Convex Synthesis of Multivariable Static Discrete-time Anti-windup via the Jury-Lee Criterion

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Abstract
Due to its ease of application, the circle criterion has been widely used to guarantee the stability of many anti-windup schemes. While the Popov criterion gives less conservative results, it has been conjectured in the literature that it cannot be used for convex anti-windup synthesis. This paper shows that the conjecture does not necessarily apply in the discrete-time setting. We show how the search for optimal parameters corresponding to the Jury-Lee criterion (a discrete counterpart of the Popov criterion) can be formulated as a convex search via a linear matrix inequality (LMI). The result is then extended to two existing multivariable static anti-windup schemes with stable open-loop plants. Two numerical examples of multivariable anti-windup controller synthesis are provided, and it is shown that in both cases the synthesis using the Jury-Lee criterion can allow better performance than existing methods which use the circle criterion alone.

1. INTRODUCTION

Most practical control systems which are designed based on linear theory have to deal with physical constraints such as saturations on the actuators. When the outputs of the controllers reach their limitations, so-called windup effects can take place. These might, in turn, cause performance degradation, large overshoots in the output and sometimes instability (Campo and Morari [1990], Kothare et al. [1994]). These phenomena have been observed since 1950's in both analog (Lozier [1956]) and digital (Fertik and Ross [1967]) control loops.

Techniques for addressing the windup effects have been widely studied in the continuous-time domain and numerous anti-windup schemes have been developed to improve the stability and performance of the controllers. Most of the traditional design techniques developed are either based on static (zero order) (Hanus et al. [1987], Wada and Saeki [1999], Saeki and Wada [2002], Mulder et al. [2001], Marcopoli and Phillips [1996]), or dynamic (low-order and full-order) (Turner and Postlethwaite [2004] Grimm et al. [2003] Zheng et al. [1994]) anti-windup compensators. Furthermore, as the extension to discrete-time setting appears straightforward, almost all existing continuous-time anti-windup schemes have their own digital versions (for example Syaichu-Rohman and Middleton [2004], Massimetti et al. [2009], Hermann et al. [2006]). It has been argued (e.g. Saeki and Wada [2002],Turner and Postlethwaite [2004]) that static anti-windup may be the most desirable structure from a practical point of view. Moreover, most practical controllers nowadays are implemented digitally using computers, which increases the importance in designing the anti-windup in discrete-time.

In early studies, a great deal of stability analysis was done for closed-loop systems having sector-bounded nonlinearities, both in continuous- and discrete-time domains. This has led to the derivation of various stability tests such as the circle, off-axis circle, and Popov criteria and the use of Zames-Falb multipliers. The extension of the theorems to stability analysis of existing anti-windup schemes has also been widely considered (see, for examples, Pittet et al. [1997], Feron et al. [1996], Kothare and Morari [1999] in continuous-time, and Cao and Lin [2003] in discrete-time). In Kothare and Morari [1999], the stability analysis of multivariable anti-windup designs is presented in a unified multiplier framework with the circle, off-axis circle, and Popov criteria and the use of Zames-Falb multipliers as special cases. It is also shown in their paper that the Zames-Falb multiplier gives the best stability margin even though the search for an optimal solution via the approach may be computationally intractable (i.e. it is non-convex).

As for the anti-windup design problems, most of the continuous-time anti-windup schemes base their synthesis on the circle criterion and/or use a quadratic Lyapunov function (Cao and Lin [2003], Weston and Postlethwaite [1998], Marcopoli and Phillips [1996], Mulder et al. [2001]). Similar techniques are also incorporated into digital anti-windup schemes (e.g. Massimetti et al. [2009], Grimm et al. [2008], Pan and Kapila [2002], Hermann et al. [2006]

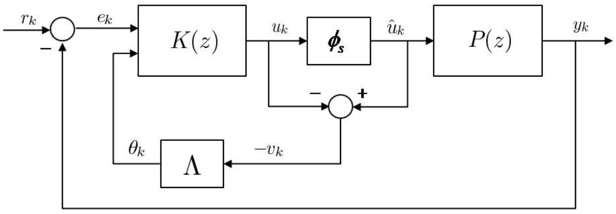


Figure 1. General static anti-windup scheme.

