible and observable or its state dimension is minimal. In the linear case those two definitions are equivalent, but not for nonlinear time-invariant systems. It is shown that definitions remain equivalent in case one is searching for minimal realization in a larger class of nonlinear time-varying systems. First, nonlinear realization theory is recasted for time-varying nonlinear systems. A necessary and sufficient realizability condition is given in terms of integrability of certain subspace. The mathematical tools used for this purpose are the algebraic approach of differential forms and the theory of the skew polynomial rings; these tools are again extended from time-invariant to time-varying systems.

Keywords: nonlinear systems, time-varying systems, state space realization, reduction, accessibility, polynomial methods

1. INTRODUCTION

There exist two possibilities to define minimality of the state space realization. First, one calls the realization minimal when it is both observable and accessible (controllable). Second, one may require minimality of the state dimension. Though in the linear time-invariant case these two definitions are equivalent, this is no longer true for nonlinear time-invariant systems, as shown in (Zhang et al., 2010). The latter points to the controversy between linear and nonlinear theories. The goal of this paper is to show that these two definitions remain equivalent when one is searching for the minimal realization in a larger class of nonlinear *time-varying* systems. That is, the minimal realization of a nonlinear time-invariant system is, in general, a time-varying system.

It has been encountered previously in nonlinear control theory that the problem, stated for time-invariant system, only has a solution in a larger class of systems. For example, the paper (Pereira da Silva and Rouchon, 2004) pointed to the fact that time-invariant systems may possess time-dependent flat outputs. The results of our paper are of the similar flavour.

To find the minimal realization it is first necessary to determine the irreducible input-output (i/o) equation of the system. If one starts from the reducible equation, application of the realization algorithms does not provide minimal realization. Reduction theory of nonlinear systems is based on the notion of autonomous variable φ_{red}, i.e. a variable, satisfying certain autonomous differential equation $F(\varphi_{\text{red}}, \varphi_{\text{red}}^{(1)}, \ldots, \varphi_{\text{red}}^{(\mu)}) = 0$, see for instance (Zhang et al., 2010; Conte et al., 2007; Zheng et al., 2001). Usually the reduced equation is formed from the assumption $F(0, \ldots, 0) = 0$, taking $\varphi_{\text{red}} = 0$ [1]. In this paper we relax this assumption and an arbitrary solution of the differential equation is taken into account in the reduction process. We will show on several examples that the non-trivial solution leads, in general, to the time-varying reduced system, even if the original system is time-invariant.

Second, one has to find the state space realization of nonlinear time-varying i/o equation. There exist numerous papers where the realization of nonlinear time-invariant systems is studied, see for instance (Conte et al., 2007; van der Schaft, 1987; Delaleau and Respondek, 1995; Crouch and Lamnabhi-Lagarrigue, 1988; Kotta and Mullari, 2006; Halás and Kotta, 2012). The authors are not aware of any papers addressing the realization problem of nonlinear time-varying systems. This is the second topic of our paper. In the present paper we follow the algebraic approach of differential one-forms (Conte et al., 2007), combined with the theory of non-commutative polynomial rings (Zhang et al., 2010; Belikov et al., 2011; Zheng et al., 2001; Halás, 2008; Zheng and Cao, 1995), adapted from time-invariant to time-varying case. More precisely, we generalize the necessary and sufficient realizability condition given in (Halás and Kotta, 2012) to the time-varying systems.

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[1] It means that zero is one of the solutions of the autonomous differential equation.

Finally, note that time-varying nonlinear systems have received much less attention compared with their time-invariant counterparts. Some results on time-varying nonlinear systems are presented in (Pereira da Silva, 2008; Pereira da Silva and Rouchon, 2004), where infinite dimensional geometric setting is used.

In Section 2 the basic notions of algebraic approach and polynomial framework are given. In Section 3 the realization of time-varying nonlinear i/o equation is studied. In Section 4 several examples are given that demonstrate the emergence of time-varying systems in the process of system reduction, and finally, Section 5 draws conclusion.

