Nonlinear Stabilization in Infinite Dimension

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Abstract: Significant advances have taken place in the last few years in the development of control designs for nonlinear infinite-dimensional systems. Such systems typically take the form of nonlinear ODEs (ordinary differential equations) with delays and nonlinear PDEs (partial differential equations). In this article we review several representative but general results on nonlinear control in the infinite-dimensional setting. First we present designs for nonlinear ODEs with constant, time-varying or state-dependent input delays, which arise in numerous applications of control over networks. Second, we present a design for nonlinear ODEs with a wave (string) PDE at its input, which is motivated by the drilling dynamics in petroleum engineering. Third, we present a design for systems of (two) coupled nonlinear first-order hyperbolic PDEs, which is motivated by slugging flow dynamics in petroleum production in off-shore facilities. Our design and analysis methodologies are based on the concepts of nonlinear predictor feedback and nonlinear infinite-dimensional backstepping. We present several simulation examples that illustrate the design methodology.

1. INTRODUCTION

1.1 Motivation and historical background

The area of control design—most notably stabilization—for nonlinear finite-dimensional systems reached relative maturity around year 2000. The method of backstepping (Krstic et al (1995)), which played the central role in this development, particularly for systems with modeling uncertainties, then became the tool of interest for stabilization of infinite-dimensional systems. However, for almost a decade, the success in that direction remained limited to linear PDE (partial differential equation) systems (Krstic and Smyshlyaev (2008)). It is not until the last few years that this development has started yielding results for nonlinear infinite-dimensional systems.

The turning point in the development of control designs for nonlinear systems was the relatively little known two-part paper by Vazquez and Krstic (2008a,b) where nonlinear infinite-dimensional operators of a Volterra type, with infinite sums of integrals in the spatial variable (rather than in time, as has been common in the input-output representation theory for ODEs for decades), were introduced for stabilization of nonlinear PDEs of the parabolic type. This design represents a proper infinite-dimensional extension of backstepping (and feedback linearization) designs for nonlinear ODEs. The design involves the construction of the Volterra transformations whose kernel functions depend on increasing numbers of spatial variables (which go to infinity), and where the kernels are governed by PDEs in an increasing number of variables, on domains whose dimension goes to infinity, with the solutions of lower-order kernels being inputs to the PDEs for the higher-order kernels. This complex formulations turns out to be constructive and provably convergent, with a well-defined feedback law and a stability result in spatial norms that are appropriate for parabolic PDEs. All subsequent backstepping developments for infinite-dimensional nonlinear systems—whether for other PDE systems (Krstic et al. (2008, 2009)) or for nonlinear delay systems (Krstic (2010a))—are conceptually based on the technique laid out in (Vazquez and Krstic (2008a,b)), although all such subsequent developments have been much less complex as they have been for less broad classes of nonlinear infinite-dimensional systems than parabolic PDEs with right-hand sides that contain spatial Volterra nonlinear operators.

Though they carry with them a wealth of mathematical challenges, nonlinear infinite-dimensional systems are not artificial mathematical inventions or esoteric generalizations of nonlinear ODEs. They are as ubiquitous in applications as ODEs. In fact, in numerous problems involving mechanics, fluids, thermal phenomena, chemistry, or telecommunications, ODE models are merely approximations of full models that incorporate PDEs and/or delay effects.

The most elementary systems in the broad class of nonlinear infinite-dimensional systems are nonlinear systems with input delays. They arise in numerous applications such as networked control systems (Cloosterman et al. (2009), Heemels et al. (2010), Hespanha et al. (2007), Montestruque and Antsaklis (2004), Witrant et al. (2007)), supply networks (Sipahi et al. (2006), Sterman (2000)), milling processes (Altinas (1999)), irrigation channels (Litrico and Fromion (2004)), engine cooling systems (Hansen et al. (2011)) and chemical processes (Kravaris and Wright (1989), Mounier and Rudolph (1998)), to name only a few (see also the survey by Richard (2003) for additional examples).

Although a nonlinear system with an input delay is as simple a problem as it gets within the realm of infinite-dimensional nonlinear systems, the design of stabilizing control laws for *general* nonlinear systems and when the input delay is *arbitrarily* large, is a highly nontrivial task (Krstic (2010a)). The situation is even more intricate when the delay is time-varying (Krstic (2010b); Bekiaris-Liberis and Krstic (2012)), and becomes formidable when the delay depends on the state of the system itself (Bekiaris-Liberis and Krstic (2013)). Several

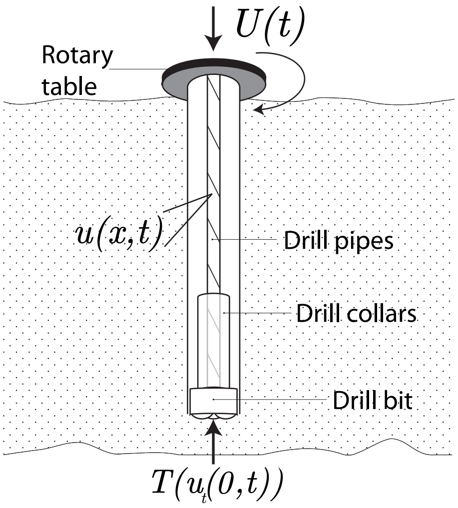

Fig. 1. A drillstring used in oil drilling. The angular displacement u of the drillstring is controlled through a torque U.

additional important results on the stabilization of nonlinear systems with input and state delays have been developed by Jankovic (2001, 2009), Karafyllis (2006, 2010), Karafyllis and Krstic (2012), Mazenc and Bliman (2006), Mazenc at al. (2004), Mazenc and Niculescu (2011).

Once the designer is equipped with the capability to overcome a delay at the input, i.e., the transport PDE process in the actuator line, there is every reason to ask whether other types of infinite-dimensional dynamics at the input can be compensated. This line of pursuit for infinite-dimensional dynamics in the actuator line of a linear ODE plant was pursued by Krstic (2009b) for diffusion-dominated (parabolic) actuator dynamics and by Krstic (2009c) for wave PDE actuator dynamics. Several extensions, all considering linear ODE plants preceded by PDE actuator dynamics, are presented by Bekiaris-Liberis and Krstic (2010), Bekiaris-Liberis and Krstic (2011b), Krstic (2009a), Ren et al. (2012), Susto and Krstic (2010), Tang and Xie (2011a,b). Extending those results from the case where the plant is a linear ODE to the case where the plant is a nonlinear ODE has proved much more challenging than for the case where the actuator dynamics are of the delay (transport PDE) type. Until recently, that is, as we show in this article and discuss next.

A representative engineering application in which wave PDE actuator dynamics are cascaded with a nonlinear ODE is oil drilling. A common type of instability in oil drilling is the so-called stick-slip oscillations (Jansen (1993)). This type of instability (which is caused by a specific composition of the ground material) results in torsional vibrations of the drillstring, which can in turn severely damage the drilling facilities (see Fig. 1 taken from Sagert et al. (2013)). The torsional dynamics of an oil drillstring are modeled as a wave PDE (that describes the dynamics of the angular displacement of the drillstring) coupled with a nonlinear ODE that describes the dynamics of the bottom angular velocity of the drill bit (Saldivar et al. (2011)). A control approach for the bottom angular velocity based on the linearization of its dynamics is presented in Sagert et al. (2013). In this article we present a design for general nonlinear ODE plants with a wave PDE as its actuator dynamics. This design solves the oil drilling problem (globally) as a special case.

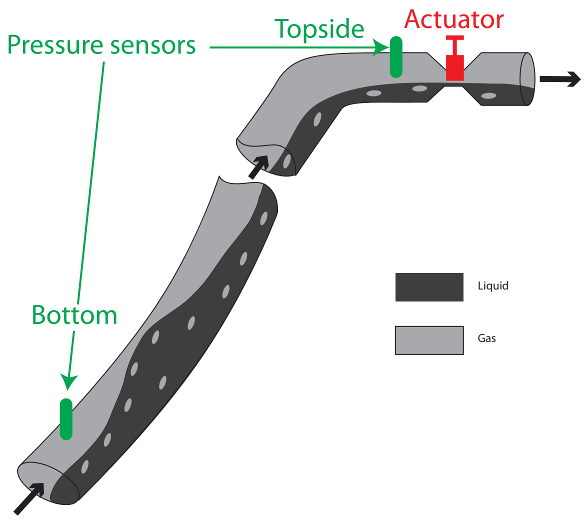

Fig. 2. An oil production pipe conveying oil and gas from a reservoir.

Once PDE-ODE cascades are systematically addressed, it is reasonable to ask a question whether interconnections of multiple PDEs can be controlled, and not only in the cascade configuration but in more general and strongly "interwoven" configurations. In fact, such problems arise in numerous physical systems and have been considered in the PDE control literature for at least a decade, albeit with limitations to the degree of open-loop instability that is permissible in the plant considered.

Systems of coupled, nonlinear first order hyperbolic PDEs model a variety of physical systems. Specifically, 2×2 systems of first order hyperbolic quasilinear PDEs model processes such as open channels (Dos Santos and Prieur (2008), Gugat and Leugering (2003), Gugat et al. (2004), Halleux et al. (2003)), transmission lines (Curro et al. (2011)), gas flow pipelines (Gugat and Dick (2011)) or road traffic models (Goatin (2006)). They also have some resemblances with systems that model the gas-liquid flow in oil production pipes (see Fig. 2 taken from Di Meglio et al. (2012b)). The problem of stabilization for some classes of 2×2 systems of first order hyperbolic quasilinear PDEs is considered by Coron et al. (2006), Dick et al. (2010), Dos Santos and Prieur (2008), Greenberg and Li (1984), Gugat and Hetry (2011), Prieur (2009), Prieur et al. (2008).

