Flatness based Control of a Gantry Crane

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Abstract: This contribution deals with flatness based control of a laboratory model of a gantry crane. The mechanical model has 3DOFs, where a trolley can be moved on a rail, the load is fixed at the end of a rope and can be lifted or lowered by coiling or uncoiling this rope on a cylinder. Under the assumption that the rope is always stretched, the underactuated system is not input to state linearisable but it is flat with the coordinates of the load as flat output. Since the flat output coincides with the variables to be controlled, a flatness based design for trajectory tracking and stabilisation is indicated. The design of the tracking control is accomplished in two steps. First, the system is exactly linearised by a quasi-static state feedback. Subsequently, for the linear system a feedback with integral parts is designed such that the motion of the load is stabilised about the reference trajectories. Moreover, the control law is extended by terms which approximately compensate for the friction occurring at the gantry crane. Finally, the setting of the controller parameters is discussed and measurement results are presented, which demonstrate an excellent tracking behaviour and disturbance attenuation.

Keywords: nonlinear tracking control; underactuated mechanical system; differential flatness; Brunovsky state; exact linearisation; quasi-static state feedback.

1. INTRODUCTION

This paper deals with the nonlinear control of the laboratory model gantry crane. Such a gantry crane can be modelled as a mechanical system with three degrees of freedom. Since there are only two control inputs, the system is underactuated, which makes the design of a nonlinear control far more challenging than in case of a fully actuated system. Fortunately, the mathematical model of the gantry crane represents a differentially flat system, see e.g. Delaleau and Rudolph [1995], Fliess et al. [1992], Fliess et al. [1995], Fliess et al. [1993], Rothfuss et al. [1997], Rouchon et al. [1993], and all the citations therein, but it is not input to state linearisable, see e.g Isidori [1995]. Furthermore the coordinates of the load are a flat output. Since the load coordinates are exactly the variables which are to be controlled, it is possible to design a flatness based tracking control combined with disturbance rejection in straightforward manner. Therefore there are several contributions on the flatness based control of a gantry crane, but according to the knowledge of the authors they all deal with the "pendulum subsystem" of the gantry crane, which necessitates the use of cascaded controllers. In Rudolph [2003b] for instance a feedforward control for the "pendulum subsystem" is considered and in Delaleau and Rudolph [1998] a flatness based tracking control for the "pendulum subsystem" is presented. In contrast, the present paper deals with the flatness based control of the gantry crane as a whole. Thus, no cascaded controllers are required.

2. MODELLING

Fig. 1 shows the laboratory model. Its functionality is as follows: A trolley is moved by a haulage cable on a rail. On the trolley a cylinder, denoted in the following as load cylinder, is mounted. By coiling or uncoiling a rope on this cylinder, the load, which is fixed at the end of this rope, can be lifted or lowered.

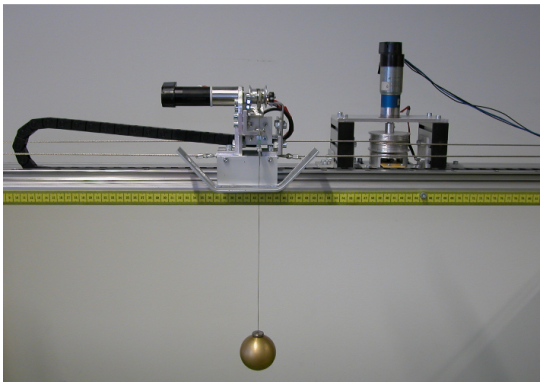

Fig. 1. Laboratory model gantry crane

For the mathematical modelling of the gantry crane, the sketch shown in Fig. 2 is used. Here the rope on which the load is fixed is supposed to be always stretched. Thus, the force F_S which is transmitted by the rope must always be positive. This assumption is valid as long as the inequality $\ddot{y}_L < g$ is fulfilled. The x-coordinate of the centre of rotation of the pendulum is denoted as x_D, the rotation

