Extremum seeking via continuation techniques for optimizing biogas production in the chemostat

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Abstract: We consider the chemostat model with the substrate concentration as the single measurement. We propose a control strategy that drives the system at a steady state maximizing the gas production without the knowledge of the specific growth rate. Our approach separates the extremum seeking problem from the feedback control problem such that each of the two subproblems can be solved with relatively simple algorithms. We are then free to choose any numerical optimization algorithm. We give a demonstration for two choices: one is based on slow-fast dynamics and numerical continuation, the other is a combination of golden-section and Newton iteration. The method copes with non-monotonic growth functions.

Keywords: adaptive control, self-optimizing control, parameter optimization, biotechnology.

1. INTRODUCTION

The control design of chemostat models have been extensively studied in the literature, with the objective to provide efficient and reliable control strategies for industrial applications, such as biotechnological or pharmaceutical processes [7, 8]. A common task is to drive a continuous stirred tank bioreactor to a set-point that optimizes an objective function, for instance the gas production rate [9, 32]. In this work, we consider the chemostat model with a single strain

$$\begin{aligned} \dot{s} &= -\mu(s)b + u(s_{\text{in}} - s) \\ \dot{b} &= \mu(s)b - ub \end{aligned} \quad (1)$$

where the state variables s and b denote the substrate and biomass concentrations, respectively (in these equations the yield coefficient has been chosen equal to unity without loss of generality). The concentration of substrate in the feed is denoted by s_{in}, and the dilution rate $u \geq 0$ is the manipulated variable. The production rate to be maximized is given by

$$r = \mu(s)b \ .$$

In such bioprocesses, if often happens that the growth kinetics $\mu(\cdot)$ is unknown (or poorly known) and possibly evolves slowly with time or changes of environment (such as temperature or pH value). Consequently, the robustness of the control strategies with respect to uncertainties on $\mu(\cdot)$ is a crucial issue for real applications. Many works have considered the on-line estimation of the growth function [16, 11, 35, 26, 6] and the robust stabilization of such processes about a given reference point [10, 18, 29, 28, 30], but there are comparatively fewer works concerned with the maximization of an objective under model uncertain-

ties [12]. Such issues are typically addressed by the design techniques of extremum seeking controls [21, 19, 3, 33, 34]. Roughly speaking, these techniques consist in adding an excitation signal to the input $u = \bar{u} + a\sin(\omega t)$, of small amplitude a and high frequency ω, and capturing an estimation of the gradient of the objective function by filtering. Three time scales (excitation signal period=fast, plant dynamics=medium, filtering=slow) then operate in the overall closed loop system. Such a technique has been developed first in [36] for unknown functions $\mu(\cdot)$ of Monod or Haldane type (the term Monod refers to monotone increasing growth rate μ, whereas growth rates of Haldane type have a maximum at some intermediate $s_{\max} < s_{\text{in}}$ and decrease for $s > s_{\max}$). Later on, an adaptive extremum-seeking scheme has been proposed to improve the performances of the transient response, for the Monod's case in [37] and the Haldane's one in [24]. In these approaches, the production rate r is assumed to be measured online, and explicit expressions of the unknown function $\mu(\cdot)$ are required. An alternative extremum-seeking scheme combined with a neural network has been proposed in [15] to get around the requirement for an analytical expression of the growth function. Nevertheless, all mentioned approaches require the on-line measurement of the whole state (s, b), except for [25] where b only is measured but the result is restricted to the Monod's kinetics. In the present paper, we consider that the single on-line measurement

$$y = s \quad (2)$$

is available. One may consider coupling the former extremum-seeking controllers with state observers [7, 14, 27], but there is an intrinsic difficulty due to the fact that "good" observers, i.e. whose speed of convergence

can be arbitrary tuned, require the exact expression of the dynamics, and that "fast" observers are often sensitive to measurement noise and input disturbances.

We present here a rather different and new approach that does not require any knowledge of the growth function. Neither do we require the ability to measure the objective function at all times. Instead of feeding an excitation signal into the plant (the usual approach in extremum seeking) we apply a classical feedback control stabilizing the plant to guarantee a uniform decay of transients. This stabilization step is then combined with classical numerical methods that treat the equilibrium output of the feedback-stabilized plant as an unknown function of the input. We demonstrate this approach with two different numerical methods. In sections 3 and 5.1 we apply a continuation method [1] tracing out the graph of equilibrium outputs (depending on the inputs) until we reach the maximum of the objective. In sections 4 and 5.2 we apply a classical optimization method to the graph, in our case a golden section iteration and Newton iterations. Our approach splits the dynamic extrmemum seeking problem into two subproblems: a feedback stabilization problem and a numerical optimization problem. This has several advantages:

First, the proof of convergence of the overall scheme is split into simpler subproblems: stability of the feedback for fixed inputs and convergence of the numerical algorithm. In the case of the chemostat with a growth rate of Haldane type the first part (stability) is globally true [31]. The second part (convergence of numerics) follows in general at least locally from the general theorems developed for numerical algorithms. For the specific case of the chemostat with growth rate of Haldane type one can again state global convergence for the continuation and the golden-section iteration because of the uniqueness of the local maximum.