2. POLYNOMIAL FRAMEWORK

The main focus of this section is to extend the polynomial framework that has been used to address many problems for nonlinear time-invariant systems (Zhang et al., 2010; Belikov et al., 2011; Zheng et al., 2001; Halás, 2008; Zheng and Cao, 1995) for the case of time-varying systems, i.e. for the case when the system coefficients are functions of time t. Formally, this means that the ground field $k = \mathbb{R}(t)$, and not just $\mathbb{R}$ as in the case of time-invariant systems. Consider the system

$$y^{(n)} = \psi(t, y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(r)}) = 0. \quad (1)$$

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the independent system variables $\{t, y, \ldots, y^{(n-1)}, u^{(k)}, k \geqslant 0\}$ with coefficients from the ground field $k = \mathbb{R}(t)$. Let $\mathrm{d}/\mathrm{d}t : \mathcal{K} \to \mathcal{K}$ be the time-derivation operator. Then the pair $(\mathcal{K}, \mathrm{d}/\mathrm{d}t)$ is differential field, (Kolchin, 1973). Over the field $\mathcal{K}$ a differential vector space $\mathcal{E} := \mathrm{span}_{\mathcal{K}} \{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$ is defined. Consider a one-form $\omega \in \mathcal{E}$ such that $\omega = \sum_i \alpha_i \mathrm{d}\varphi_i$, $\alpha_i, \varphi_i \in \mathcal{K}$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i (\dot{\alpha}_i \mathrm{d}\varphi_i + \alpha_i \mathrm{d}\dot{\varphi}_i)$.

The differential field $\mathcal{K}$ and the differentiation operator $\mathrm{d}/\mathrm{d}t$ induce a non-commutative ring of left differential polynomials $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$. A polynomial $p \in \mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ can be uniquely written as

$$p = p_m s^m + p_{m-1} s^{m-1} + \ldots + p_1 s + p_0, \quad (2)$$

where s is a formal variable and $p_i \in \mathcal{K}$ for $i = 0, \ldots m$. Polynomial $p \neq 0$ iff at least one of the functions p_i is nonzero. If $p_m \not\equiv 0$, then the integer m is called the *degree* of p and denoted by $\deg(p)$. We set additionally $\deg(0) = -\infty$. The addition of the polynomials is defined in the standard way. However, for $a \in \mathcal{K} \subset \mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ the multiplication is defined by the commutation rule

$$s \cdot a := a\, s + \dot{a}. \quad (3)$$

It is easy to see that for $s^2 \cdot a = as^2 + 2\dot{a}s + \ddot{a}$, $a \in \mathcal{K}$, and in general, for $n \geqslant 0$ we obtain $s^n \cdot a = \sum_{i=0}^{n} \binom{n}{i} a^{(n-i)} s^i$, where $\binom{n}{i}$ is binomial coefficient.

Definition 1. Abramov et al. (2005) The *adjoint* of the skew polynomial ring $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ is defined as the skew polynomial ring $\mathcal{K}[s^*; (\mathrm{d}/\mathrm{d}t)^*]$, where $(\mathrm{d}/\mathrm{d}t)^* = -\mathrm{d}/\mathrm{d}t$.

From Definition 1 it follows that multiplication in the adjoint ring is defined by the commutation rule $s^* a = a\, s^* - \dot{a}$ for $a \in \mathcal{K}$. If

$$p = p_m s^m + \cdots + p_1 s + p_0$$

is a polynomial in $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ then the *adjoint polynomial* p^* is defined by the formula

$$p^* = s^{*m} p_m + \cdots + s^* p_1 + p_0 \in \mathcal{K}[s^*; (\mathrm{d}/\mathrm{d}t)^*],$$

where the products $s^{*i} p_i$ must be computed in $\mathcal{K}[s^*; (\mathrm{d}/\mathrm{d}t)^*]$, to yield $p^* = p_m^* s^{*m} + \cdots + p_1^* s^* + p_0^*$. Application of the adjoint operator may be considered as reversing the order of the polynomial variable and polynomial coefficient.