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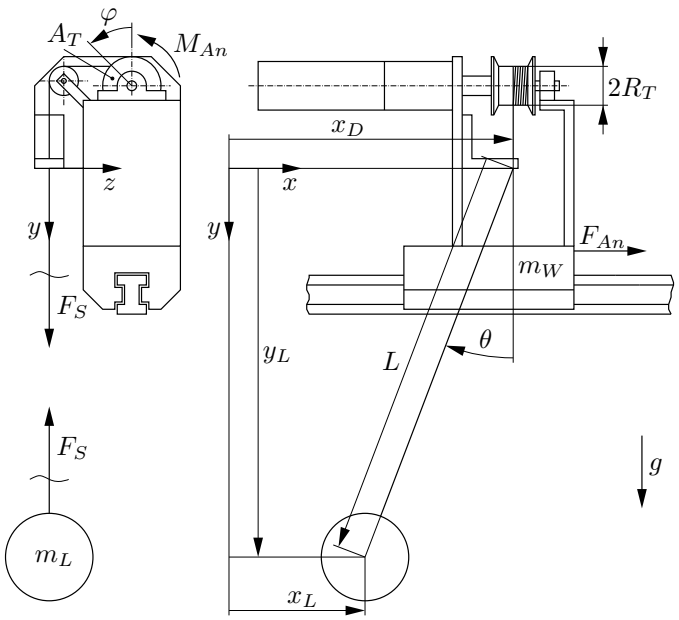

Fig. 2. Sketch showing the front view (right) and a side view (left) of the gantry crane

angle of the load cylinder is denoted as φ and θ describes the pendulum angle. The radius of the load cylinder is given by R_T. The length of the pendulum can be calculated from $L = L_0 + R_T\varphi$, where L_0 is the pendulum length for $\varphi = 0$. The position of the load is given by x_L and y_L. The parameter m_W describes the mass of the trolley and A_T represents the moment of inertia of the load cylinder. The load has the mass m_L and the gravitational acceleration g points in the positive y-direction. The driving force F_{An} which acts on the trolley and the driving torque M_{An} which acts on the load cylinder are the control inputs.

Since we assumed that the rope is always stretched, we are dealing with a rigid multi-body system with holonomic constraints. Therefore the equations of motion can be derived from the Euler-Lagrange equations (see e.g. Spong and Vidyasagar [1989]), which read

$$\tfrac{d}{dt}\left(\partial_{\dot{\mathbf{q}}} T\right) - \partial_{\mathbf{q}} T + \partial_{\mathbf{q}} V = \mathbf{Q}^T. \tag{1}$$

Here T denotes the system's kinetic energy, V is the potential, and $\mathbf{Q}$ represents the generalised forces. The variables $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the generalised coordinates resp. the generalised velocities. For the generalised coordinates the choice $\mathbf{q}^T = [x_D, \varphi, \theta]$ is made. The system's kinetic energy is given by

$$T = \tfrac{1}{2} m_W \dot{x}_D^2 + \tfrac{1}{2} A_T \dot{\varphi}^2 + \tfrac{1}{2} \mathbf{v}_L^T m_L \mathbf{v}_L, \tag{2}$$

where

$$\mathbf{v}_L = \begin{bmatrix} \dot{x}_D - R_T\dot{\varphi}\sin(\theta) - (L_0 + R_T\varphi)\,\dot{\theta}\cos(\theta) \\ R_T\dot{\varphi}\cos(\theta) - (L_0 + R_T\varphi)\,\dot{\theta}\sin(\theta) \\ 0 \end{bmatrix} \tag{3}$$

is the velocity of the load. The potential of the gantry crane is given by

$$V = -m_L g\,(L_0 + R_T\varphi)\cos(\theta), \tag{4}$$

and the driving force F_{An} and the driving torque M_{An} result in the generalised forces

$$\mathbf{Q}^T = [F_{An}, M_{An}, 0]\,. \tag{5}$$

Plugging T, V, and $\mathbf{Q}$ into (1) results in the equations of motion of the crane, which read as

$$\begin{aligned} &(m_W + m_L)\,\ddot{x}_D - m_L R_T \sin(\theta)\ddot{\varphi} \\ &\qquad - m_L\,(L_0 + R_T\varphi)\cos(\theta)\ddot{\theta} \\ &+ m_L\dot{\theta}\left((L_0 + R_T\varphi)\,\dot{\theta}\sin(\theta) - 2R_T\dot{\varphi}\cos(\theta)\right) = F_{An} \end{aligned} \tag{6}$$

$$\begin{aligned} &- m_L R_T \sin(\theta)\ddot{x}_D + \left(A_T + m_L R_T^2\right)\ddot{\varphi} \\ &\qquad - m_L R_T\left((L_0 + R_T\varphi)\,\dot{\theta}^2 + g\cos(\theta)\right) = M_{An} \end{aligned} \tag{7}$$

$$\begin{aligned} &- m_L\,(L_0 + R_T\varphi)\cos(\theta)\ddot{x}_D + m_L\,(L_0 + R_T\varphi)^2\,\ddot{\theta} \\ &\qquad + m_L\,(L_0 + R_T\varphi)\left(2R_T\dot{\varphi}\dot{\theta} + g\sin(\theta)\right) = 0. \end{aligned} \tag{8}$$