Second, the number of different time scales is reduced to two: the decay time of the feedback controlled system and the iteration process of the numerical algorithm. In theory the decay time can be made arbitrarily short by adjusting the feedback gains. In practice this is limited by intrinsic noise, measurement noise and sampling time. The numerical convergence speed can be superlinear (quadratic) in the ideal case for Newton iterations.

Third, one only needs to evaluate the objective at equilibrium. One can see the importance of this aspect in the chemostat example: the rate $r = \mu(s)b$ depends on the unknown μ and the internal state b, which is not measured. However, in equilibrium μ equals the input u and b equals $s_{\text{in}} - s$ such that we can apply the continuation or optimization to $D(s_{\text{in}} - s)$ instead.

Our extremum seeking scheme is an extension of a former work [31] that gives such a method for the reconstruction of the graph of $\mu(\cdot)$ with the single measurement of the substrate, without any a priori knowledge on μ (excepted to be a smooth function). In particular, this technique can cope with non-monotonic growth functions and allows to identify unstable states of the open-loop system. The paper is organized as follows. In Section 2 we first present our general methodology, and then show how to apply it on the chemostat model in Section 5. Section 6 is devoted to numerical demonstrations.

2. ASSUMPTIONS IN THE GENERAL FRAMEWORK

We consider a single-input/single-output dynamical system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{3}$$

($x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$) and an objective function

$$z = \phi(y) \tag{4}$$

to be optimized at steady state, i.e., we are looking for an output feedback controller that steers the state to an operating point $(x^\star, u^\star)$ that fulfills

$$\begin{aligned} &f(x^\star, u^\star) = 0 \\ &\phi(h(x)) \text{ is locally max. w.r.t. } u \in \mathbb{R} \text{ at } x^\star \end{aligned} \tag{5}$$

where the functions f and ϕ are unknown or partially known. We follow the statement of the general nonlinear problem of extremum seeking given in [3], that we adapt here to an output feedback framework.

Assumption A1 (Existence of stabilizing output feedback). There exists a smooth output feedback $u(t) = \alpha(y(t), \bar{u}, \bar{y})$ parametrized by the parameter pair $(\bar{u}, \bar{y}) \in \mathcal{U} \times \mathcal{Y}$ (a pair of reference input and output), such that

$$\alpha(\bar{y}, \bar{u}, \bar{y}) = \bar{u} , \tag{6}$$

and the closed loop system

$$\dot{x} = f(x, \alpha(h(x), \bar{u}, \bar{y})) \tag{7}$$

admits a unique equilibrium $x_{eq}(\bar{u}, \bar{y})$, that is locally asymptotically stable for any $(\bar{u}, \bar{y}) \in \mathcal{U} \times \mathcal{Y}$.

For example, for the chemostat $(1), (2)$ the output feedback could be of the form $u(t) = \bar{u} + G_1(\bar{s} - s(t))$ with a sufficiently large G_1 [31].

Under Assumption A1, we consider the extremum-seeking problem for the closed-loop dynamics (7) as if the pair $(\bar{u}, \bar{y})$ was a new control, and look for pairs such that $h(x_{eq}(\bar{u}, \bar{y})) = \bar{y}$. For this purpose, we shall consider the output-input characteristics defined as follows.

Assumption A2 (Output-input characteristics). There exists a smooth function $\psi : \mathcal{Y} \mapsto \mathcal{U}$ such that

$$\bar{y} = h(x_{eq}(\bar{u}, \bar{y})) \iff \bar{u} = \psi(\bar{y}) \tag{8}$$

for any $\bar{y} \in \mathcal{Y}$.

This assumption means that for each parameter $\bar{y} \in \mathcal{Y}$, there exists an unique input $\bar{u} \in \mathcal{U}$ that is a zero of the function $u \mapsto h(x_{eq}(u, \bar{y})) - \bar{y}$. For instance, in the chemostat model, positive equilibriums have to fulfill $\alpha(s_{eq}, \bar{u}, \bar{s}) = \mu(s_{eq})$, and then having $s_{eq} = \bar{s}$, along with (6), amounts to write $\bar{u} = \mu(\bar{y})$.

Assumption A3 (Existence of local maximum). There exists $y^\star \in \mathcal{Y}$ such that the function $\phi(\cdot)$ has a local strict maximum at $y^\star$.