1.2 Contents of the article

In this paper we present some recent results on the *compensation* of input delays in nonlinear systems employing predictor-based control laws. Predictor feedback was developed originally for unstable linear plants with input delays, see the early paper by Artstein (1982) that conceptualizes the results of the preceding decade generalizes them in several mathematically interesting directions. Yet, a nonlinear counterpart of predictor feedback was unavailable until recently (Krstic (2010a)). The design by Krstic (2010a) is based on the introduction of a nonlinear infinite-dimensional backstepping transformation, which provides a Lyapunov functional for studying the stability of the closed-loop system. Although for linear systems with a time-varying input delay the formula of the predictor feedback law was provided by Nihtila (1991), for general nonlinear systems, predictor-based control laws were provided only recently by Bekiaris-Liberis and Krstic (2012). One of the most challenging problems in delay systems is the control of systems with state-dependent delays, as highlighted by Richard (2003). The first systematic approach for designing stabilizing controllers for nonlinear systems with state-dependent delays introduced by Bekiaris-Liberis and Krstic (2013). The design is based on predictor feedback. The key challenge that is resolved in Bekiaris-Liberis and Krstic (2013) is the definition of the predictor state: The state-dependence of the delay makes the prediction horizon dependent on future values of the state which are unavailable.

We also consider finite-dimensional nonlinear plants which are controlled through a string and we design a predictor-based feedback law that *compensates* the string (wave) dynamics in the input of the plant. Our design is based on a preliminary transformation which allows one to convert the problem of the compensation of the wave PDE, to a problem of the compensation of a 2×2 system of first order transport equations which convect in opposite directions (see, for example, Vazquez et al. (2011a)), for an augmented (by one integrator) plant. We then introduce the infinite-dimensional backstepping transformations for the two transport states, which transform the new, augmented system to a target system. With the aid of the backstepping transformations we prove global asymptotic stability of the closed-loop system by constructing a Lyapunov functional.

Finally, we review some recent results on the local exponential H_2 stabilization of a 2×2 system of first order hyperbolic quasilinear PDEs using backstepping developed by Coron et al. (2012) and Vazquez et al. (2011b). Specifically, we present the design of a control law that stabilizes the linearized system using the recently developed backstepping technique of Vazquez et al. (2011a) for 2×2 systems of linear hyperbolic PDEs (see also Di Meglio et al. (2012a) for an extension to $n\times n$ systems). We then prove the local exponential stability of the closed-loop system in the H_2 norm by constructing a strict Lyapunov functional with the aid of the backstepping transformations.

1.3 Oganization

Section 2 is devoted to nonlinear systems with input delays. We introduce the predictor-based design for constant delays in Section 2.1 For time-varying delays the predictor feedback design is presented in Section 2.2. State-dependent delays are treated in Section 2.3. In Section 3 we present a design that compensates the wave actuator dynamics in nonlinear systems. In Section 4 we are dealing with a 2×2 system of first order quasilinear PDEs for which we design a control law that achieves local exponential stability.

2. NONLINEAR SYSTEMS WITH INPUT DELAYS

One of the main obstacles in designing globally stabilizing control laws for nonlinear systems with long input delays is the finite escape phenomenon. The input delay may be so large that the control signal can not reach the plant before its state escapes to infinity. Therefore, in the following we assume that the plant $\dot{X} = f(X,\omega)$ is forward complete, that is, for every initial condition and every bounded input signal the corresponding solution is defined for all $t\geq 0$.

Our predictor-based designs are based on a (possibly time-varying) feedback law $\kappa(t,X(t))$, which is assumed to be periodic in its first argument and locally Lipschitz, that globally stabilizes the delay-free plant, i.e., $\dot{X}(t) = f(X(t),\kappa(t,X(t)))$ is globally asymptotically stable.

2.1 Constant delay

In this section we focus on nonlinear systems with constant input delay, i.e, systems of the form

$$\dot{X}(t) = f(X(t),U(t-D)). \tag{1}$$

The predictor-based control law for plant (1) is

$$U(t) = \kappa(t+D,P(t)) \tag{2}$$

$$P(t) = X(t) + \int_{t-D}^{t} f(P(\theta),U(\theta))\,d\theta, \tag{3}$$

where the initial condition for the integral equation for $P(t)$ is defined for all $\theta\in[t_0-D,t_0]$ (t_0 is the initial time which must be given because the closed-loop system is time-varying) as

$$P(\theta) = X(t_0) + \int_{t_0-D}^{\theta} f(P(\sigma),U(\sigma))\,d\sigma. \tag{4}$$

The signal $P(t)$ represents the D time-units ahead predictor of X, i.e., $P(t) = X(t+D)$. In the case of linear systems the predictor $P(t)$ is given explicitly using the variation of constants formula, with the initial condition $P(t-D) = X(t)$, as $P(t) = e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta$. For systems that are nonlinear, $P(t)$ cannot be written explicitly, for the same reason as a nonlinear ODE cannot be solved explicitly. So we represent $P(t)$ implicitly using the nonlinear integral equation (3). The computation of $P(t)$ from (3) is straightforward with a discretized implementation in which $P(t)$ is assigned values based on the right-hand side of (3), which involves earlier values of P and the values of the input U.

Together with the predictor-based control law (2) we define the infinite-dimensional backstepping transformation of the actuator state given by

$$W(t) = U(t) - \kappa(t+D,P(t)), \tag{5}$$

together with its inverse

$$U(t) = W(t) + \kappa(t+D,\Pi(t)), \tag{6}$$

where [1]

$$\Pi(t) = X(t) + \int_{t-D}^{t} f(\Pi(\theta),\kappa(\theta+D,\Pi(\theta))+W(\theta))\,d\theta, \tag{7}$$

with initial condition for all $\theta\in[t_0-D,t_0]$

$$\Pi(\theta) = X(t_0) + \int_{t_0-D}^{\theta} f(\Pi(\sigma),\kappa(\sigma+D,\Pi(\sigma))+W(\sigma))d\sigma. \tag{8}$$

The backstepping transformation maps the original system (1) into the "target system" given by

[1] The quantities P in (3) and Π in (7) are identical. However, we use two distinct symbols for the same quantity because, in one case, P is expressed in terms of X and U, for the direct backstepping transformation, while, in the other case, Π is expressed in terms of X and W, for the inverse backstepping transformation.

$$\dot{X}(t) = f(X(t), \kappa(t, X(t)) + W(t-D)) \quad (9)$$

$$W(t) = 0, \quad \text{for } t \geq t_0. \quad (10)$$

We have the following result. Its proof can be found in Krstic (2010a).

Theorem 1. Let $\dot{X} = f(X, \omega)$ be forward complete and $\dot{X}(t) = f(X(t), \kappa(t, X(t)))$ globally uniformly asymptotically stable. Consider the closed-loop system consisting of the plant (1) and the control law (2), (3). There exists a class $\mathcal{KL}$ function β such that for all initial conditions $X(t_0) \in \mathbb{R}^n$, $U(t_0 + \theta); \theta \in [-D, 0] \in L^\infty[-D, 0]$ the following holds

$$\Omega(t) \leq \beta(\Omega(t_0), t - t_0) \quad (11)$$

$$\Omega(t) = |X(t)| + \sup_{t-D \leq \theta \leq t} |U(\theta)|, \quad (12)$$

for all $t \geq t_0 \geq 0$.

If the global asymptotic stability assumption in Theorem 1 is strengthened with an input-to-state stability assumption of the plant $\dot{X}(t) = f(X(t), \kappa(t, X(t)) + \omega(t))$ with respect to ω, one can construct a Lyapunov functional [2] for the closed-loop system. Towards that end we observe from the "target system" (9), (10) that $W(t-D)$ vanishes in finite time (in D time-units). Hence, under the input-to-state stability assumption on the plant $\dot{X}(t) = f(X(t), \kappa(t, X(t)) + \omega(t))$ with respect to ω one can construct a Lyapunov functional for the system in the (X, W) variables. Using Malisoff and Mazenc (2005) there exists a C^1 function $S: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions α_1, α_2, α_3, α_4 such that

$$\alpha_3(|X(t)|) \leq S(t, X(t)) \leq \alpha_4(|X(t)|) \quad (13)$$

$$\dot{S}(t, X(t)) \leq -\alpha_1(|X(t)|) + \alpha_2(|W(t-D)|), \quad (14)$$

$$\dot{S}(t, X(t)) = \frac{\partial S(t, X(t))}{\partial t} + \frac{\partial S(t, X(t))}{\partial X} \times f(X(t), \kappa(t, X(t)) + W(t-D)). \quad (15)$$

The Lyapunov functional for the "target system" is then

$$V(t) = S(t, X(t)) + \frac{2}{c} \int_0^{L(t)} \frac{\alpha_2(r)}{r} dr, \quad (16)$$

where $\frac{\alpha_2(r)}{r}$ is a class $\mathcal{K}$ function or α_2 has been appropriately majorized so this is true (with no generality loss), $c > 0$ is arbitrary and

$$L(t) = \sup_{t-D \leq \theta \leq t} \left| e^{c(\theta - t + D)} W(\theta) \right|. \quad (17)$$

Using the inverse backstepping transformation (6) one can then prove stability in the original variables (X, U). The functional L can be also written directly in terms of the original variables (X, U) as

$$L(t) = \sup_{t-D \leq \theta \leq t} \left| e^{c(\theta - t + D)} (U(\theta) - \kappa(\theta + D, P(\theta))) \right|, \quad (18)$$

where P is given in terms of (X, U) from (3). The two different representations of the functional L, namely, representations (17) and (18), reveal one of the benefits of the backstepping transformation: If the construction of the functional L in terms of the transformed actuator state W appears to be non-trivial, its form in terms of the original variables (X, U), i.e., relation (18), is rather impossible to guess without the backstepping and predictor transformations.