These equations can be written in the form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{Q} \tag{9}$$

with the symmetric and positive definite mass matrix $\mathbf{M}(\mathbf{q})$ and the vector $\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}})$. Because of its positive definiteness the mass matrix can be inverted. Hence the equations of motion can be solved for $\ddot{\mathbf{q}}$, which yields

$$\ddot{\mathbf{q}} = \mathbf{M}(\mathbf{q})^{-1}\left(\mathbf{Q} - \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}})\right). \tag{10}$$

By introducing the state $\mathbf{x}^T = [\mathbf{q}, \dot{\mathbf{q}}]$ and the input $\mathbf{u}^T = [F_{An}, M_{An}]$, a system of first order ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{11}$$

with

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{M}(\mathbf{q})^{-1}\left(\mathbf{Q} - \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}})\right) \end{bmatrix} \tag{12}$$

can be derived. Since the generalised coordinates $\mathbf{q}$ can be measured and the generalised velocities $\dot{\mathbf{q}}$ are calculated by numerical differentiation, the state $\mathbf{x}$ is known and therefore available for the controller design.

3. DIFFERENTIAL FLATNESS

The gantry crane is a differentially flat system and the coordinates of the load are a flat output. For the exact argumentation and a general definition of differentially flat systems see for instance Rudolph [2003b]. Therefore, we confine us here to show, how all system variables, i.e. all variables that were used in the mathematical model, can be expressed by the flat output $\mathbf{y}^T = [x_L, y_L]$ and its time derivatives.

3.1 Parameterisation of the System Variables by the Flat Output

In addition to the equations of motion (6), (7), and (8), the gantry crane is subject to the equations

$$x_L = x_D - (L_0 + R_T\varphi)\sin(\theta) \tag{13}$$

and

$$y_L = (L_0 + R_T\varphi)\cos(\theta). \tag{14}$$

Solving (13) and (14) for φ and θ we derive

$$\varphi = \frac{\sqrt{(x_D - x_L)^2 + y_L^2} - L_0}{R_T} \tag{15}$$

and

$$\theta = \arctan\left(\frac{x_D - x_L}{y_L}\right). \tag{16}$$

Here it is assumed that the length of the pendulum $L = (L_0 + R_T\varphi)$ is always positive. The calculation of the time derivative of (15) results in

$$\dot{\varphi} = \frac{(x_D - x_L)(\dot{x}_D - \dot{x}_L) + y_L\dot{y}_L}{R_T\sqrt{(x_D - x_L)^2 + y_L^2}}, \tag{17}$$

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and twofold differentiation of (16) yields

$$\dot{\theta} = \frac{(\dot{x}_D - \dot{x}_L) y_L - (x_D - x_L)\dot{y}_L}{(x_D - x_L)^2 + y_L^2} \tag{18}$$

$$\ddot{\theta} = f\left(x_D, \dot{x}_D, \ddot{x}_D, x_L, \dot{x}_L, \ddot{x}_L, y_L, \dot{y}_L, \ddot{y}_L\right). \tag{19}$$

By plugging (15) to (19) into (8), one obtains the relation

$$x_D = x_L + \frac{\ddot{x}_L y_L}{g - \ddot{y}_L}. \tag{20}$$

By consideration of (20), the equations (15) and (16) result in

$$\varphi = \frac{\sqrt{\left(\frac{\ddot{x}_L y_L}{g - \ddot{y}_L}\right)^2 + y_L^2} - L_0}{R_T} \tag{21}$$

and

$$\theta = \arctan\left(\frac{\ddot{x}_L}{g - \ddot{y}_L}\right). \tag{22}$$

With (20), (21), and (22), the generalised variables $\mathbf{q}$ are given as functions of $\mathbf{y}$ and its time derivatives. Now these expressions are differentiated two times. One obtains

$$\dot{x}_D = \dot{x}_L + \frac{x_L^{(3)} y_L + \ddot{x}_L \dot{y}_L}{g - \ddot{y}_L} + \frac{\ddot{x}_L y_L \dot{y}_L^{(3)}}{(g - \ddot{y}_L)^2} \tag{23}$$

$$\ddot{x}_D = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right) \tag{24}$$

as well as

$$\dot{\varphi} = f\left(\ddot{x}_L, x_L^{(3)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}\right) \tag{25}$$

$$\ddot{\varphi} = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right) \tag{26}$$

and

$$\dot{\theta} = \frac{\ddot{x}_L y_L^{(3)} + x_L^{(3)}(g - \ddot{y}_L)}{\ddot{x}_L^2 + (g - \ddot{y}_L)^2} \tag{27}$$

$$\ddot{\theta} = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right). \tag{28}$$

Plugging (21), (22), and (24) to (28) into (6) and (7) yields

$$F_{An} = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right) \tag{29}$$

$$M_{An} = f\left(\ddot{x}_L, x_L^{(3)}, x_L^{(4)}, y_L, \dot{y}_L, \ddot{y}_L, y_L^{(3)}, y_L^{(4)}\right), \tag{30}$$

which is the parameterisation of the control input by the flat output and its time derivatives.