In (McConnel and Robson, 1987) it has been shown that the ring $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ is an integral domain. Therefore, one can construct from the ring $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ a non-commutative *field of fractions*, which will be required later to prove Lemma 2. Let $\mathcal{W}$ be a multiplicative subset of $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$. Consider the set of left fractions, denoted by $\mathcal{K}\langle s; \mathrm{d}/\mathrm{d}t\rangle$. The elements of $\mathcal{K}\langle s; \mathrm{d}/\mathrm{d}t\rangle$ have the form $p^{-1} q$, where $p \in \mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$ and $q \in \mathcal{W}$.

The time-varying nonlinear system (1) can be represented by two non-commutative polynomials from the ring $\mathcal{K}[s; \mathrm{d}/\mathrm{d}t]$. By applying the operator d to equation (1) we obtain

$$\mathrm{d}y^{(n)} - \sum_{i=0}^{n-1} \frac{\partial \psi}{\partial y^{(i)}} \mathrm{d}y^{(i)} - \sum_{j=0}^{r} \frac{\partial \psi}{\partial u^{(j)}} \mathrm{d}u^{(j)} - \frac{\partial \psi}{\partial t} \mathrm{d}t = 0. \quad (4)$$

The latter equation can be rewritten as

$$p(s)\mathrm{d}y + q(s)\mathrm{d}u + r(s)\mathrm{d}t = 0, \quad (5)$$

where

$$p = s^n - \sum_{i=0}^{n-1} p_i s^i, \; q = -\sum_{j=0}^{r} q_j s^j, \; r = -r_0, \quad (6)$$

whereas

$$p_i = \frac{\partial \psi}{\partial y^{(i)}} \in \mathcal{K}, \; q_j = \frac{\partial \psi}{\partial y^{(j)}}, \; r_0 = \frac{\partial \psi}{\partial t} \in \mathcal{K}.$$

3. REALIZATION

The realization problem is defined as follows. For a time-varying nonlinear system (1), find, if possible, the state coordinates $x \in X \subseteq \mathbb{R}^n$, $x = \psi(t, y, \dot{y}, \ldots, y^{(n-1)}, u, \ldots, u^{(r)})$ such that in these coordinates the system takes the classical state-space form

$$\begin{aligned} \dot{x} &= f(t, x, u) \\ y &= h(x) \end{aligned} \quad (7)$$

and the sequences $\{u(t), y(t), t \geqslant 0\}$, generated by (7) (for different initial states), coincide with those, satisfying equation (1). Then (7) will be called a realization of (1). A system (1) is said to be realizable if there exists a realization of the form (7) for it.

The proofs presented in this section are adapted from (Halás and Kotta, 2012), where the realization of time-invariant systems was studied. In order to prove the main theorem, we need the following lemma, which establishes the relation between the tangent linearized i/o and state equations.

Lemma 2. Assume we have given the i/o equation (1) together with its tangent linearized system (5). Let

$$p^* = \sum_{i=0}^{n} p_i^* s^{*i}, \quad q^* = \sum_{j=0}^{r} q_j^* s^{*j} \quad (8)$$

be respectively the adjoint polynomials of p and q. Then the realization of the linearized system (5) has the form

$$\begin{aligned} \mathrm{d}\dot{x} &= A\mathrm{d}x + B\mathrm{d}u + D\mathrm{d}t \\ \mathrm{d}y &= C\mathrm{d}x, \end{aligned} \tag{9}$$

where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_0^* \\ 1 & 0 & \dots & 0 & -p_1^* \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -p_{n-1}^* \end{pmatrix}, \quad B = \begin{pmatrix} -q_0^* \\ \vdots \\ -q_{n-1}^* \end{pmatrix}, D = \begin{pmatrix} -r_0^* \\ \vdots \\ -r_{n-1}^* \end{pmatrix}, \quad C = (0, \dots, 0, 1), \tag{10}$$

and $r_0^*, \dots, r_{n-1}^*$ are arbitrary functions from $\mathcal{K}$, satisfying the condition

$$r_0^* + \dot{r}_1^* + \dots + r_{n-1}^{*(n-1)} = r_0. \tag{11}$$

Proof. It is sufficient to show that the equations (9) can be transformed into the form (5). From (9) one may compute