[2] The availability of a Lyapunov functional enables one in principle, to study, robustness of the predictor feedback to parametric uncertainties, its disturbance attenuation properties, and the inverse-optimal re-design problem.

2.2 Time-varying delay

In this section we consider plants of the form

$$\dot{X}(t) = f(X(t), U(t - D(t))), \quad (19)$$

where D is a positive-valued continuously differentiable function of time. We define the functions

$$\phi(t) = t - D(t) \quad (20)$$

$$\sigma(t) = \phi^{-1}(t), \quad (21)$$

and we refer to the quantity $t - \phi(t) = D(t)$ as the delay time. This is the time interval that indicates how long *ago* the control signal that is currently affects the plant was actually applied. The main goal of this section is to determine the predictor state, i.e., the quantity P such that $X(\sigma(t)) = P(t)$. From now on we refer to the quantity $\sigma(t) - t$ as the prediction horizon. This is the time interval which indicates *after* how long an input signal that is currently applied affects the plant. In the constant delay case, the prediction horizon is equal to the delay time, i.e., $t - \phi(t) = D = \sigma(t) - t$. The predictor-based control law is

$$U(t) = \kappa(\sigma(t), P(t)) \quad (22)$$

$$P(t) = X(t) + \int_{t-D(t)}^{t} \frac{f(P(\theta), U(\theta))\, d\theta}{\phi'(\phi^{-1}(\theta))}, \quad (23)$$

with an initial condition for all $\theta \in [t_0 - D(t_0), t_0]$ as

$$P(\theta) = X(t_0) + \int_{t_0 - D(t_0)}^{\theta} \frac{f(P(\sigma), U(\sigma))\, d\sigma}{\phi'(\phi^{-1}(\sigma))}. \quad (24)$$

The fact that $P(t) = X(\sigma(t))$ can be established by applying the change of variables $t = \sigma(\tau)$ in (19).

From (23) one can observe that the function $\frac{d\sigma(\theta)}{d\theta} = \frac{1}{\phi'(\phi^{-1}(\theta))}$ is employed in the control law. Therefore, one has to appropriately restrict the delay time $D(t)$ such that $\phi'(t) \neq 0$ for all $t \geq 0$. Actually, we impose the condition $\phi'(t) > 0$ for all $t \geq 0$. The reason is that if $\phi'(t) > 0$ for all $t \geq 0$ then the control signal is able to reach the plant and it does not change the direction of propagation of the control signal (the plant keeps receiving control inputs that are never older than the ones it has already received). Besides the condition $\phi'(t) > 0$ for all $t \geq 0$, which can be also expressed in terms of the delay function as $\dot{D}(t) < 1$, for all $t \geq 0$, we also assume that the delay can not disappear instantaneously, i.e., ϕ' (or $\dot{D}$) is bounded. Also, the delay has to be positive (to guarantee the causality of the system) and bounded (such that the control signal eventually reaches the plant).

We are now ready to state the following theorem, the proof of which can be found in Bekiaris-Liberis and Krstic (2012).

Theorem 2. Let $\dot{X} = f(X, \omega)$ be forward complete and $\dot{X}(t) = f(X(t), \kappa(t, X(t)))$ globally uniformly asymptotically stable. Let the delay time $D(t) = t - \phi(t)$ be positive and uniformly bounded from above, and its rate $\dot{D}(t)$ be smaller than one and uniformly bounded from below. Consider the closed-loop system consisting of the plant (19) and the control law (22),

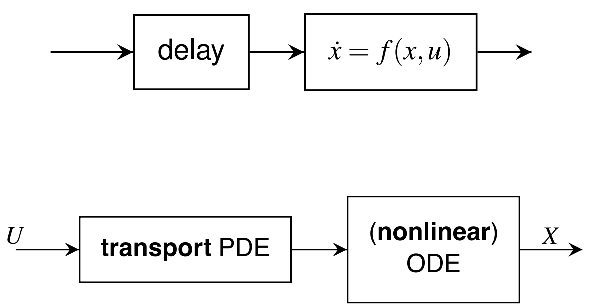

Fig. 3. Top: A nonlinear system with a delay in the input. Bottom: The equivalent representation of the delay/nonlinear ODE cascade using a transport PDE for the actuator state.

(23). There exists a class $\mathscr{KL}$ function $\beta_{\rm v}$ such that for all initial conditions $X(t_0) \in \mathbb{R}^n$ and $U(t_0+\theta); \theta \in [-D(t_0), 0] \in L^\infty[-D(t_0), 0]$ the following holds

$$\Omega_{\rm v}(t) \leq \beta_{\rm v}\left(\Omega_{\rm v}(t_0), t-t_0\right) \tag{25}$$

$$\Omega_{\rm v}(t) = |X(t)| + \sup_{t-D(t)\leq\theta\leq t} |U(\theta)|, \tag{26}$$

for all $t \geq t_0 \geq 0$.

The proof of this result is based on the following equivalent representation of the plant (19) using a transport PDE representation for the actuator state (see also Fig. 3) as

$$\dot{X}(t) = f\left(X(t), u(0,t)\right) \tag{27}$$

$$u_t(x,t) = \pi(x,t) u_x(x,t), \quad x \in [0,1] \tag{28}$$

$$u(1,t) = U(t), \tag{29}$$

where

$$\pi(x,t) = \frac{1 + x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t}, \tag{30}$$

and $\phi(t)$ is defined in (20). The choice of the transport speed $\pi(x,t)$ is guided by the fact that we seek a representation for the infinite-dimensional actuator state $u(x,t)$ such that relations (29) and

$$u(0,t) = U(\phi(t)), \tag{31}$$

are satisfied. One can verify that $u(x,t)$ is given by

$$u(x,t) = U\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right), \tag{32}$$

and consequently both (29) and (31) are satisfied. For a more detailed discussion about the choice of the transport speed $\pi(x,t)$ we refer the reader to Krstic (2009a). Analogously with representation (27)–(30) of the plant, an equivalent representation of the predictor defined in (23) is as

$$p(1,t) = \left(\phi^{-1}(t) - t\right) \int_0^1 f\left(p(y,t), u(y,t)\right) dy + X(t), \tag{33}$$

where for all $x \in [0,1]$

$$p(x,t) = P\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right). \tag{34}$$

With this representation for the predictor state we are able to define the backstepping transformation of the actuator state as

$$w(x,t) = u(x,t) - \kappa\left(t + x\left(\phi^{-1}(t) - t\right), p(x,t)\right). \tag{35}$$

Noting that the predictor state $p(x,t)$ satisfies

$$p(x,t) = \left(\phi^{-1}(t) - t\right) \int_0^x f\left(p(y,t), u(y,t)\right) dy + X(t), \tag{36}$$

and using the control law (22), system (27)–(29) is mapped to the following "target system"

$$\dot{X}(t) = f\left(X(t), \kappa(t, X(t)) + w(0,t)\right) \tag{37}$$

$$w_t(x,t) = \pi(x,t) w_x(x,t), \quad x \in [0,1] \tag{38}$$

$$w(1,t) = 0. \tag{39}$$

One can then construct a Lyapunov functional for the target system, as in the constant delay case, under the assumption that the plant $\dot{X}(t) = f\left(X(t), \kappa(t, X(t)) + \omega(t)\right)$ is input-to-state stable with respect to ω (instead of just globally asymptotically stable when $\omega = 0$). The Lyapunov functional is given in terms of the transformed actuator state as

$$V_{\rm v}(t) = S(t, X(t)) + \frac{2b}{c} \int_0^{L_{\rm v}(t)} \frac{\alpha_2(r)}{r} dr, \tag{40}$$

where $c > 0$ is arbitrary, $b > 0$ is a constant that depends on the delay D, and S, α_2 are defined in (14) and

$$\begin{aligned} L_{\rm v}(t) &= \sup_{x\in[0,1]} \left|e^{cx} w(x,t)\right| \\ &= \lim_{n\to\infty} \left(\int_0^1 e^{2ncx} w^{2n}(x,t) dx\right)^{\frac{1}{2n}}. \end{aligned} \tag{41}$$

2.3 *State-dependent delay*

In this section we concentrate on nonlinear systems with state-dependent input delay, i.e.,

$$\dot{X}(t) = f\left(X(t), U\left(t - D\left(X(t)\right)\right)\right), \tag{42}$$

where D is a nonnegative-valued continuously differentiable function. The main challenge in the case of systems with state-dependent delays is the determination of the predictor state. For systems with constant delays, $D = \text{const}$, the predictor of the state $X(t)$ is simply defined as $P(t) = X(t+D)$. For systems with state-dependent delays finding the predictor $P(t)$ is much trickier. The time when U reaches the system depends on the value of the state at that time, namely, the following implicit relationship holds $P(t) = X(t + D(P(t)))$ (and $X(t) = P(t - D(X(t)))$).