4. FLATNESS BASED TRACKING CONTROL

For differentially flat systems it is possible to design a flatness based tracking controller, see for instance Rudolph [2003a]. With such a control the flat output can be forced to track preset reference trajectories. The design is accomplished in two steps. First, the system is exactly linearised by defining an appropriate new input. In the present paper, this linearisation is performed by means of a quasi-static state feedback as discussed in Delaleau and Rudolph [1998]. Then the new input is chosen such that the tracking-error systems are rendered linear and stable.

4.1 Exact Linearisation by Quasi-Static State Feedback

According to Rudolph [2005], the tuple

$$\tilde{\mathbf{x}}^T = \left[x_L, \dot{x}_L, \ddot{x}_L, x_L^{(3)}, y_L, \dot{y}_L\right] \tag{31}$$

represents a Brunovsky state for the gantry crane. Therefore, the new input $v_1 = x_L^{(4)}$, $v_2 = \ddot{y}_L$ defines a quasi-static state feedback which obviously results in the linear system

$$\begin{aligned} x_L^{(4)} &= v_1 \\ \ddot{y}_L &= v_2. \end{aligned} \tag{32}$$

Information on Brunovsky states and quasi-static state feedback can be found for instance in Delaleau and Rudolph [1998] and Rudolph [2005].

For the realisation of the feedback, the control inputs F_{An} and M_{An} must be expressed by functions which depend on the known state $\mathbf{x}$, the new input $\mathbf{v}$, and time derivatives of $\mathbf{v}$ only. Now, we present the derivation of the relations for the control input in detail, because one cannot find them in the literature according to the knowledge of the authors. Let us consider (13) and (14). Differentiation with respect to the time gives

$$\dot{x}_L = \dot{x}_D - R_T \sin(\theta)\dot{\varphi} - (L_0 + R_T\varphi)\cos(\theta)\dot{\theta} \tag{33}$$

$$\dot{y}_L = R_T \cos(\theta)\dot{\varphi} - (L_0 + R_T\varphi)\sin(\theta)\dot{\theta}. \tag{34}$$

Calculation of the time derivatives of (33) and (34) yields expressions that contain $\ddot{\mathbf{q}}$. By replacing $\ddot{\mathbf{q}}$ with (10) one obtains

$$\ddot{x}_L = f\left(\mathbf{x}, \mathbf{u}\right) \tag{35}$$

$$\ddot{y}_L = f\left(\mathbf{x}, \mathbf{u}\right). \tag{36}$$

Equation (35) is now differentiated another two times, each time replacing $\ddot{\mathbf{q}}$ with (10). This results in

$$x_L^{(3)} = f\left(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}\right) \tag{37}$$

and

$$x_L^{(4)} = f\left(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}\right). \tag{38}$$

Now the right-hand side of (38) is equated with v_1 and the right-hand side of (36) is equated with v_2. This gives the equations

$$v_1 = f\left(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}\right) \tag{39}$$

$$v_2 = f\left(\mathbf{x}, \mathbf{u}\right). \tag{40}$$

Solving (40) for M_{An} one gets

$$M_{An} = f\left(\mathbf{x}, F_{An}, v_2\right). \tag{41}$$

This expression is now differentiated with respect to the time. By replacing $\ddot{\mathbf{q}}$ with (10) and subsequently M_{An} with (41), one obtains

$$\dot{M}_{An} = f\left(\mathbf{x}, F_{An}, \dot{F}_{An}, v_2, \dot{v}_2\right). \tag{42}$$

Repeating this procedure yields

$$\ddot{M}_{An} = f\left(\mathbf{x}, F_{An}, \dot{F}_{An}, \ddot{F}_{An}, v_2, \dot{v}_2, \ddot{v}_2\right).$$

Plugging the above expressions for M_{An}, $\dot{M}_{An}$, and $\ddot{M}_{An}$ into (39) gives an equation of the form

$$v_1 = f\left(\mathbf{x}, F_{An}, v_2, \dot{v}_2, \ddot{v}_2\right). \tag{43}$$

It is important to state that in this equation there does not occur any time derivative of F_{An}. Solving for F_{An} yields

$$F_{An} = f\left(\mathbf{x}, v_1, v_2, \dot{v}_2, \ddot{v}_2\right). \tag{44}$$

Eventually, by plugging (44) into (41) one obtains

$$M_{An} = f\left(\mathbf{x}, v_1, v_2, \dot{v}_2, \ddot{v}_2\right). \tag{45}$$

The equations (44) and (45) represent the required functions for the control inputs.