$$\mathrm{d}x = (sI - A)^{-1}(B\mathrm{d}u + D\mathrm{d}t),$$

where I is the identity matrix, and

$$\mathrm{d}y = C(sI - A)^{-1}(B\mathrm{d}u + D\mathrm{d}t). \tag{12}$$

The inverse matrix $(sI - A)^{-1}$ has rather complex entries, therefore it is better to avoid computing it directly. However, the product $C(sI - A)^{-1} =: E$ is much easier to find. The elements of $E = (e_1, \dots, e_n)$ are fractions from $\mathcal{K}\langle s, \mathrm{d}/\mathrm{d}t\rangle$. In (Halás and Kotta, 2012) it is shown that $E = (p^{-1}, p^{-1}s, \dots, p^{-1}s^{n-1})$, where the polynomial p is defined by (6). From above and (12),

$$\begin{aligned} \mathrm{d}y &= E(B\mathrm{d}u + D\mathrm{d}t) \\ &= -\sum_{i=0}^{n-1} p^{-1}(s)s^i q_i^* \mathrm{d}u - \sum_{i=0}^{n-1} p^{-1}(s)s^i r_i^* \mathrm{d}t \\ &= -p^{-1}(s)q(s)\mathrm{d}u - p^{-1}(s)r'(s)\mathrm{d}t. \end{aligned}$$

Note that

$$s^i \mathrm{d}t = 0, \ i > 1,$$

since $\mathrm{d}\dot{t} = \mathrm{d}\ddot{t} = \dots \mathrm{d}t^{(i)} = 0$. Due to restriction (11) the constant term of $r'(s)$ equals to r_0. Thus, $r'(s)\mathrm{d}t = r_0\mathrm{d}t$ and

$$\mathrm{d}y = -p^{-1}(s)q(s)\mathrm{d}u - p^{-1}(s)r_0\mathrm{d}t,$$

or alternatively,

$$p(s)\mathrm{d}y + q(s)\mathrm{d}u + r_0\mathrm{d}t = 0.$$

□

Note that $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ in Lemma 2 are just symbols denoting the one-forms from $\mathcal{E}$, not necessarily exact. If $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ are exact, one can integrate the equations (9) immediately, and obtain the realization of (1) in the observer form. In case the one-forms $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ are not exact, one may ask whether there exist a set of n linear combinations of $\mathrm{d}x_1, \dots, \mathrm{d}x_n$, that are exact, or said in the other words, whether the subspace $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\}$ is integrable. If so, we can apply the coordinate transformation to (9) and integration of the transformed one-forms yields the realization of (1) in a general form. To conclude, the system (1) is realizable iff the subspace $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\}$ is integrable. Therefore, our aim is to find the expressions for the one-forms $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ in (9). These expressions are given in Lemma 3. For that we first introduce the set of one-forms

$$\tilde{\omega}_l := p_l^* \mathrm{d}y + q_l^* \mathrm{d}u + r_l^* \mathrm{d}t, \ l = 1, \dots, n, \tag{13}$$

where p_l^* and q_l^* are respectively the coefficients of s^l of the adjoint polynomials p^* and q^*, defined by (8), and $r_0^*, \dots, r_n^* \in \mathcal{K}$ are arbitrary functions satisfying the restriction (11).

Lemma 3. The differentials of the state coordinates expressed in linearized system equations (9) are given by the formulas

$$\mathrm{d}x_i = \omega_i := \sum_{k=i}^{n} \tilde{\omega}_k^{(k-i)}, \ i = 1, \dots, n. \tag{14}$$

Proof. In terms of the one-forms $\tilde{\omega}_i$, $i = 0, \dots, n$ the system equations (9) take the form

$$\begin{aligned} \mathrm{d}\dot{x}_1 &= -\tilde{\omega}_0 \\ \mathrm{d}\dot{x}_2 &= \mathrm{d}x_1 - \tilde{\omega}_1 \\ &\dots \\ \mathrm{d}\dot{x}_{n-1} &= \mathrm{d}x_{n-2} - \tilde{\omega}_{n-2} \\ \mathrm{d}\dot{x}_n &= \mathrm{d}x_{n-1} - \tilde{\omega}_{n-1} \\ \mathrm{d}y &= \mathrm{d}x_n \end{aligned} \tag{15}$$