Gomes da Silva and Tarbouriech [2006]) to achieve stability. This is mainly due to their ease of applications and the convexity obtained during the synthesis. Hence, the focus of the anti-windup design in both time domains has been mostly devoted to satisfying certain $l_2(\mathcal{L}_2)$ or H_∞ performance requirements (Wada and Saeki [1999], Marcopoli and Phillips [1996], Mulder et al. [2001], Hermann et al. [2006]). On the other hand, it has been observed that other less conservative criteria such as the Popov criterion do not lead to convex formulation when used for synthesis (see Gomes da Silva and Tarbouriech [2006], Kapila et al. [2001], Weston and Postlethwaite [1998] and Feron et al. [1996] for discussion on this). This has also been stressed in Grimm et al. [2003] if an attempt is made to use the quadratic-plus-integral Lyapunov function (which is often associated with Popov criterion) for synthesis purposes. In Kapila et al. [2001], however, the Popov criterion is used in the anti-windup schemes but only sub-optimal solutions can be found. In summary, to the best of the authors knowledge, there is not much attention being directed towards applying criteria other than the circle for anti-windup synthesis due to the difficulties in achieving the optimal solutions.

In this paper, the focus is on the synthesis of static discrete-time anti-windup schemes with stable open-loop plants. The novelty of this paper is that the static anti-windup controller synthesis problem using the Jury-Lee criterion (Jury and Lee [1964a], Jury and Lee [1964b]), which is a discrete-time counterpart of the Popov criterion, is formulated into a convex search over an LMI where an optimal solution can be found. This directly shows that the conjecture of the Popov criterion leading to nonconvex solution does not necessarily apply in the discrete-time setting. The new stability criterion is then extended to existing anti-windup schemes in the literature which follow the conventional two-step paradigm: the linear controller is designed first ignoring the saturation and the anti-windup compensation is added to attenuate the performance degradation resulting from the saturation (Wada and Saeki [1999], Marcopoli and Phillips [1996]).

This paper is structured as follows: Section 2 presents the problem formulation of a standard static anti-windup scheme where the static gain is fed back into the controller's input. Section 3 formulates the anti-windup stability conditions via the Jury-Lee criterion into an LMI. The result of Section 3 is extended to two existing static anti-windup schemes (Wada and Saeki [1999], Marcopoli and Phillips [1996]) in Section 4. In Section 5, we provide some numerical examples of multivariable anti-windup controller synthesis to compare the performance of the Jury-Lee criterion and the circle criterion under given performance requirements. The conclusion is given in the last section.

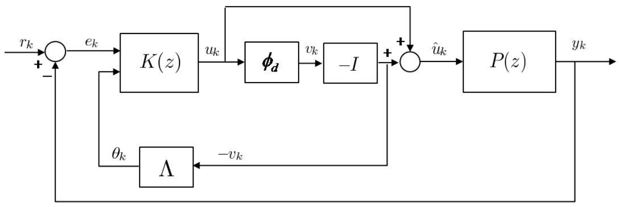


Figure 2. Equivalent representation of Figure 1.

The notation used in this paper is standard throughout. We denote x_k for $x(k)$ and $G^*(z)$ as the complex conjugate of $G(z)$. If $M \in \mathbb{C}^{p\times p}$, we write $\mathrm{He}(M) = M + M^*$. We omit the upper triangle half of the Hermitian matrix M as it is always the complex conjugate of the lower triangle half.

2. PROBLEM FORMULATION

Figure 1 shows a standard static anti-windup scheme (Wada and Saeki [1999], Hermann et al. [2006]) with a stable, strictly proper plant $P(z)$

$$\begin{aligned} x^p_{k+1} &= A_p x^p_k + B_p \hat{u}_k \\ y^p_k &= C_p x^p_k \end{aligned} \tag{1}$$

and a controller $K(z)$

$$x^c_{k+1} = A_c x^c_k + B_c e_k + B_c \theta_k \tag{2}$$

$$u_k = C_c x^c_k + D_c e_k. \tag{3}$$

where $x^p_k \in \mathbb{R}^{np}$, $x^c_k \in \mathbb{R}^{nc}$,$u_k \in \mathbb{R}^{nu}$ and $y_k \in \mathbb{R}^{ny}$. The saturation nonlinearity is described as $\hat{u}_k = \phi_s(u_k)$ where

$$(\phi_s(u_k))_i = \begin{cases} -1 & \text{for } u^i_k < -1 \\ u^i_k & \text{for } -1 \le u^i_k \le 1 \\ 1 & \text{for } u^i_k > 1. \end{cases} \tag{4}$$