4.2 Stabilisation about the Reference Trajectories

In order to receive linear tracking-error systems, the input $\mathbf{v}$ of the linear system (32) is chosen as

$$\begin{aligned} v_1 = x_{L,d}^{(4)} &- a_{1,3}\left(x_L^{(3)} - x_{L,d}^{(3)}\right) - a_{1,2}\left(\ddot{x}_L - \ddot{x}_{L,d}\right) \\ &- a_{1,1}\left(\dot{x}_L - \dot{x}_{L,d}\right) - a_{1,0}\left(x_L - x_{L,d}\right) - a_{1,I} e_{1,I} \end{aligned} \tag{46}$$

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and

$$v_2 = \ddot{y}_{L,d} - a_{2,1}\left(\dot{y}_L - \dot{y}_{L,d}\right) - a_{2,0}\left(y_L - y_{L,d}\right) - a_{2,I} e_{2,I}, \quad (47)$$

where

$$e_{1,I} = \int_{t_0}^{t} \left(x_L(\tau) - x_{L,d}(\tau)\right) d\tau \quad (48)$$

and

$$e_{2,I} = \int_{t_0}^{t} \left(y_L(\tau) - y_{L,d}(\tau)\right) d\tau \quad (49)$$

are the integrated tracking errors in x- and y-direction. The variables $x_{L,d}$ and $y_{L,d}$ denote the x- and the y-coordinate of the reference trajectory $\mathbf{y}_d$ of the flat output. With the tracking error in x-direction $e_1 = (x_L - x_{L,d})$ and

$$\mathbf{e}_1^T = \left[e_{1,I}, e_1, \dot{e}_1, \ddot{e}_1, e_1^{(3)}\right], \quad (50)$$

the resulting tracking-error system for the x-direction reads

$$\dot{\mathbf{e}}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_{1,I} & -a_{1,0} & -a_{1,1} & -a_{1,2} & -a_{1,3} \end{bmatrix} \mathbf{e}_1. \quad (51)$$

Likewise, with the tracking error $e_2 = (y_L - y_{L,d})$ and

$$\mathbf{e}_2^T = [e_{2,I}, e_2, \dot{e}_2] \quad (52)$$

one obtains the tracking-error system

$$\dot{\mathbf{e}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{2,I} & -a_{2,0} & -a_{2,1} \end{bmatrix} \mathbf{e}_2 \quad (53)$$

for the y-direction. The eigenvalues of the error systems can be placed freely by setting the values of the parameters $a_{i,j}$ and $a_{i,I}$.

For the calculation of the control inputs from (44) and (45), in addition to v_1 and v_2 the first and the second time derivative of v_2 are also required. The differentiation of (47) yields

$$\dot{v}_2 = y_{L,d}^{(3)} - a_{2,1}\left(\ddot{y}_L - \ddot{y}_{L,d}\right) - a_{2,0}\left(\dot{y}_L - \dot{y}_{L,d}\right) - a_{2,I}\left(y_L - y_{L,d}\right). \quad (54)$$

Because of (32), $\ddot{y}_L$ can be replaced with v_2, which results in

$$\dot{v}_2 = y_{L,d}^{(3)} - a_{2,1}\left(v_2 - \ddot{y}_{L,d}\right) - a_{2,0}\left(\dot{y}_L - \dot{y}_{L,d}\right) - a_{2,I}\left(y_L - y_{L,d}\right). \quad (55)$$

Further differentiation and subsequent replacement of $\ddot{y}_L$ gives

$$\ddot{v}_2 = y_{L,d}^{(4)} - a_{2,1}\left(\dot{v}_2 - y_{L,d}^{(3)}\right) - a_{2,0}\left(v_2 - \ddot{y}_{L,d}\right) - a_{2,I}\left(\dot{y}_L - \dot{y}_{L,d}\right). \quad (56)$$