Expressing $\mathrm{d}y, \dots, \mathrm{d}y^{(n-1)}$ from (15) yields

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}x_n \\ \mathrm{d}\dot{y} &= \mathrm{d}x_{n-1} - \tilde{\omega}_{n-1} \\ \mathrm{d}\ddot{y} &= \mathrm{d}x_{n-2} - \tilde{\omega}_{n-2} - \tilde{\omega}_{n-1}^{(1)} \\ &\dots \\ \mathrm{d}y^{(n-1)} &= \mathrm{d}x_1 - \tilde{\omega}_1 - \tilde{\omega}_2^{(1)} - \dots - \tilde{\omega}_{n-1}^{(n-2)} \end{aligned} \tag{16}$$

From the definition (13) we obtain $\omega_n = \mathrm{d}y$. Thus, (16) can be rewritten as

$$\begin{aligned} \mathrm{d}x_n &= \omega_n \\ \mathrm{d}x_{n-1} &= \tilde{\omega}_{n-1} + \tilde{\omega}_n^{(1)} \\ \mathrm{d}x_{n-2} &= \tilde{\omega}_{n-2} + \tilde{\omega}_{n-1}^{(1)} + \tilde{\omega}_n^{(2)} \\ &\dots \\ \mathrm{d}x_1 &= \tilde{\omega}_1 + \tilde{\omega}_2^{(1)} + \dots + \tilde{\omega}_{n-1}^{(n-2)} + \tilde{\omega}_n^{(n-1)} \end{aligned}$$

Regarding the definition (14) it is easy to see that $\mathrm{d}x_1 = \omega_1, \dots, \mathrm{d}x_n = \omega_n$. □

The following theorem gives the realizability criterion in terms of the adjoint polynomials.

Theorem 4. The i/o equation (1) has the state space realization in the form (7) iff the subspace

$$\mathcal{V} = \mathrm{span}_{\mathcal{K}}\{\omega_1, \dots, \omega_n\}$$

is completely integrable.

Proof. After linearizing the i/o equation (1), its realization, by Lemma 2, is expressed in the form (9). From (9), by Lemma 3, the differentials of the state coordinates $\mathrm{d}x_i = \omega_i$, $i = 1, \dots, n$. If the differentials of the state coordinates are integrable then the system is realizable and vice versa. □

The fact that the functions $r_1^*, \dots, r_{n-1}^* \in \mathcal{K}$ in definition (13) are arbitrary, gives additional freedom to find the exact basis of $\mathcal{V}$.

Remark 5. For the realizable i/o model (1) the differentials of the state coordinates dx_i can be calculated as the integrable linear combination of the one-forms $\omega_1, \ldots, \omega_n$.

Remark 6. The one-forms ω_i in (14) can be alternatively computed by recursive formula, starting from ω_n:

$$\omega_n := \mathrm{d}y,\ \omega_i := \dot{\omega}_{i+1} + \tilde{\omega}_i, \tag{17}$$

where $i = n-1, \ldots, 1$.

Example 7. Consider the system

$$\ddot{y} + te^t\dot{y} + \frac{1}{t}y - \ln\dot{u} - tu = 0 \tag{18}$$

After differentiating (18) we obtain the polynomial representation in the form (5), where

$$\begin{aligned} p &= s^2 + e^t ts + \frac{1}{t},\ q = -\ln ts - t, \\ r &= -r_0 = te^t\dot{y} + e^t\dot{y} - \frac{1}{t^2}y - \frac{1}{t}\dot{u}. \end{aligned} \tag{19}$$

Computation of the adjoint polynomials of (19) yields

$$\begin{aligned} p^* &= (s^*)^2 + e^t ts^* + \frac{1}{t} - e^t(t+1) \\ q^* &= -\ln ts^* + \frac{1}{t} - t. \end{aligned}$$