The predictor-based controller for the plant (42) is

$$U(t) = \kappa\left(\sigma(t), P(t)\right), \tag{43}$$

where the predictor state P and the prediction time σ are

$$P(t) = X(t) + \int_{t-D(X(t))}^{t} \frac{f\left(P(s), U(s)\right) ds}{1 - \nabla D\left(P(s)\right) f\left(P(s), U(s)\right)} \tag{44}$$

$$\sigma(t) = t + D\left(P(t)\right), \tag{45}$$

respectively. The initial predictor $P(\theta)$, $\theta \in [t_0 - D\left(X(t_0)\right), t_0]$, is

$$P(\theta) = X(t_0) + \int_{t_0-D(X(t_0))}^{\theta} \frac{f\left(P(s), U(s)\right) ds}{1 - \nabla D\left(P(s)\right) f\left(P(s), U(s)\right)}. \tag{46}$$

The fact that $P(t)$ given in (44) is the $\sigma(t)-t = D(P(t))$ time units ahead predictor of $X(t)$, i.e., $P(t) = X(\sigma(t))$, can be established by performing a change of variables $t = \sigma(\tau)$ in the ODE for $X(t)$ given in (42) and noting from relations $\phi(t) = t - D(X(t))$ and $\sigma(t) = \phi^{-1}(t)$ that $D(X(\sigma(t))) = \sigma(t) - t$, which implies in particular that

$$\frac{d\sigma(t)}{dt} = \frac{1}{1-\nabla D(P(t)) f(P(t),U(t))}. \tag{47}$$

As in the case of time-varying delays ϕ' and D must be positive and bounded. The positiveness of ϕ' (or equivalently of σ') is guaranteed by imposing the following condition on the solutions

$$\mathscr{F}_c: \quad \nabla D(P(\theta)) f(P(\theta),U(\theta)) < c, \quad \text{for all } \theta \geq t_0 - D(X(t_0)), \tag{48}$$

for $c \in (0,1]$. We refer to $\mathscr{F}_1$ as the *feasibility condition* of the controller (43)–(45). Due to this condition, we obtain a local result. Boundness of ϕ' and D is then guaranteed by the boundness of the system's norm. We obtain the following result. Its proof can be found in Bekiaris-Liberis and Krstic (2013).

Theorem 3. Let $\dot{X} = f(X,\omega)$ be forward complete and $\dot{X}(t) = f(X(t),\kappa(t,X(t)))$ globally uniformly asymptotically stable. Consider the closed-loop system consisting of the plant (42) and the control law (43)–(45). There exist a class $\mathscr{K}$ function ψ_{RoA} and a class $\mathscr{KL}$ function β_{s} such that for all initial conditions $X(t_0) \in \mathbb{R}^n$ such that U is locally Lipschitz on the interval $[t_0 - D(X(t_0)), t_0)$ and which satisfy

$$\Omega_{\mathrm{s}}(t_0) < \psi_{\mathrm{RoA}}(c), \tag{49}$$

for some $0 < c < 1$, where

$$\Omega_{\mathrm{s}}(t) = |X(t)| + \sup_{t-D(X(t)) \leq \theta \leq t} |U(\theta)|, \tag{50}$$

the following holds

$$\Omega_{\mathrm{s}}(t) \leq \beta_{\mathrm{s}}(\Omega_{\mathrm{s}}(t_0), t-t_0), \tag{51}$$

for all $t \geq t_0 \geq 0$. Furthermore, there exists a class $\mathscr{K}$ function δ^* such that, for all $t \geq t_0 \geq 0$,

$$D(X(t)) \leq D(0) + \delta^*(c) \tag{52}$$

$$\left|\dot{D}(X(t))\right| \leq c. \tag{53}$$

A Lyapunov functional for the closed-loop system consisting of the plant (42) and the control law (43)–(45) is

$$V_{\mathrm{s}}(t) = S(t,X(t)) + \frac{2}{g}\int_0^{L_{\mathrm{s}}(t)} \frac{\alpha_2(r)}{r} dr, \tag{54}$$

where $g > 0$ is arbitrary, S, α_2 are defined in (14) and

$$L_{\mathrm{s}}(t) = \sup_{t-D(X(t)) \leq \theta \leq t} \left| e^{g(\theta + D(P(\theta)) - t)} W(\theta) \right| \tag{55}$$

$$W(\theta) = U(\theta) - \kappa(\theta + D(P(\theta)), P(\theta)), \tag{56}$$

where P is given in terms of (X,U) in (44).

The following example illustrates the fact that global stabilization is not possible even for linear systems.

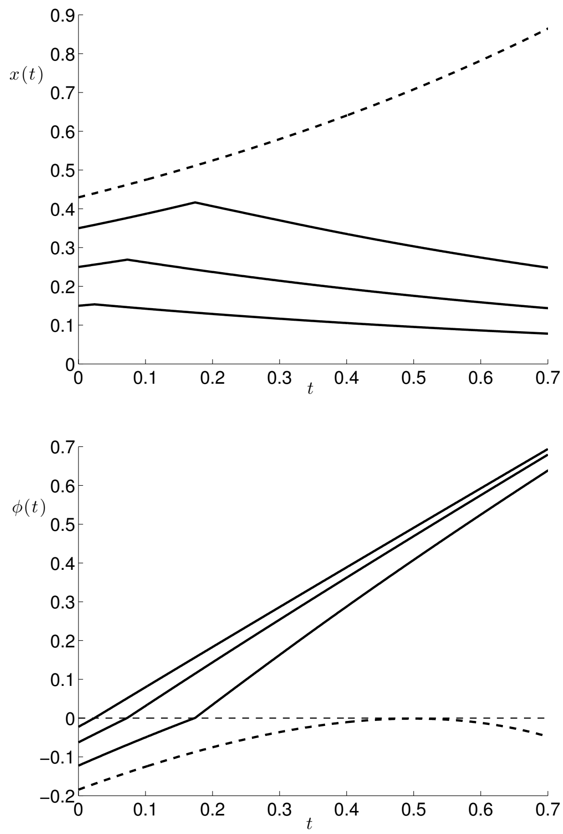

Fig. 4. Response of system (57) with the controller (58)–(59) with initial conditions $U(\theta) = 0$, $-X(0)^2 \leq \theta \leq 0$ and four different initial conditions for the state $X(0) = 0.15, 0.25, 0.35, 0.43$.

Example 1: We consider a scalar unstable system with a Lyapunov-like delay

$$\dot{X}(t) = X(t) + U\left(t - X(t)^2\right). \tag{57}$$

The delay-compensating controller is

$$U(t) = -2P(t), \tag{58}$$

where for all $\theta \geq -X(0)^2$

$$P(\theta) = X(t) + \int_{t-X(t)^2}^{\theta} \frac{(P(s)+U(s))\,ds}{1-2P(s)(P(s)+U(s))}. \tag{59}$$

In Fig. 4 we show the response of the system and the function $\phi(t) = t - X(t)^2$ for four different initial conditions of the state and with the initial conditions for the input chosen as $U(\theta) = 0$, $-X(0)^2 \leq \theta \leq 0$. We choose $X(0) = 0.15, 0.25, 0.35, X^*$. With X^* we denote the critical value of $X(0)$ for the given initial condition of the input, such that, for any $X(0) \geq X^*$, the control inputs produced by the feedback law (58), (59) for positive t never reach the plant. We calculate this time as follows: The function $\phi(t) = t - X(0)^2 e^{2t}$ has a maximum at t^* if $\log\left(\frac{1}{\sqrt{2X(0)^2}}\right) = t^* > 0$. Since $\phi(t^*) = \log\left(\frac{1}{\sqrt{2X(0)^2}}\right) - \frac{1}{2}$ has to be positive for the control to reach the plant, it follows $X^* = \frac{1}{\sqrt{2e}} = 0.43$.

In the following example we consider the stabilization problem of a mobile robot with an input delay that grows with the distance of the robot from then reference position.