For the realisation of the stabilising feedback, v_1, v_2, $\dot{v}_2$, and $\ddot{v}_2$ have to be expressed by known variables, more precisely meaning the state $\mathbf{x}$, the integrated tracking errors $e_{1,I}$ and $e_{2,I}$ plus the desired trajectories $\mathbf{y}_d$ and their time derivatives. By means of (14) and (34), the variables y_L and $\dot{y}_L$ can be calculated from $\mathbf{x}$. Thus, v_2 and its time derivatives are given by expressions of the form

$$\begin{aligned} v_2 &= f\left(\mathbf{x}, e_{2,I}, y_{L,d}, \dot{y}_{L,d}, \ddot{y}_{L,d}\right) \\ \dot{v}_2 &= f\left(\mathbf{x}, e_{2,I}, y_{L,d}, \dot{y}_{L,d}, y_{L,d}^{(3)}\right) \\ \ddot{v}_2 &= f\left(\mathbf{x}, e_{2,I}, y_{L,d}, \dot{y}_{L,d}, y_{L,d}^{(4)}\right). \end{aligned} \quad (57)$$

For the computation of v_1, the time derivatives of x_L up to the order of three are required. Via (13) and (33), the variables x_L and $\dot{x}_L$ can be expressed by $\mathbf{x}$. The variables $\ddot{x}_L$ and $x_L^{(3)}$ additionally depend on the control input $\mathbf{u}$ and its time derivative $\dot{\mathbf{u}}$. However, because of the quasi-static state feedback the components of $\mathbf{u}$ and $\dot{\mathbf{u}}$ are related by (41) and (42). The substitution of (41) for M_{An} in (35) yields

$$\ddot{x}_L = (g - v_2)\tan(\theta). \quad (58)$$

Similarly, the insertion of (41) and (42) into (37) results in

$$x_L^{(3)} = \frac{(g - v_2)\dot{\theta} - \dot{v}_2 \sin(\theta)\cos(\theta)}{\cos(\theta)^2}. \quad (59)$$

Since it has already been shown that v_2 and $\dot{v}_2$ can be calculated, $\ddot{x}_L$ und $x_L^{(3)}$ are also known and therefore available for the computation of v_1. One obtains an expression of the form

$$v_1 = f\left(\mathbf{x}, e_{1,I}, e_{2,I}, \mathbf{y}_d, \dot{\mathbf{y}}_d, \ddot{\mathbf{y}}_d, \mathbf{y}_d^{(3)}, \mathbf{y}_d^{(4)}\right). \quad (60)$$

With (60) and (57), the required expressions for $\mathbf{v}$ and its time derivatives are given.

4.3 Approximate Compensation of the Friction

Although the mechanical quality of the laboratory setup is rather high, viscous and even more important Coulomb friction are not negligible. For the compensation of the friction which acts on the trolley, a correction term

$$F_{R,W,d} = \begin{cases} r_{v,Wp}\dot{x}_{D,d} + r_{C,Wp} & \text{for} \quad \dot{x}_{D,d} > 0 \\ r_{v,Wn}\dot{x}_{D,d} - r_{C,Wn} & \text{for} \quad \dot{x}_{D,d} < 0 \\ 0 & \text{for} \quad \dot{x}_{D,d} = 0 \end{cases} \quad (61)$$

is added to the value of F_{An} resulting from (44). Similarly, to compensate for the friction acting on the load cylinder, we add the term

$$M_{R,T,d} = \begin{cases} r_{v,Tp}\dot{\varphi}_d + r_{C,Tp} & \text{for} \quad \dot{\varphi}_d > 0 \\ r_{v,Tn}\dot{\varphi}_d - r_{C,Tn} & \text{for} \quad \dot{\varphi}_d < 0 \\ 0 & \text{for} \quad \dot{\varphi}_d = 0 \end{cases} \quad (62)$$

to the value of M_{An} resulting from (45). For the evaluation of (61) and (62), the variables $\dot{x}_{D,d}$ resp. $\dot{\varphi}_d$ are required. They can be calculated easily by insertion of the reference trajectories into (23) resp. (25), which yields

$$\dot{x}_{D,d} = \dot{x}_{L,d} + \frac{x_{L,d}^{(3)} y_{L,d} + \ddot{x}_{L,d}\dot{y}_{L,d}}{g - \ddot{y}_{L,d}} + \frac{\ddot{x}_{L,d} y_{L,d} y_{L,d}^{(3)}}{\left(g - \ddot{y}_{L,d}\right)^2} \quad (63)$$

and

$$\dot{\varphi}_d = f\left(\ddot{x}_{L,d}, x_{L,d}^{(3)}, y_{L,d}, \dot{y}_{L,d}, \ddot{y}_{L,d}, y_L^{(3)}\right). \quad (64)$$

The parameters $r_{v,Wp}$, $r_{v,Wn}$, $r_{C,Wp}$, and $r_{C,Wn}$ resp. $r_{v,Tp}$, $r_{v,Tn}$, $r_{C,Tp}$, and $r_{C,Tn}$ are the viscous and the Coulomb coefficients of friction. Since experiments have shown that the friction depends slightly on the direction of movement, a distinction between positive and negative direction is made.