The polynomial r^* is an arbitrary polynomial of degree $n-1=1$, so we may assume it has a form

$$r^* = r_1^* s^* + r_0^*.$$

Though by Lemma 2, $r_0^* + \dot{r}_1^* = r_0$, it is not necessary to know the precise values of r_1^* and r_0^* at this stage of computations, they have rather to be chosen later, during the integration process in order to make the subspace $\mathcal{V}$ integrable, if possible. By formula (13) we obtain

$$\begin{aligned} \tilde{\omega}_1 &= p_1^*\mathrm{d}y + q_1^*\mathrm{d}u + r_1^*\mathrm{d}t = e^t t\mathrm{d}y - \ln t\mathrm{d}u + r_1^*\mathrm{d}t \\ \tilde{\omega}_2 &= p_2^*\mathrm{d}y = \mathrm{d}y, \end{aligned}$$

and from (17)

$$\begin{aligned} \omega_2 &= \tilde{\omega}_2 = \mathrm{d}y \\ \omega_1 &= \dot{\omega}_2 + \tilde{\omega}_1 = \mathrm{d}\dot{y} + e^t t\mathrm{d}y - \ln t\mathrm{d}u + r_1^*\mathrm{d}t \end{aligned}$$

The next task is to find the polynomial r^* so that $\mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2\}$ would be integrable, if possible. Observe that $\omega_1 = \mathrm{d}(\dot{y} + e^t ty - u\ln t)$, if we take $r_1^* := e^t t + e^t - \frac{u}{t}$. It means ω_1 and ω_2 are both exact and one may choose differentials of the state coordinates as

$$\begin{aligned} \mathrm{d}x_1 &= \omega_1 = \mathrm{d}(\dot{y} + e^t ty - u\ln t) \\ \mathrm{d}x_2 &= \omega_2 = \mathrm{d}y \end{aligned}$$

yielding the state-space equations

$$\begin{aligned} \dot{x}_1 &= \frac{1}{t}(t^2-1)u + \frac{1}{t}(e^t t^2 + e^t t - 1)x_2 + \ln\dot{u} - \dot{u}\ln t \\ \dot{x}_2 &= x_1 - \ln tu - e^t tx_2 \\ y &= x_2 \end{aligned}$$

where $x_1 = \dot{y} + e^t ty - u\ln t$ and $x_2 = y$.

4. MINIMAL REALIZATION

Like in (Zhang et al., 2010), we call the realization minimal if it has the smallest order (i.e. the smallest number of state variables) among all realizations having the same generalized transfer function. To obtain the minimal realization, the i/o equation (1) has to be in the irreducible form. Irreducibility is defined using the notion of autonomous element.

Definition 8. (Conte et al., 2007) A non-constant function $\varphi_{\mathrm{red}} \in \mathcal{K}$ is said to be an *autonomous variable* for system (1) if there exist an integer $\mu \geqslant 1$ and a non-zero meromorphic function F such that

$$F(\varphi_{\mathrm{red}}, \varphi_{\mathrm{red}}^{(1)}, \ldots, \varphi_{\mathrm{red}}^{(\mu)}) = 0. \tag{20}$$

The equation (20) is called an *autonomous differential equation* of the system (1).

Definition 9. The system (1) is said to be *irreducible* if there does not exist any non-constant autonomous variable in $\mathcal{K}$.

In case the system admits an autonomous variable, the system can be reduced. That is, one may find the new, lower order system, which is transfer equivalent with the original system.

Example 10. Consider the following motivating example

$$\varphi := \ddot{y}u - \dot{y}\dot{u} + \dot{y}u = 0. \tag{21}$$

According to the results of (Zhang et al., 2010) the system (21) is not reducible though it admits an autonomous variable $\varphi_{\mathrm{red}} := \dot{y}/u$, satisfying the relation

$$\varphi = kF(\varphi_{\mathrm{red}}, \dot{\varphi}_{\mathrm{red}}) = u^2\left[\varphi_{\mathrm{red}} + \dot{\varphi}_{\mathrm{red}}\right]. \tag{22}$$

The source of discrepancy is that $\varphi_{\mathrm{red}} = \dot{y}/u = 0$ yields a degenerate system $\dot{y} = 0$. This again yields that the minimal realization of (21), i.e.

$$\begin{aligned} \dot{x}_1 &= ux_2 \\ \dot{x}_2 &= -x_2 \\ y &= x_1 \end{aligned}$$

is not an accessible system being in contradiction with the classical control theory.