Example 2: We consider the problem of stabilizing a mobile robot modeled as

$$\dot{x}(t) = v(t - D(x(t), y(t)))\cos(\theta(t)) \tag{60}$$

$$\dot{y}(t) = v(t - D(x(t), y(t)))\sin(\theta(t)) \tag{61}$$

$$\dot{\theta}(t) = \omega(t - D(x(t), y(t))), \tag{62}$$

subject to an input delay that grows with the distance relative to the reference position as

$$D(x(t), y(t)) = x(t)^2 + y(t)^2, \tag{63}$$

where $(x(t), y(t))$ is position of the robot, $\theta(t)$ is heading, $v(t)$ is speed and $\omega(t)$ is turning rate. When $D = 0$ a time-varying stabilizing controller is proposed in Pomet (1992) as

$$\omega(t) = -5P(t)^2\cos\left(3\phi^{-1}(t)\right) - P(t)Q(t) \times \left(1 + 25\cos^2\left(3\phi^{-1}(t)\right)\right) - \Theta(t) \tag{64}$$

$$v(t) = -P(t) + 5Q(t)\left(\sin\left(3\phi^{-1}(t)\right) - \cos\left(3\phi^{-1}(t)\right)\right) + Q(t)\omega(t) \tag{65}$$

$$P(t) = X(t)\cos(\Theta(t)) + Y(t)\sin(\Theta(t)) \tag{66}$$

$$Q(t) = X(t)\sin(\Theta(t)) - Y(t)\cos(\Theta(t)), \tag{67}$$

with

$$X = x, \quad Y = y, \quad \Theta = \theta, \quad \phi^{-1}(t) = t. \tag{68}$$

The proposed method replaces (68) with

$$X(t) = x(t) + \int_{t-D(x(t),y(t))}^{t} \frac{d\sigma(s)}{ds} v(s)\cos(\Theta(s))\,ds \tag{69}$$

$$Y(t) = y(t) + \int_{t-D(x(t),y(t))}^{t} \frac{d\sigma(s)}{ds} v(s)\sin(\Theta(s))\,ds \tag{70}$$

$$\Theta(t) = \theta(t) + \int_{t-D(x(t),y(t))}^{t} \frac{d\sigma(s)}{ds} \omega(s)ds \tag{71}$$

$$\sigma(t) = t + D(X(t), Y(t)) \tag{72}$$

$$\dot{\sigma}(s) = \frac{1}{1 - 2v(s)\left(X(s)\cos(\Theta(s)) + Y(s)\sin(\Theta(s))\right)}. \tag{73}$$

The initial conditions are chosen as $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \le s \le 0$. From the given initial conditions we get the initial conditions for the predictors (69)–(71) as $X(s) = Y(s) = \Theta(s) = 1$ for all $-2 \le s \le 0$. From the above initial conditions for the predictors one can verify that the system initially lies inside the feasibility region. The controller "kicks in" at the time instant t_0 at which $t_0 = x(t_0)^2 + y(t_0)^2$. Since $v(s) = \omega(s) = 0$ for $s < 0$ we conclude that $x(t) = y(t) = \theta(t) = 1$ for all $0 \le t \le t_0$ and hence, $t_0 = 2$. In Fig. 5 we show the trajectory of the robot in the xy plane, whereas in Fig. 6 we show the resulting state-dependent delay and the controls $v(t)$ and $\omega(t)$. In the case of the uncompensated controller (64)–(68), the system is unstable, the delay grows approximately linearly in time, and the vehicle's trajectory is a divergent Archimedean spiral. The compensated controller (64)–(67), (69)–(73) recovers the delay-free behavior after 2 seconds. From Fig. 5 one can also conclude that the heading $\theta(t)$ in the case of the compensated controller converges to zero with damped oscillations, whereas in the case of the uncompensated controller it increases towards negative infinity (the robot moves clockwise on a spiral).

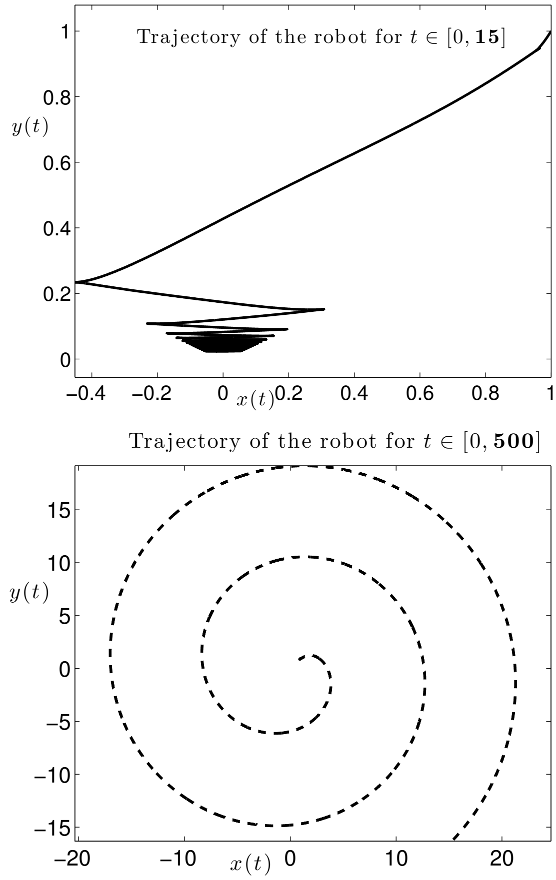

Fig. 5. The trajectory of the robot model (60)–(62), with the compensated controller (64)–(67), (69)–(73) (solid line) and the uncompensated controller (64)–(68) (dashed line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \le s \le 0$.

3. NONLINEAR SYSTEMS WITH A WAVE PDE IN THE INPUT

In this section we consider the following system

$$\dot{X}(t) = f(X(t), u(0,t)) \tag{74}$$

$$u_{tt}(x,t) = u_{xx}(x,t) \tag{75}$$

$$u_x(0,t) = 0 \tag{76}$$

$$u_x(1,t) = U(t), \tag{77}$$

where $X \in \mathbb{R}^n$, $U \in \mathbb{R}$, $t \in \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$. Our controller design is based on converting the wave equation to a 2×2 system of first order transport equations which convect in opposite directions (see Fig. 7). To achieve this we define the following transformations

$$\zeta(x,t) = u_t(x,t) + u_x(x,t) \tag{78}$$

$$\omega(x,t) = u_t(x,t) - u_x(x,t), \tag{79}$$

together with their inverses given by

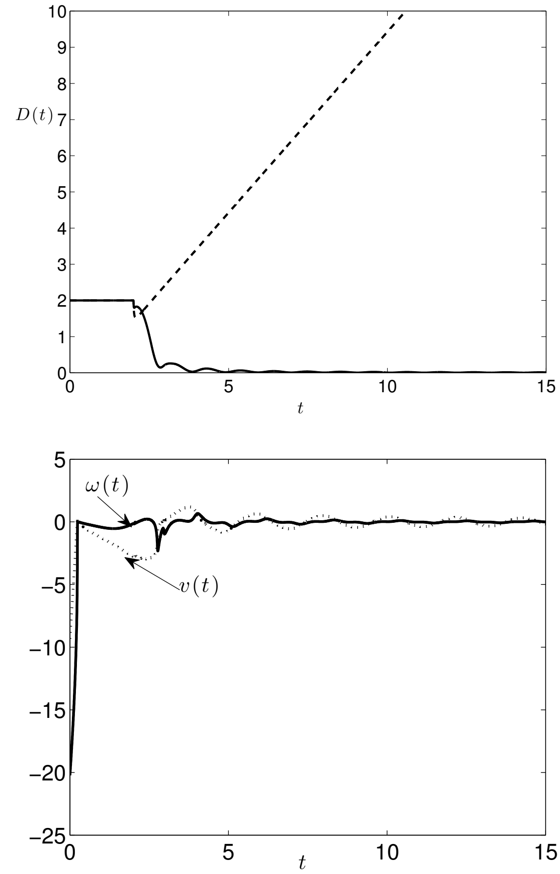

Fig. 6. Top: The delay (63) for the robot model (60)–(62) with the controller (64)–(67), (69)–(73) (solid line) and the controller (64)–(68) (dashed line) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$. Bottom: The control efforts $v(t)$ and $\omega(t)$ for the robot model (60)–(62) with the controller (64)–(67), (69)–(73) with initial conditions $x(0) = y(0) = \theta(0) = 1$ and $\omega(s) = v(s) = 0$ for all $-x(0)^2 - y(0)^2 \leq s \leq 0$.

$$u_t(x,t) = \frac{\zeta(x,t) + \omega(x,t)}{2} \tag{80}$$
$$u_x(x,t) = \frac{\zeta(x,t) - \omega(x,t)}{2}. \tag{81}$$

Noting from (76) that $\zeta(0,t) = u_t(0,t)$ and defining

$$\xi(t) = u(0,t), \tag{82}$$

system (74)–(77) can be written as

$$\dot{Z}(t) = g(Z(t), \zeta(0,t)) \tag{83}$$
$$\omega_t(x,t) = -\omega_x(x,t) \tag{84}$$
$$\omega(0,t) = \zeta(0,t) \tag{85}$$
$$\zeta_t(x,t) = \zeta_x(x,t) \tag{86}$$
$$\zeta(1,t) = U(t) + u_t(1,t), \tag{87}$$

where

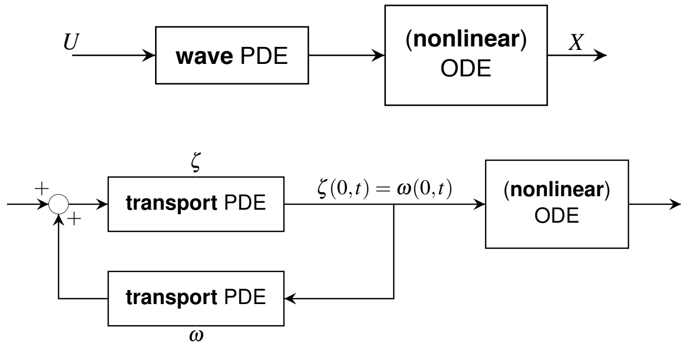

Fig. 7. Top: A nonlinear system with a wave PDE in the input. Bottom: The equivalent representation of the wave PDE/nonlinear ODE cascade using the change of variables (78), (79).

$$Z = \begin{bmatrix} X \\ \xi \end{bmatrix} \tag{88}$$
$$g(Z,v) = \begin{bmatrix} f(X,\xi) \\ v \end{bmatrix}. \tag{89}$$

Our feedback design, that compensates the wave actuator dynamics, is based on a nominal feedback law $\mu : \mathbb{R}^{n+1} \to \mathbb{R}$ that stabilizes the plant $\dot{Z} = g(Z,U)$ defined in (83), i.e., that stabilizes the following system (which is identical to the original system (74) augmented by one integrator)

$$\dot{X}(t) = f(X(t), \xi(t)) \tag{90}$$
$$\dot{\xi}(t) = U(t). \tag{91}$$

Note that such a nominal control law for the augmented system (90), (91) can be constructed, using backstepping, if there exists a control law κ that stabilizes the plant $\dot{X} = f(X,U)$, i.e., $\dot{X} = f(X, \kappa(X))$ is globally asymptotically stable. A choice of the feedback law μ is then as

$$\mu(X(t), \xi(t)) = -c_1(\xi(t) - \kappa(X(t))) + \frac{\partial \kappa(X(t))}{\partial X} \times f(X(t), \xi(t)). \tag{92}$$

Noting that the input to the Z system is the delayed version of the signal $\zeta(1,t) = U(t) + u_t(1,t)$ we conclude that our control law has to employ the prediction of Z.