When there is no saturation, the system will act linearly since the controller output u_k will be the same as the plant input $\hat{u}_k$. However, when the controller output reaches the saturation levels, the difference between u_k and $\hat{u}_k$ will be fed back into the input of the controller via a static gain $\Lambda \in \mathbb{R}^{nc\times nu}$ as $\theta_k = -\Lambda v_k = \Lambda(\hat{u}_k - u_k)$. It is standard to represent the loop around the saturation ϕ_s as the deadzone nonlinearity ϕ_d as shown in Figure 2 (Wada and Saeki [1999] Mulder et al. [2001] Marcopoli and Phillips [1996]). The deadzone function ϕ_d can be expressed as $\phi_d(u_k) = u_k - \phi_s(u_k)$. Hence $\phi_d(u_k) = \left[(\phi_d(y_k))_1, \cdots, (\phi_d(y_k))_p\right]^T$.

Since we assume the plant is given and the controller has already been designed first to achieve acceptable performance in the unsaturated region, the only design parameter is the static gain Λ. Therefore, the problem formulation is to optimize the static gain which can minimize the effect of the nonlinearity (in some sense) while preserving the stability. In the next section, we will show how the Jury-Lee criterion can be formulated into a convex search in the anti-windup synthesis problem.

3. ANTI-WINDUP STABILITY CONDITIONS

The closed-loop system as shown in Figure 3 consists of a stable, strictly proper LTI plant $\tilde{G}(z)$ in negative feedback with a static nonlinearity ϕ. To guarantee the stability of the system, we begin with the set of nonlinearities described as follows:

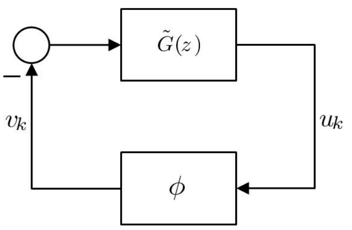


Figure 3. Discrete-time Lure system

Let $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a memoryless (static) diagonal nonlinearity $\phi(u_k) = \left[\phi_1(u_k^1), \ldots, \phi_p(u_k^p)\right]^T$. Each ϕ_i lies within the sector bound $[0, 1]$, i.e.

$$\phi_i(u^i)(\phi_i(u^i) - u^i) \leq 0 \quad \forall u^i, \quad i = 1, \ldots, p. \tag{5}$$

The nonlinearity is also monotonic and slope-restricted by $[0, 1]$, which can be described as

$$0 \leq \frac{\phi_i(u^i) - \phi_i(\hat{u}^i)}{u^i - \hat{u}^i} \leq 1 \quad \forall \hat{u^i} \neq u^i, \quad i = 1, \ldots, p. \tag{6}$$

Now define Φ to be the set of ϕ which satisfies both (5) and (6). To provide the stability conditions for the closed-loop system (1)-(4), Figure 2 can be transformed into a Lur'e system framework as shown in Figure 3 where $\phi_d = \phi \in \Phi$, and

$$\tilde{G}(z) \sim \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & 0 \end{array}\right] \tag{7}$$

with

$$A = \begin{bmatrix} A_p - B_p D_c C_p & B_p C_p \\ -B_c C_p & A_c \end{bmatrix} \tag{8}$$

$$B_1 = B_a \Lambda - B_v \text{ with } \quad B_v = \begin{bmatrix} -B_p \\ 0 \end{bmatrix} \text{ and } B_a = \begin{bmatrix} 0 \\ B_c \end{bmatrix}; \tag{9}$$

$$C_1 = \begin{bmatrix} -D_c C_p & C_c \end{bmatrix}. \tag{10}$$

The following theorem corresponds to the Jury-Lee criterion in the frequency domain.