Note that the convention $\varphi_{\mathrm{red}} = 0$ comes from the assumption that $F(0, \ldots, 0) = 0$, meaning that zero is one (constant) solution of the homogeneous differential equation. Though this is an assumption often made in nonlinear control theory, for instance in (Conte et al., 2007; Zhang et al., 2010) and is also valid in (22), it is far from being the only possible choice and may be relaxed.

Observe that from $\varphi = 0$ (and $u \neq 0$) we obtain

$$\dot{\varphi}_{\mathrm{red}} + \varphi_{\mathrm{red}} = 0.$$

The solution of this homogeneous differential equation is $\varphi_{\mathrm{red}} = C\,e^{-t}$, where C is an integrating constant. Therefore, the reduced i/o equation, in general, is

$$\varphi_{\mathrm{red}} = \frac{\dot{y}}{u} = C\,e^{-t}, \tag{23}$$

where $C \in \mathbb{R}$ is an arbitrary constant. If we choose $C = 0$, we arrive at the situation described above. However, the general realization of the i/o equation (23) is

$$\begin{aligned} \dot{x} &= C\,e^{-t}u \\ y &= x \end{aligned} \tag{24}$$

that is a linear time-varying system. The realization (24) is not unique but depends on constant C. It is because we have abandoned the standard assumption $\varphi(0, \ldots, 0) = 0$. If we fix the initial conditions of the autonomous differential equation (20), then C will acquire the specific value. Note also that the choice $C = 0$ is the only one that does not take us out from the set of time-invariant systems. For every other choice the reduced i/o equation will depend explicitly on time t. To conclude, the typical assumption $F(0,\dots,0)=0$ allows in majority of cases to simplify the exposition of the theory, and is often made by this reason. However, in the nonlinear context this assumption is not always natural or even not a valid choice.

Example 11. Consider for example the system

$$\varphi = \ddot{y}\dot{y}u - \dot{y}^2\dot{u} + u^3 = 0 \tag{25}$$

yielding

$$\varphi = u^2\dot{y}\left[\dot{\varphi}_{\rm red} + \frac{1}{\varphi_{\rm red}}\right], \varphi_{\rm red} = \frac{\dot{y}}{u}.$$

Since the assumption $F(0,0)=0$ can not be used here, one has to solve the differential equation $\dot{\varphi}_{\rm red} + \frac{1}{\varphi_{\rm red}}$ yielding $\varphi_{\rm red} = \pm\sqrt{2(C_1 - t)}$.

$$\varphi_{\rm red} = \frac{\dot{y}}{u} = \pm\sqrt{2(C_1 - t)}$$

The realization of (25) has the form

$$\begin{aligned}\dot{x} &= \pm\sqrt{2(C_1 - t)}u\\ y &= x.\end{aligned}$$

Example 12. Consider the equation

$$\begin{aligned}\varphi := -9u\dot{u}^2 - \dot{u}^2\ddot{u} + 2u\ddot{u}^2 - 9\dot{u}^2\ddot{y} + 2\ddot{u}^2\ddot{y} -\\ u\dot{u}u^{(3)} - \dot{u}\ddot{y}u^{(3)} - 2\dot{u}\ddot{u}y^{(3)} + \dot{u}^2y^{(4)} = 0.\end{aligned} \tag{26}$$

The latter equation can be represented in the form

$$\varphi = \dot{u}^3[\ddot{\varphi}_{\rm red} - 9\varphi_{\rm red}] = 0, \tag{27}$$

where

$$\varphi_{\rm red} = \frac{u + \ddot{y}}{\dot{u}}.$$

In (27) the function $F = \ddot{\varphi}_{\rm red} - 9\varphi_{\rm red}$ is a special case of the linear differential equation with constant coefficients

$$a_n\varphi_{\rm red}^{(n)} + \dots + a_1\dot{\varphi}_{\rm red} + a_0\varphi_{\rm red} = 0. \tag{28}$$