The control law that compensates the wave dynamics is given by

$$U(t) = -u_t(1,t) - c_1(p_2(1,t) - \kappa(p_1(1,t))) + \frac{\partial \kappa(p_1(1,t))}{\partial p_1} \times f(p_1(1,t), p_2(1,t)), \tag{93}$$

where $c_1 > 0$ is arbitrary, and $p_1 \in \mathbb{R}^n$ and $p_2 \in \mathbb{R}$, the predictors of $X(t)$ and $u(0,t)$ respectively, are given for all $x \in [0,1]$ by

$$p_1(x,t) = X(t) + \int_0^x f(p_1(y,t), p_2(y,t))\, dy \tag{94}$$
$$p_2(x,t) = u(x,t) + \int_0^x u_t(y,t) dy, \tag{95}$$

with initial conditions for all $x \in [0,1]$ as

$$p_1(x,0) = X(0) + \int_0^x f(p_1(y,0), p_2(y,0))\,dy \quad (96)$$

$$p_2(x,0) = u(x,0) + \int_0^x u_t(y,0)dy. \quad (97)$$

The name "predictors" for p_1 and p_2 is chosen to emphasize that $p_1(1,t)$ and $p_2(1,t)$ are actually the 1-time units ahead predictors of $X(t)$ and $u(0,t)$ respectively, i.e., it holds that $p_1(1,t) = X(t+1)$ and $p_2(1,t) = u(0,t+1)$. This fact is shown in the next section [3]. Note that the control law (93) is directly implementable. To see this note that the predictors $p_1(1,t)$, $p_2(1,t)$ are computed, at each time t, based on the numerical integration of the integrals in relation (94), (95) on the triangular domain $0 \le y \le x$, starting from the initial condition (in x) $p_1(0,t) = X(t)$, $p_2(0,t) = u(0,t)$.

Defining for any $\theta \in L^\infty[0,1]$ its supremum norm

$$\sup_{x\in[0,1]} |\theta(x,t)| = \|\theta(t)\|_\infty, \quad (98)$$

we are able to state the following result.

Theorem 4. Consider the closed-loop system consisting of the plant (74)–(77) and the control law (93), (94), (95). Let the plant $\dot{X} = f(X,v)$ be complete and the "disturbed" closed-loop system $\dot{X} = f(X, \mu(X)+v)$ input-to-state stable and backward complete. There exist a class $\mathcal{KL}$ function β such that

$$\Omega(t) \le \beta(\Omega(0),t) \quad (99)$$

$$\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty, \quad (100)$$

for all $t \ge 0$.

The proof of this result is based on the introduction of the following invertible backstepping transformations of ω and ζ defined for all $x \in [0,1]$ as

$$z(x,t) = \omega(x,t) - \mu(r(x,t)) \quad (101)$$

$$w(x,t) = \zeta(x,t) - \mu(p(x,t)), \quad (102)$$

respectively, where for all $x \in [0,1]$

$$r(x,t) = Z(t) - \int_0^x g(r(y,t), \omega(y,t))\,dy \quad (103)$$

$$p(x,t) = Z(t) + \int_0^x g(p(y,t), \zeta(y,t))\,dy, \quad (104)$$

and μ is defined in (92). Transformation (101), (102) and the control law (93)–(95) transform system (83)–(87) to the "target system" given by

$$\dot{Z}(t) = g(Z(t), \mu(Z(t)) + w(0,t)) \quad (105)$$

$$z_t(x,t) = -z_x(x,t) \quad (106)$$

$$z(0,t) = w(0,t) \quad (107)$$

$$w_t(x,t) = w_x(x,t) \quad (108)$$

$$w(1,t) = 0. \quad (109)$$

The stability of the "target system" can be then studied using the following Lyapunov functional

$$V(t) = S(Z(t)) + \frac{2}{c}\int_0^{\|v(t)\|_{c,\infty}} \frac{\alpha_2(r)}{r}dr, \quad (110)$$

where $c > 0$ is arbitrary, S and α_2 are defined in (14) (note that in the present case the closed-loop system is autonomous so S can be chosen independent of t), and the new variable $v(x,t)$, $x \in [-1,1]$ is defined as

$$v(x,t) = \begin{cases} w(x,t), & \text{for all } x \in [0,1] \\ z^*(x,t), & \text{for all } x \in [-1,0] \end{cases}, \quad (111)$$

where $\|v(t)\|_{c,\infty} = \sup_{x\in[-1,1]} e^{c(1+x)}|v(x,t)|$, $z^*(x,t) = z(-x,t)$.

3.1 Example

We consider the following system

$$\dot{X}_1(t) = X_2(t) - X_2(t)^2 u(0,t) \quad (112)$$

$$\dot{X}_2(t) = u(0,t) \quad (113)$$

$$u_{tt}(x,t) = u_{xx}(x,t) \quad (114)$$

$$u_x(0,t) = 0 \quad (115)$$

$$u_x(1,t) = U(t). \quad (116)$$

System (112)–(113) is in the strict-feedforward form, and hence, is complete with respect to the input $u(0,t)$. The nominal control law (i.e., in the case where $u(0,t) \equiv U(t)$)

$$U(t) = -X_1(t) - 2X_2(t) - \frac{1}{3}X_2(t)^3, \quad (117)$$

renders the closed-loop system input-to-state stable and backward complete [4]. The control design that compensates the wave dynamics is

$$\begin{aligned} U(t) = & -u_t(1,t) - 2(p_3(1,t) + p_1(1,t) + 2p_2(1,t)) \\ & -\frac{2}{3}p_2(1,t)^3 - p_2(1,t) + p_2(1,t)^2 p_3(1,t) \\ & -(2 + p_2(1,t)^2)\,p_3(1,t), \end{aligned} \quad (118)$$

where

$$\begin{aligned} p_1(1,t) = & X_1(t) + X_2(t) + \int_0^1 (1-x)u(x,t)dx \\ & + \int_0^1 (1-x)^2 u_t(x,t)dx \\ & - \int_0^1 dx\left(u(x,t) + \int_0^x u_t(y,t)dy\right) \\ & \times\left(X_2(t) + \int_0^x (u(y,t) + (1-y)u_t(y,t))\,dy\right)^2 \end{aligned} \quad (119)$$

$$p_2(1,t) = X_2(t) + \int_0^1 u(x,t)dx + \int_0^1 (1-x)u_t(x,t)dx \quad (120)$$

$$p_3(1,t) = u(1,t) + \int_0^1 u_t(x,t)dx. \quad (121)$$

We choose the initial conditions for the system as $X_1(0) = 1$, $X_2(0) = 0$ and the initial conditions for the actuator state as $u(x,0) = u_t(x,0) = 1$, for all $x \in [0,1]$. In Fig. 8 we show the

[3] Another way to see this is as follows. Construct first the standard 1-time unit ahead predictor for Z satisfying (83) as $P(t) = Z(t) + \int_{t-1}^t g(P(\theta), \Xi(\theta))\,d\theta$, where $\Xi(t+x-1) = \zeta(x,t)$. Defining $P(t+x-1) = p(x,t)$ we rewrite the predictor as $p(1,t) = Z(t) + \int_0^1 g(p(x,t), \zeta(x,t))\,dx$. Using definitions (88), (89) and noting that $p_2(1,t) = u(0,t) + \int_0^1 u_x(x,t)dx + \int_0^1 u_t(x,t)dx$, we get after integrating u_x relations (94), (95) for $x = 1$.

[4] This fact follows from the fact that the control law (117) can be written as $U = -\phi_1 - \phi_2$, where ϕ is the linearizing diffeomorphic transformation $\phi_1 = X_1 + X_2 + \frac{1}{3}X_2^3$, $\phi_2 = X_2$, which transforms system (112)–(113) to $\dot{\phi}_1 = \phi_2 + U$, $\dot{\phi}_2 = U$ (see Krstic (2004)) .