Theorem 3.1. (Jury and Lee [1964b, 1966]) Let the LTI system $\tilde{G}(z)$ which is stable and strictly proper be connected to a monotonic, sector- and slope-restricted nonlinearity ϕ as shown in Figure 3. The closed-loop system is stable if there exist $\delta > 0$ and $N_+ = \text{diag}(n_+^1, \ldots, n_+^p)$ with $n_+^i \geq 0$ such that the following frequency domain condition is satisfied:

$$\text{He}\big[I + [I + (z-1)N_+]\tilde{G}(z) - \frac{1}{2}|z-1|^2 \tilde{G}^*(z) N_+ \tilde{G}(z)\big] \geq \delta I \tag{11}$$

$\forall |z| = 1$.

Proof. The proof (and its graphical interpretation) for the SISO case can be found in Jury and Lee [1964b], with the extension to the MIMO cases in Jury and Lee [1966] and Ahmad et al. [2013].

The next corollary provides a generalized Jury-Lee condition and its equivalent time-domain condition in matrix inequality form.

Corollary 3.1. Define $W = \text{diag}(w_1, \ldots, w_p)$ and $N_+ = \text{diag}(n_+^1, \ldots, n_+^p)$ with $w_i > 0, n_+^i \geq 0$ and $N_1 = W N_+$. Consider the system in Figure 3 with $\tilde{G}(z)$ described in (7)-(9), in negative feedback with $\phi \in \Phi$. The closed-loop system (1)-(4) is stable if there exist $W > 0$, $\delta > 0$ and $N_1 \geq 0$ such that the frequency-domain condition below is satisfied:

$$\text{He}\big[W + [W + (z-1)N_1]\tilde{G}(z) - \frac{1}{2}|z-1|^2 \tilde{G}^*(z) N_1 \tilde{G}(z)\big] \geq \delta I \tag{12}$$

$\forall |z| = 1$, or equivalently, in the time domain, if there exist $P > 0$, $W > 0$, $\delta > 0$ and $N_1 \geq 0$ such that

$$M_0 + \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 < 0 \tag{13}$$

with

$$M_0 = \begin{bmatrix} A^T P A - P & A^T P B_1 \\ B_1^T P A & B_1^T P B_1 \end{bmatrix}; \quad \tilde{M}_1 = \begin{bmatrix} 0 & -C_1^T W \\ -W C_1 & \delta I - 2W \end{bmatrix} \tag{14}$$

$$\tilde{M}_2 = \begin{bmatrix} (A-I)^T C_1^T N_1 C_1 (A-I) & (A-I)^T C_1^T N_1 C_1 B_1 \\ B_1^T C_1^T N_1 C_1 (A-I) & B_1^T C_1^T N_1 C_1 B_1 \end{bmatrix} \tag{15}$$

$$\tilde{M}_3 = \begin{bmatrix} 0 & -(A-I)^T C_1^T N_1 \\ -N_1 C_1 (A-I) & -N_1 C_1 B_1 - B_1^T C_1^T N_1 \end{bmatrix}. \tag{16}$$

In this case, the associated Lur'e-Lyapunov function of the system can be formed as

$$V(x_k) = x_k^T P x_k + 2 \int_0^{u_k} \phi(\sigma)^T N_1 \, d\sigma. \tag{17}$$

Proof. See Premaratne and Jury [1994], Ahmad [2012] and the references therein for the matrix inequality derivation.

For analysis purposes, when Λ (or B_1) is fixed, the matrix inequality (13)-(16) is convex as it is an LMI in variables P, W, N_1 and δ. However, for synthesis, it is not convex since there are products of variables (P and B_1 in (14), N_1 and B_1 in (15)-(16)). Therefore the search for an optimal static gain Λ is not straightforward. The next lemma shows how the matrix inequality (13)-(16) can be formulated into a convex search.