4.4 Measurement Results

The control law was implemented on a dSPACE® real-time system. The sampling time was set to 10 ms. The

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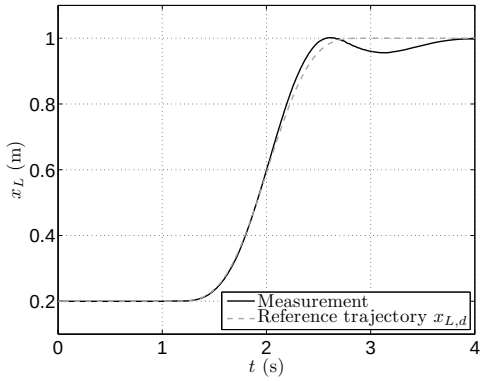
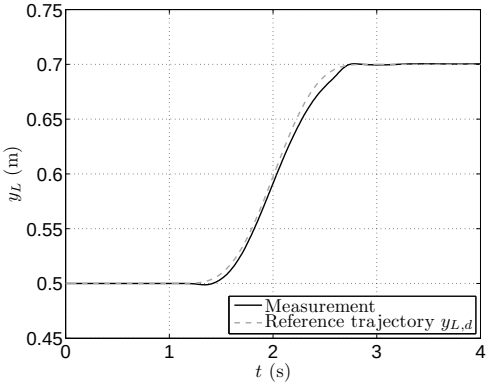
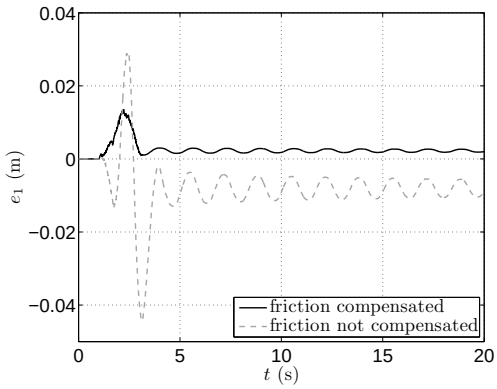


(a) x-coordinate of the load

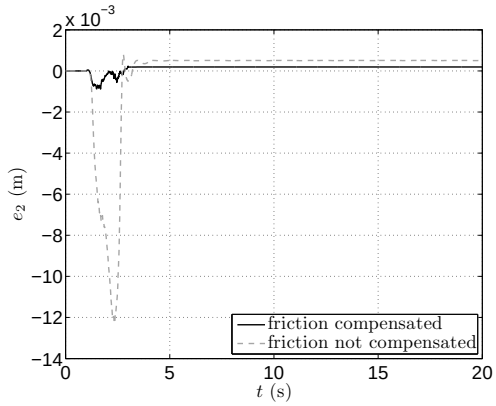


(b) y-coordinate of the load

Fig. 3. Tracking behaviour without integral parts and without compensation of the friction



(a) x-coordinate of the load



(b) y-coordinate of the load

Fig. 4. Tracking error without integral parts

reference trajectories $x_{L,d}$ and $y_{L,d}$ for the x- and the y-coordinate of the load are polynomials of degree nine. Their coefficients were calculated from the initial and the final values of $\mathbf{y}_d$ and its time derivatives. Since the load should be transferred from one equilibrium to another one, the initial and the final values of the time derivatives of $\mathbf{y}_d$ are set to zero. To show the impact of the integral parts and the compensation of the friction on the performance of the control circuit, the measurements were made with different controller settings. First, the integral parts were not used and the terms compensating for the friction were also omitted. The integral parts can be disabled by setting $a_{1,I} = a_{2,I} = 0$. As a consequence, the order of both error systems is decremented by one. To obtain stable tracking-error systems, the coefficients $a_{i,j}$ of the feedback must be chosen such that the resulting eigenvalues have negative real parts. By placing the eigenvalues sufficiently far left in the complex plane, excellent simulation results can be achieved (of course, friction is considered in the simulation model). In contrast, experiments with the gantry crane have shown that the eigenvalues may not be placed too far left because otherwise there occur oscillations in the haulage cable that moves the trolley. The characteristic polynomials of the error systems were set as

$$p_1(s) = (s + 5.5)^4 \tag{65}$$

$$p_2(s) = (s + 15)^2. \tag{66}$$

It may be remarked that of course it is not necessary to set all eigenvalues of an error system to the same value, but since other settings did not result in a better performance of the control circuit, the above, very simple choice was made. The measurement results are shown in Fig. 3. Obviously, there is a significant deviation between the reference and the actual trajectory of the load both in x- and y-direction. This is due to the fact that it was not possible to place the eigenvalues of the error systems farther left because of the problems with the haulage cable.