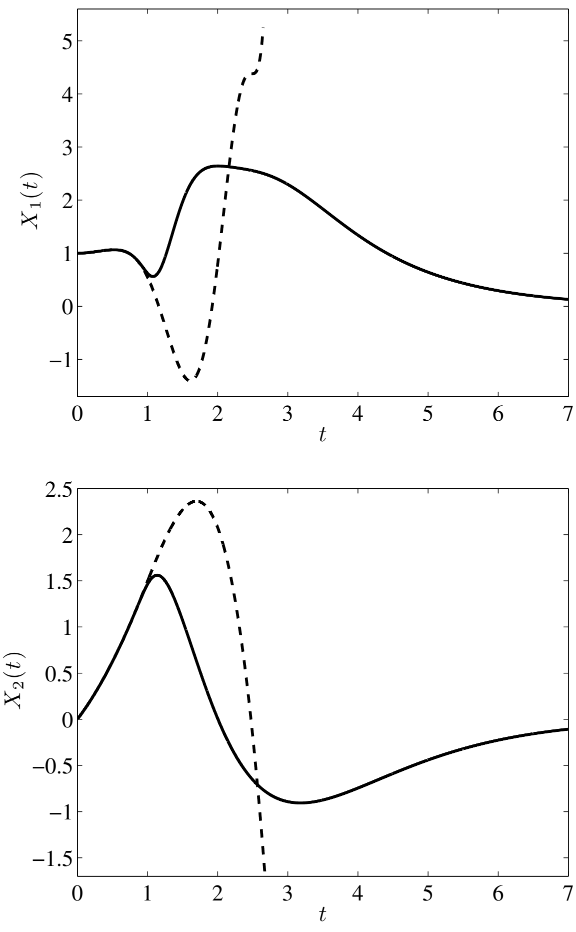

Fig. 8. The response of the states of the plant (112)–(113) with the control law (118)–(121) (solid line) and with the nominal control law (117) (dashed line) for initial conditions as $X_1(0) = 1, X_2(0) = 0$ and $u(x,0) = u_t(x,0) = 1$, for all $x \in [0,1]$.

response of the states of the plant (112)–(113) for the case of the uncompensated nominal control law (117) and the case of the proposed control law (118)–(121). As one can observe, in the latter case stabilization is achieved, whereas the states grow unbounded in the former case, in which a control law that does not take into account the wave dynamics is employed. In Fig. 9 we show the response of the actuator state and the control effort in the case of the proposed control law (118)–(121). As one can observe, both the actuator state and the control effort converge to zero.

4. SYSTEMS OF NONLINEAR HYPERBOLIC PDES

In this section we present the results developed by Coron et al. (2012) and Vazquez et al. (2011b). We consider the following system

$$z_t(x,t) + \Lambda(z(x,t),x)\, z_x(x,t) + f(z(x,t),x) = 0, \quad (122)$$

with the following boundary conditions

$$z_1(0,t) = G_0(z_2(0,t)) \quad (123)$$
$$z_2(1,t) = U(t), \quad (124)$$

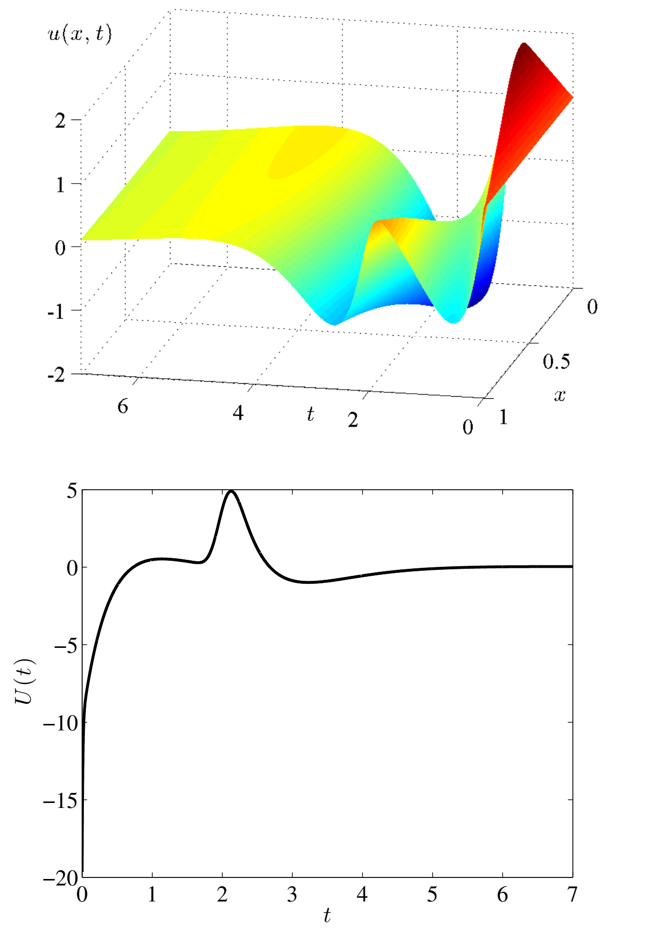

Fig. 9. The response of the actuator state (top) and the control effort (bottom) of the plant (112)–(116) with the control law (118)–(121) for initial conditions as $X_1(0) = 1$, $X_2(0) = 0$ and $u(x,0) = u_t(x,0) = 1$, for all $x \in [0,1]$.

where $x \in [0,1]$, $z : [0,1] \times [0,\infty) \to \mathbb{R}^2$, $\Lambda : \mathbb{R}^2 \times [0,1] \to \mathscr{M}_{2,2}(\mathbb{R})$, $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$, with $\mathscr{M}_{2,2}$ denoting the set of 2×2 real matrices. We further assume that $\Lambda(z,x)$ is twice continuously differentiable with respect to z and x, and we assume that (possibly after an appropriate state transformation) $\Lambda(0,x)$ is a diagonal matrix with nonzero eigenvalues $\Lambda_1(x)$, $\Lambda_2(x)$ which are, respectively, positive and negative, i.e., for all $x \in [0,1]$

$$\Lambda(0,x) = \operatorname{diag}(\Lambda_1(x), \Lambda_2(x)), \quad \Lambda_1(x) > 0, \Lambda_2(x) < 0, \quad (125)$$

where $\operatorname{diag}(\Lambda_1, \Lambda_2)$ denotes the diagonal matrix with Λ_1 in the first position of the diagonal and Λ_2 in the second. We also assume that $f(0,x) = 0$, implying that there is an equilibrium at the origin, and that f is twice continuously differentiable with respect to z. Denote

$$\frac{\partial f}{\partial z}(0,x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix} \quad (126)$$

and assume that $f_{ij} \in C^1([0,1])$. Finally, we assume that $G_0(x)$ is twice differentiable and vanishes at the origin. We seek a control law $U(t)$ that makes the origin of (122)–(124) locally exponentially stable. Our control design is based on the linearization of system (122)–(124). Before we linearize system (122)–(124) around the origin we rescale the variable z so that we make the linear part of f antidiagonal since we present our linear design for the case of an antidiagonal linear f (with no generality loss). Defining the new variable w as

$$w = \Phi(x)z \tag{127}$$

$$\Phi(x) = \text{diag}\left(\phi_1(x), \phi_2(x)\right), \tag{128}$$

where

$$\phi_1(x) = e^{\int_0^x \frac{f_{11}(y)}{\Lambda_1(y)} dy} \tag{129}$$

$$\phi_2(x) = e^{-\int_0^x \frac{f_{22}(y)}{\Lambda_2(y)} dy}, \tag{130}$$

we rewrite system (122)–(124) in the new variables as (see Fig. 10)

$$w_t(x,t) - \Sigma(x)w_x(x,t) - C(x)w(x,t) + \Lambda_{\text{NL}}(w(x,t),x)w_x(x,t) + f_{\text{NL}}(w(x,t),x) = 0, \tag{131}$$

with boundary conditions as

$$w_1(0,t) = qw_2(0,t) + G_{\text{NL}}\left(w_2(0,t)\right) \tag{132}$$

$$w_2(1,t) = V(t), \tag{133}$$

where

$$\Sigma(x) = -\Lambda(0,x) \tag{134}$$

$$C(x) = \begin{bmatrix} 0 & -f_{12}(x) \\ -f_{21}(x) & 0 \end{bmatrix} \tag{135}$$

$$V(t) = \phi_2(1)U(t) \tag{136}$$

$$q = \frac{dG_0(0)}{dz}, \tag{137}$$

and the nonlinear perturbation terms Λ_{NL} and f_{NL} are such that $\Lambda_{\text{NL}}(0,x) = 0$, $f_{\text{NL}}(0,x) = \frac{\partial f_{\text{NL}}}{\partial w}(0,x) = 0$, $G_{\text{NL}}(0) = 0$.