Lemma 3.1. Let M_0, $\tilde{M}_1$, $\tilde{M}_2$, and $\tilde{M}_3$ be defined as in (14)-(16) and let $\tilde{B} = B_a X - B_v M$, $Q = P^{-1} > 0$. For $N_1 > 0$, let $R_1 = N_1^{-1} > 0$, $M = \text{diag}(M_1, \ldots, M_p) = W^{-1} > 0$, $U = (\delta I - N_1)^{-1} > 0$ and $X = \Lambda M$. Then (13) may be expressed as:

$$\begin{bmatrix} -Q & * & * & * & * \\ -C_1 Q & -2M & * & * & * \\ AQ & \tilde{B} & -Q & * & * \\ C_1(A-I)Q & C_1\tilde{B} - M & 0 & -R_1 & * \\ 0 & M & 0 & 0 & -U \end{bmatrix} < 0. \tag{18}$$

This is an LMI in variables Q, R_1, U, X and M.

Proof. Let

$$M_N = \begin{bmatrix} 0 & 0 \\ 0 & -N_1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} A & B_1 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} C_1(A-I) & C_1 B_1 - I \end{bmatrix}.$$

From (13), taking the Schur complements gives

$$\begin{bmatrix} \tilde{M}_1 + M_N & V_1^T & V_2^T \\ V_1 & -Q & 0 \\ V_2 & 0 & -R_1 \end{bmatrix} < 0. \tag{19}$$

Applying a congruence transformation with $\text{diag}(Q, M, I, I)$ and another Schur complement to remove the quadratic term leads to (18).

The result in Lemma 3.1 is the foundation of the anti-windup synthesis in the next section.

Remark 3.1. In Lemma 3.1, N_1 needs to be positive definite to ensure the existence of R_1. Even though it can be made sufficiently small, it may lead to some numerical errors in the computation. Therefore, we need to compute the LMI for $N_1 > 0$ and the LMI for $N = 0$ separately for the application of the Jury-Lee criterion.

Remark 3.2. In the continuous-time domain, the Lyapunov function corresponding to the Popov criterion takes a form similar to (17), but the resulting matrix inequality is quite different from (13)-(16), which is given as follows (Haddad and Bernstein [1993])

$$\begin{bmatrix} A^T P + PA & PB - C^T W - A^T C^T N \\ B^T P - WC - NCA & \delta I - 2W - NCB - B^T C^T N \end{bmatrix} < 0. \tag{20}$$

In the anti-windup strategy, it is a standard approach to apply the congruence transformation in order to obtain a convex search (see Mulder et al. [2001] for example). However, if the congruence transformation is applied to the matrix inequality in (20), the result will be in bilinear matrix inequaliy (BMI) (see Kapila et al. [2001] for details). This is not the case in the discrete-time setting because the LMI (13) has an extra term ($\tilde{M}_2$) and hence the convex formulation (18).

4. ANTI-WINDUP STABILITY AND PERFORMANCE

The circle criterion has been used to guarantee the stability of many anti-windup schemes in the literature. Most of the schemes differ by their performance requirements which depend on the goals of the controllers. In the following subsections, we extend the stability condition in Lemma 3.1. which is based on the Jury-Lee criterion to two existing anti-windup designs: (i) Design 1 - based on H_∞ norm performance condition (Wada and Saeki [1999]) and (ii) Design 2 - based on the induced l_2 norm performance objective (Marcopoli and Phillips [1996], Mulder et al. [2001]).

4.1 Anti-windup Design 1

In this design (Wada and Saeki [1999]), the performance requirement is to attenuate the error resulting from the windup effect, which is done by making the transfer function from v_k to y_k small (see Figure 2). Let $G_p(z)$ be the transfer function between y_k and v_k; the goal is to make sure that $G_p(z)$ is always bounded by using the H_∞ norm. This is given in the next lemma which is based on the bounded-real lemma (Boyd et al. [1994]).

Lemma 4.1. (Performance condition) Given a stable transfer

$$G_p(z) \sim \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & 0 \end{array}\right], \tag{21}$$

where A and B_1 are defined as in (8) and (9) respectively, and $C_2 = [-C_p \quad 0]$. The following statements are then equivalent:

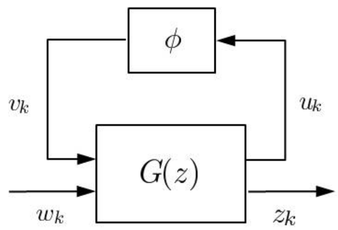


Figure 4. Generalized problem for stability and performance of static anti-windup scheme.