The solution of it is determined by the roots of the characteristic equation $a_n\lambda^n + \dots + a_1\lambda + a_0 = 0$. If all the roots $\lambda_1,\dots,\lambda_n$ are real and distinct, then the solution of (28) has the form

$$\varphi_{\rm red} = C_1e^{\lambda_1 t} + \dots + C_ne^{\lambda_n t},$$

where $C_1,\dots,C_n$ are arbitrary constants. Thus,

$$\varphi_{\rm red} = \frac{u + \ddot{y}}{u} = C_1e^{3t} + C_2e^{-3t},$$

then the minimal realization of i/o equation (26) is

$$\begin{aligned}\dot{x}_1 &= (C_1e^{3t} + C_2e^{-3t})u + x_2\\ \dot{x}_2 &= (-1 - 3C_1e^{3t} + 3C_2e^{-3t})u\\ y &= x_1,\end{aligned}$$

where $x_1 = y$ and $x_2 = -(C_1e^{3t} + C_2e^{-3t})u + \dot{y}$.

Example 13. If F is a nonlinear differential equation with respect to $\varphi_{\rm red}$, then the closed-form solution, in general, may be more complicated or does not exist at all, though some simple cases the analytic solution for $\varphi_{\rm red}$ may still be found. For instance, consider the equation

$$\begin{aligned}\varphi := 2\dot{u}^2\ddot{y} + 2\dot{u}\ddot{y}^2 - (1+u)(\ddot{u}\ddot{y} + 2\dot{u}y^{(3)} + 2\ddot{y}y^{(3)})\\ +(1+u)^2y^{(4)} = 0,\end{aligned} \tag{29}$$

yielding

$$\varphi = (1+u)^3\left[\ddot{\varphi}_{\rm red} - 2\dot{\varphi}_{\rm red}\varphi_{\rm red}\right] = 0,\ \varphi_{\rm red} = \frac{\ddot{y}}{1+u}.$$

Then

$$\varphi_{\rm red} = C_1\tan[C_1(t + C_2)],$$

and the realization of (29) is

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= C_1\tan[C_1(t + C_2)](1+u)\\ y &= x,\end{aligned}$$

where $x_1 = y$ and $x_2 = \dot{y}$.

Example 14. Consider the Example 5.4 from (Zhang et al., 2010):

$$\varphi := u\ddot{y} - \dot{u}\dot{y} = 0. \tag{30}$$

It is easy to check that

$$\varphi = u^2\dot{\varphi}_{\rm red},\ \text{ where }\ \varphi_{\rm red} = \frac{\dot{y}}{u}.$$

Solving the equation $\dot{\varphi}_{\rm red} = 0$ yields $\varphi_{\rm red} = C$, thus the reduced i/o equation has the form $\dot{y} = Cu$, $u \neq 0$, and the minimal state equations are

$$\begin{aligned}x &= Cu\\ y &= x.\end{aligned}$$

The latter system is time-invariant, depending on arbitrary constant C. This example shows that the general solution of the autonomous differential equation does not always cause the reduced equation to be time-varying.

Note that taking $\varphi_{\rm red} = 0$ (see (Zhang et al., 2010)), yields $\dot{y}/u = 0$ and we obtain the degenerate i/o equation $\dot{y} = 0$. The paper (Zhang et al., 2010) demonstrates via this example that minimality (in terms of the state dimension n) does not necessarily yield accessibility. The last property may be related to the following fact. Though the i/o equation may be reducible generically (like in case of (30)), it may happen that at a specific 'singular' point the i/o equation is not reducible. We conjecture that extending the assumption $F(0,\dots,0) = 0$ as done in this paper, one may abandon the above discrepancy, and generically, the minimal realization is accessible.

5. CONCLUSION

The reduction process of the i/o equation is related to the solutions of the autonomous differential equation (20). In earlier papers usually only trivial solution was considered, yielding $\varphi_{\rm red} = 0$. Allowing the nonzero solution points to a somewhat surprising fact that the reduced equation of the time-invariant nonlinear system may be, in general, a time-varying system. However, it is important to stress that in case of arbitrary differential equation (20) the close-form solution can not be found. Example 14 solves the seeming contradiction in (Zhang et al., 2010), where it was claimed that the minimal realization is not necessarily accessible. The approach used in our paper allows to unravel this discrepancy with the linear case.

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