Our design is based on a backstepping design for the linear part of system (131). Defining $w = [\begin{array}{cc} u & v \end{array}]^T$, $\Lambda_1 = \varepsilon_1$, $\Lambda_2 = -\varepsilon_2$, $f_{12} = -c_1$ and $f_{21} = -c_2$ we rewrite the linear part of system (131) as

$$u_t(x,t) = -\varepsilon_1(x)u_x(x,t) + c_1(x)v(x,t) \tag{138}$$

$$v_t(x,t) = \varepsilon_2(x)v_x(x,t) + c_2(x)u(x,t) \tag{139}$$

$$u(0,t) = qv(0,t) \tag{140}$$

$$v(1,t) = V(t). \tag{141}$$

System (138)–(141) is mapped to the following "target system"

$$\alpha_t(x,t) = -\varepsilon_1(x)\alpha_x(x,t) \tag{142}$$

$$\beta_t(x,t) = \varepsilon_2(x)\beta_x(x,t) \tag{143}$$

$$\alpha(0,t) = q\beta(0,t) \tag{144}$$

$$\beta(1,t) = 0, \tag{145}$$

using the invertible backstepping transformation

$$\alpha(x,t) = u(x,t) - \int_0^x K^{uu}(x,\xi)u(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)v(\xi,t)dy\xi \tag{146}$$

$$\beta(x,t) = v(x,t) - \int_0^x K^{vu}(x,\xi)u(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)v(\xi,t)d\xi, \tag{147}$$

and the control law

$$V(t) = \int_0^1 K^{vu}(1,x)u(x,t)dx + \int_0^1 K^{vv}(1,x)v(x,t)dx. \tag{148}$$

The kernels of the backstepping transformation satisfy the following 2×2 system of linear hyperbolic PDEs on the triangular domain $\mathscr{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$ which can be shown to be well-posed (Vazquez et al. (2011a))

$$\varepsilon_1(x)K_x^{uu} + \varepsilon_1(\xi)K_\xi^{uu} = -\varepsilon_1'(\xi)K^{uu} - c_2(\xi)K^{uv} \tag{149}$$

$$\varepsilon_1(x)K_x^{uv} - \varepsilon_2(\xi)K_\xi^{uv} = \varepsilon_2'(\xi)K^{uv} - c_1(\xi)K^{uu} \tag{150}$$

$$\varepsilon_2(x)K_x^{vu} - \varepsilon_1(\xi)K_\xi^{vu} = \varepsilon_1'(\xi)K^{vu} + c_2(\xi)K^{vv} \tag{151}$$

$$\varepsilon_2(x)K_x^{vv} + \varepsilon_2(\xi)K_\xi^{vv} = -\varepsilon_2'(\xi)K^{vv} + c_2(\xi)K^{vu}, \tag{152}$$

with boundary conditions

$$K^{uu}(x,0) = \frac{\varepsilon_2(0)}{q\varepsilon_1(0)}K^{uv}(x,0) \tag{153}$$

$$K^{uv}(x,x) = \frac{c_1(x)}{\varepsilon_1(x) + \varepsilon_2(x)} \tag{154}$$

$$K^{vu}(x,x) = -\frac{c_2(x)}{\varepsilon_1(x) + \varepsilon_2(x)} \tag{155}$$

$$K^{vv}(x,0) = \frac{\varepsilon_1(0)}{q\varepsilon_2(0)}K^{vu}(x,0). \tag{156}$$

Using definition (127) and (136), the control law for the original nonlinear system (122)–(124) is

$$U(t) = \frac{1}{\phi_2(1)}\int_0^1 K^{vu}(1,x)\phi_1(x)z_1(x,t)dx + \frac{1}{\phi_2(1)}\int_0^1 K^{vv}(1,x)\phi_2(x)z_2(x,t)dx. \tag{157}$$

With the control law (157) the boundary condition (124) for the closed-loop system is written as

$$z_1(1,t) = \frac{1}{\phi_2(1)}\int_0^1 K^{vu}(1,x)\phi_1(x)z_1(x,t)dx + \frac{1}{\phi_2(1)}\int_0^1 K^{vv}(1,x)\phi_2(x)z_2(x,t)dx. \tag{158}$$

Defining the H_2 norm of $z = [\begin{array}{cc} z_1 & z_2 \end{array}]^T$ as

$$\|z(t)\|_{H_2} = \int_0^1 z(x,t)^T z(x,t)dx + \int_0^1 z_x(x,t)^T z_x(x,t)dx + \int_0^1 z_{xx}(x,t)^T z_{xx}(x,t)dx, \tag{159}$$

and imposing the following compatibility conditions

$$0 = z_1(0,0) - G_0(z_2(0,0)) \tag{160}$$

$$0 = z_2(1,0) - \frac{1}{\phi_2(1)}\int_0^1 K^{vu}(1,x)\phi_1(x)z_1(x,0)dx - \frac{1}{\phi_2(1)}\int_0^1 K^{vv}(1,x)\phi_2(x)z_2(x,0)dx \tag{161}$$

$$0 = -\Lambda_1(z(0,0),0)z_{1,x}(0,0) - f_1(z(0,0),0) + G_0'(z_2(0,0)) \times (\Lambda_2(z(0,0),0)z_{2,x}(0,0) + f_2(z(0,0),0)) \tag{162}$$

$$0 = \int_0^1 \frac{K^{vu}(1,x)\phi_1(x)}{\phi_2(1)}\Lambda_1(z(x,0),x)z_{1,x}(x,0)dx + \int_0^1 \frac{K^{vu}(1,x)\phi_1(x)}{\phi_2(1)}f_1(z(x,0),x)dx + \int_0^1 \frac{K^{vv}(1,x)\phi_2(x)}{\phi_2(1)} \times (\Lambda_2(z(x,0),x)z_{2,x}(x,0) + f_2(z(x,0),x))dx - \Lambda_2(z(1,0),1)z_{2,x}(1,0) - f_2(z(1,0),1), \tag{163}$$

we obtain the following result.

Theorem 5. Consider the closed-loop system (122), (123), (158). Under the assumptions that $\Lambda \in C^2(\mathbb{R}^2 \times [0,1])$, $f(\cdot,x) \in C^2(\mathbb{R}^2)$, $\frac{\partial f(0,\cdot)}{\partial z} \in C^1([0,1])$, $G_0 \in C^2(\mathbb{R})$, for all initial condition $z_0 \in H_2([0,1])$ that satisfy the compatibility conditions (161)–(163), there exist $\delta > 0$, $\lambda > 0$ and $c > 0$ such that if $\|z(0)\|_{H_2} < \delta$, then for all $t \geq 0$

$$\|z(t)\|_{H_2} \leq ce^{-\lambda t}\|z(0)\|_{H_2}. \tag{164}$$

Note that the compatibility conditions (161) and (163) depend on our feedback laws and therefore are not natural. They can be omitted by considering a dynamical extension (see Coron et al. (2012)). The proof of Theorem 5 is based on employing the linear backstepping transformation (146), (147) on the rescaled nonlinear system (131), which results in the following target system

$$\gamma_t - \Sigma(x)\gamma_x + F_3[\gamma,\gamma_x] + F_4[\gamma] = 0, \tag{165}$$

where $\gamma = [\begin{array}{cc}\alpha & \beta\end{array}]^T$ and F_3, F_4 are nonlinear functionals of γ and γ_x (see Coron et al. (2012) for details). The H_2 local exponential stability of the target system can be then studied with the following Lyapunov functional

$$S(t) = U(t) + V(t) + W(t) \tag{166}$$

$$U(t) = \int_0^1 \gamma^T(x,t)D(x)\gamma(x,t)dx \tag{167}$$

$$V(t) = \int_0^1 \gamma_t^T(x,t)R[\gamma](x)\gamma_t(x,t)dx \tag{168}$$

$$W(t) = \int_0^1 \gamma_{tt}^T(x,t)R[\gamma](x)\gamma_{tt}(x,t)dx, \tag{169}$$

where $D(x) = \mathrm{diag}(D_1(x), D_2(x))$ is positive definite for all $x \in [0,1]$ and $R[\gamma]$ is a symmetric and positive definite matrix for all $\sup_{x\in[0,1]}|\gamma(x,t)| < \delta$.

5. CONCLUSIONS

In our development we assume that the nonlinear plant under consideration is forward complete and globally stabilizable. However, our predictor-based design can be applied to systems that are not forward complete (but they are globally stabilizable in the absence of the input delay) Krstic (2008) and to systems that are only locally stabilizable Bekiaris-Liberis and Krstic (2013). One of the topics of ongoing research is to extend the predictor idea to nonlinear systems with distributed input and state delays (see Bekiaris-Liberis and Krstic (2011a,b) for linear results) and to systems with input-dependent delay.

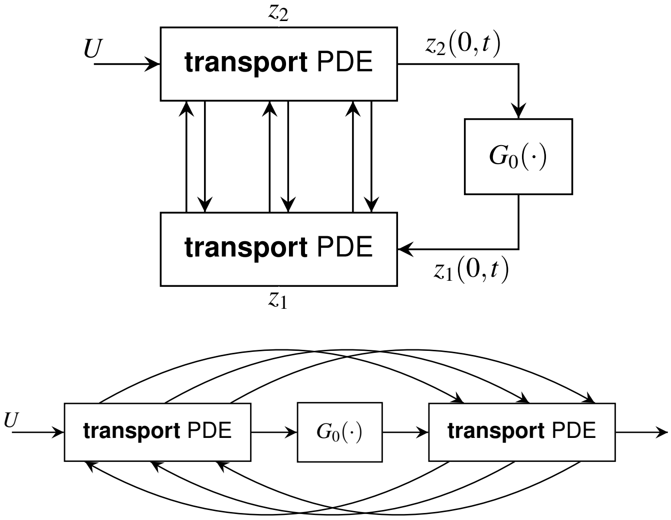

Fig. 10. Top: A 2×2 quasilinear system of transport PDEs. Bottom: An equivalent representation of the system as a nonlinear transport PDE/nonlinear transport PDE cascade with boundary and in domain coupling.

Although we focus on the stabilization of a wave PDE/nonlinear ODE cascade, our results opens an opportunity to tackle stabilization problems of other PDE/nonlinear ODE cascades, for example, when the PDE is of diffusive type.

We present results on the stabilization of 2×2 systems of first order hyperbolic quasilinear PDEs assuming measurement of the full state. Yet, we remove this requirement in Vazquez et al. (2012) where we design an output feedback control law. In the future we would like to extend the present methodology to the case of $n \times n$ systems. For the linear case an extension to $n \times n$ systems is presented in Di Meglio et al. (2012a) for system that have n positive and one negative transport speeds, with actuation only on the state corresponding to the negative velocity.

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