(i) $||G_p(z)||_\infty < \gamma^2$
(ii) there exists $Q_p > 0$, $M = diag(M_1, \ldots, M_p) = W^{-1} > 0$, and $\mu = \gamma^2 > 0$ such that the following LMI is satisfied.

$$\begin{bmatrix} -Q_p & * & * & * \\ 0 & -M^T M & * & * \\ C_2 Q_p & 0 & -\mu I & * \\ AQ_p & B_a X - B_v M & 0 & -Q_p \end{bmatrix} < 0. \tag{22}$$

If the LMI is feasible then $\Lambda = XM^{-1}$ is the static anti-windup compensator that can satisfy the performance requirement of the system.

Proof. See Wada and Saeki [1999].

To guarantee the stability and to satisfy the performance condition of the closed-loop system, both Lemmas 3.1 and 4.1 are combined in the following theorem.

Theorem 4.1. Let $F_1(Q, X)$ be the LMI in (18), and $F_2(Q_p, X)$ be the LMI in (22). Under the conditions of Lemma 3.1 and Lemma 4.1, if there exist $Q > 0$, $Q_p > 0$ and $M > 0$ such that the following LMI is satisfied

$$\begin{bmatrix} F_1(Q, X) & 0 \\ 0 & F_2(Q_p, X) \end{bmatrix} < 0, \tag{23}$$

then $\Lambda = XM^{-1}$ is the static anti-windup compensator that satisfies the stability and performance requirements of the closed-loop system in Figure 2 with $\phi_{dz} \in \Phi$.

Proof. The proof is straightforward.

Note that the parameter $M > 0$ in (23) is a prespecified parameter; therefore (23) is an LMI with respect to $Q > 0$, $Q_p > 0$ and X only.

4.2 Anti-windup Design 2

In this design (Marcopoli and Phillips [1996], Mulder et al. [2001]), the performance condition is based on the minimization of the induced l_2 norm objective. From Figure 4, the aim is to attenuate the error z_k with respect to the exogenous input, w_k as $\sup_{||w_k||_2 \neq 0} \frac{||z_k||_2}{||w_k||_2} \leq \gamma$. Here the error and the exogenous input are defined as $z_k = e_k$ and $w_k = r_k$ respectively (refer to Figure 2). From the generalized problem as shown in Figure 4 which is the transformation from Figure 2 with $\phi_{dz} = \phi$, the state space of $G(z)$ takes the form

$$G(z) \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \\ C_3 & D_{31} & D_{32} \end{array}\right], \tag{24}$$

with A, B_1, and C_1 defined as in (8)-(9) and

$$B_2 = \begin{bmatrix} B_p D_c \\ B_c \end{bmatrix}; \quad C_2 = \begin{bmatrix} -C_p & 0 \end{bmatrix}, \tag{25}$$

and $C_3 = D_{11} = D_{21} = D_{32} = 0$; $D_{12} = D_c$; $D_{22} = D_{32} = I$. The next theorem provides the anti-windup stability and performance which is derived using the quadratic-plus-integral Lyapunov function and the S-procedure technique.

Theorem 4.2. If there exist a matrix $Q > 0$, a scalar $\mu > 0$, and positive definite digonal matrices M, R_1, U with $Q, M, R_1, U \in \mathbb{R}^{p\times p}$ such that the following LMI is satisfied:

$$\begin{bmatrix} -Q & * & * & * & * & * & * \\ -C_1 Q & -2M & * & * & * & * & * \\ 0 & -D_c^T & -\mu I & * & * & * & * \\ AQ & \tilde{B} & B_2 & -Q & * & * & * \\ C_1 \tilde{A} Q & C_1 \tilde{B} - M & 0 & 0 & -R_1 & * & * \\ C_2 Q & 0 & I & 0 & 0 & -I & * \\ 0 & M & 0 & 0 & 0 & 0 & -U \end{bmatrix} < 0 \tag{26}$$

where $\tilde{B} = B_a X - B_v M$, $N_1^{-1} = R_1$, and $\tilde{A} = A - I$, then the system (1)-(4) is l_2 stabilizable for all $\phi(y_k) \in \Phi$ and has a weighted induced l_2 gain less than $\sqrt{\mu} = \gamma$. The static anti-windup compensator which stabilizes the closed-loop system is given by $\Lambda = XM^{-1}$.