In the following it shall be shown how the tracking behaviour can be improved by use of the integral parts and the compensation of the friction. In order to facilitate the comparison of the measurement results which were achieved with the different controller settings, only the tracking error is shown. The underlying reference trajectory is the same as in Fig. 3, i.e. the trajectory leaves the initial rest position at $t = 1$ s and reaches the final rest position at $t = 3$ s. Fig. 4 shows a comparison between the measured tracking error with and without compensation of the friction. It can be seen that compensating for the friction considerably reduces the tracking error. Since the integral parts were not used, there remains a steady state control deviation. As for the integral parts, it turned out to be useful to activate them only after the reference trajectory has reached the final rest position. As long as the integrators are not activated, the tracking-error systems are of order four resp. two as above. Thus, it is straightforward to use the same values for the coefficients $a_{i,j}$. Once the integrators are activated, the tracking-error systems are given by (51) resp. (53). Hence, the coefficients $a_{i,I}$ must be set such that these error systems are stable. Experiments have shown that the choice $a_{1,I} = 1500$ and $a_{2,I} = 200$ yields very satisfying results. Fig. 5 shows the measured tracking error with and without compensation

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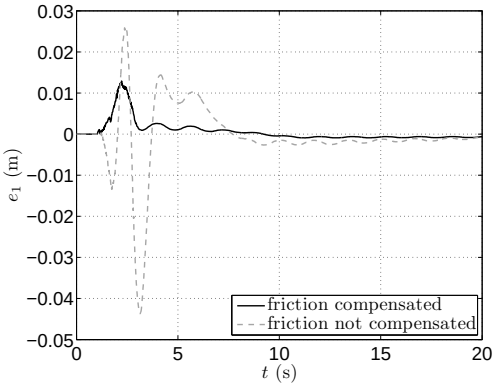
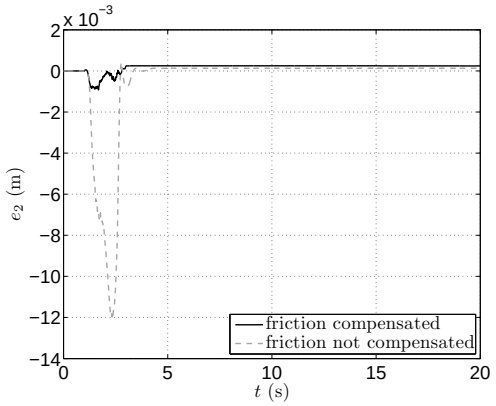
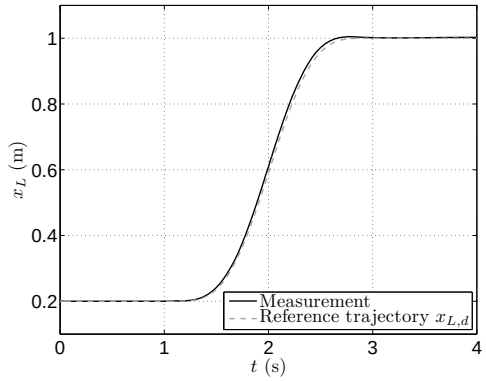


(a) x-coordinate of the load

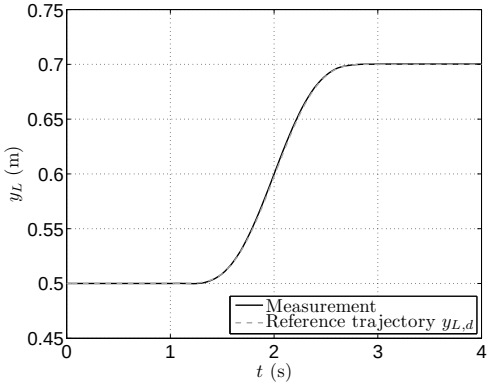


(b) y-coordinate of the load

Fig. 5. Tracking error with integral parts



(a) x-coordinate of the load



(b) y-coordinate of the load

Fig. 6. Tracking behaviour with integral parts and compensation of the friction

of the friction. In both cases, a comparison with Fig. 4 shows that thanks to the integral parts, the final position of rest is reached with a significantly increased accuracy.

The comparison of the presented measurements clearly shows that the best results are achieved by the application of both the integral parts and the compensation of the friction. Fig. 6 shows the measured and the desired trajectories for this case. In conclusion it can be stated that thanks to the integral parts and the compensation of the friction, the presented control algorithm yields excellent results.

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