Proof. See Ahmad [2012].

5. NUMERICAL EXAMPLES

Within the framework of Figure 1, we synthesize and compare anti-windup compensators for MIMO cases based on the Jury-Lee and the circle criteria.

5.1 Example 1: Application of Theorem 4.1

For the first example, we use the continuous-time plant $P(s)$ and the PI controller $K(s)$ as follows:

$$P(s) = \frac{10}{100s+1} \begin{bmatrix} 2 & 2.5 \\ 1.5 & 2 \end{bmatrix}, \quad K(s) = \frac{200s+2}{100s} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}.$$

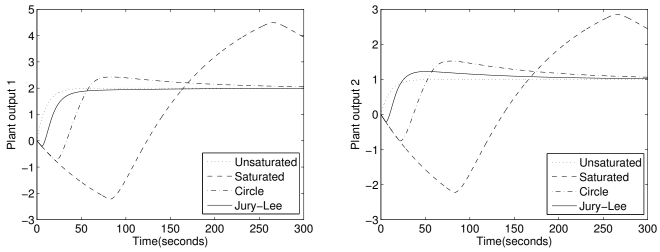
The discrete-time models $P(z)$ and $K(z)$ are obtained via the zero-order-hold method with a sampling time of $T = 0.01s$, and a delay of z^{-2} is imposed on the plant $P(z)$. Theorem 4.1 is applied to the closed-loop system with $M = 100I$ (since it is feasible), giving an H_∞ norm bound of $\gamma_J = 3.5274$. The circle criterion alone gives $\gamma_C = 3.6904$. The responses are shown in Figure 5. As can be observed, although the norm bounds γ_C and γ_J are quite similar, there is a clear improvement in the response.

5.2 Example 2: Application of Theorem 4.2

The continuous-time plant $P(s)$ and the PI controller $K(s)$ are chosen as follows:

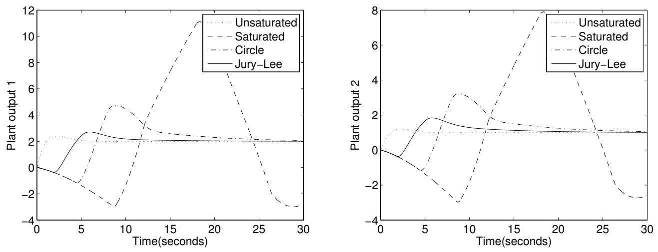
$$P(s) = \frac{10s+1}{100s^2+2s} \begin{bmatrix} 2 & 2.5 \\ 1.5 & 2 \end{bmatrix}, \quad K(s) = \frac{300s+9}{100s} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

and the discrete-time models $P(z)$ and $K(z)$ are obtained via the zero-order-hold method with a sampling time of $T = 0.05s$, with a delay of z^{-5} imposed on the plant $P(z)$. Applying Theorem 4.2 with LMI (26) where $N_1 \geq 0$ gives $\gamma_J = 0.2656$ whereas when the circle criterion alone



(a) Plant output 1. (b) Plant output 2.

Figure 5. Example 1: Application of Theorem 4.1 with $M = 40I$



(a) Plant output 1. (b) Plant output 2.

Figure 6. Example 2: Application of Theorem 4.2.

gives $\gamma_C = 0.4475$. The outputs of the plant are shown in Figure 6. Once again there is a clear improvement in the response.

6. CONCLUSIONS

In this paper, the static discrete-time anti-windup synthesis problem via the Jury-Lee criterion is formulated into a convex search over an LMI where an optimal solution can be found. We have shown that the conjecture of Popov criterion leading to nonconvex solution does not necessarily apply to the discrete-time setting. The result is then extended to two existing anti-windup schemes available in the literature. The circle criterion is a special case and we have demonstrated (in the examples) that the Jury-Lee criterion can give considerable improvement in the synthesis problem. The convex formulation in this paper may also be applied to the direct approach which accounts for the saturation nonlinearity throughout the design procedure (see Tyan and Bernstein [1995] for example), and also to the robustness of static anti-windup schemes such as those presented in Turner et al. [2007].

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