Finally, we shall assume that the unknown objective function $\phi(\cdot)$ possesses a known structure.

Assumption A4 (Structure of objective function). There exists a smooth function $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\phi(y) = \varphi(y, \psi(y)) , \forall y \in \mathcal{Y} \tag{9}$$

Assumptions A2 and A4 imply that the extremum seeking problem amouts to optimize a known function of y and u at steady state. Thus, the problem consists then in finding

an output feedback strategy that drives the state of the system to $x^\star$ such that $h(x^\star) = y^\star$ using the knowledge of the feedback law α and the objective φ only. The following two sections present two approaches.

3. CONTINUOUS-TIME ADAPTATION

The continuous-time approach first constructs a differential equation for $\bar{u}$ that achieves $\bar{u} = \psi(\bar{y})$ (and, thus, $\bar{y} = h(\bar{x})$) asymptotically, and then applies a gradient search (again through a differential equation) along the curve of $(\bar{u}, \bar{y})$ given implicitly by $\bar{y} = h(\bar{x}(\bar{u}, \bar{y}))$. Note that for feedback laws of the form $u(t) = \bar{u} + G[\bar{y} - y(t)]$ this implicit curve corresponds to the curve of open-loop equilibrium outputs.

3.1 Step 1: an adaptive scheme for $\bar{u}$

We look for a dynamics
$$\dot{\bar{u}} = \beta(y, \bar{u}, \bar{y}) \qquad (10)$$
with $\beta(\bar{y}, \bar{u}, \bar{y}) = 0$, such that
$$\bar{E}(\bar{y}) = \begin{bmatrix} x_{eq}(\psi(\bar{y}), \bar{y}) \\ \psi(\bar{y}) \end{bmatrix} \qquad (11)$$
is a locally asymptotically stable equilibrium of the coupled dynamics
$$\begin{aligned} \dot{x} &= f(x, \alpha(h(x), (\bar{u}, \bar{y}))) \\ \dot{\bar{u}} &= \beta(h(x), \bar{u}, \bar{y}) \end{aligned} \qquad (12)$$
Note that Assumptions A1 and A2 ensures that $E(\bar{y})$ is an equilibrium of (12) for any $\bar{y} \in \mathcal{Y}$, and that one has
$$\lim_{t \to +\infty} y(t) = \bar{y} \ . \qquad (13)$$
This is a classical adaptive output control problem, for which several techniques are available in the literature [5, 20]. We simply require the convergence
$$\lim_{t \to +\infty} \bar{u}(t) = \psi(\bar{y}) \qquad (14)$$
to be uniform w.r.t. $\bar{y} \in \mathcal{Y}$.

3.2 Step 2: a continuation method for $\bar{y}$

Step 1 with Assumption A4 provides an approximation of the unknown objective function $\phi(\cdot)$ at a fixed $\bar{y}$:
$$\phi(\bar{y}) \simeq \bar{\phi}(\bar{y}) = \varphi(\bar{y}, \bar{u}) \ . \qquad (15)$$
The continuation consists in having $\bar{y}$ evolving slowly (step-wise or continuously) with a steepest descent based on an estimation of the gradient of the function ϕ. A continuous adaptation can be written as
$$\dot{\bar{y}} = -\epsilon \left[\partial_1 \varphi(\bar{y}, \bar{u}) + \partial_2 \varphi(\bar{y}, \bar{u}) \hat{\psi}'(\bar{y}, \bar{u}) \right] \qquad (16)$$
where $\epsilon > 0$ is small compared to the time scale of dynamics (12), and $\hat{\psi}'(\bar{y}, \bar{u})$ is an estimation of the derivative of the function ψ at $\bar{y}$, for which $\bar{u}$ is an estimation of $\psi(\bar{y})$. Several gradient estimations techniques are available in the literature [23, 13]. In the scalar case, an estimation of the sign of the derivative of the function ϕ is enough to choose the steepest descent. We can use for instance a dynamics with delay:
$$\begin{aligned} \frac{1}{\epsilon} \dot{\bar{y}} &= \delta(\bar{y}, \bar{u}, \bar{y}(t-\tau), \bar{u}(t-\tau)) \\ &= \operatorname{sgn}\left[(\varphi(\bar{y}, \bar{u}) - \varphi(\bar{y}(t-\tau), u(t-\tau))(\bar{y} - \bar{y}(t-\tau)) \right] \ . \end{aligned} \qquad (17)$$

Then, the overall dynamics
$$\begin{aligned} \dot{x} &= f(x, \alpha(h(x), \bar{u}, \bar{y})) \\ \dot{\bar{u}} &= \beta(h(x), \bar{u}, \bar{y}) \\ \dot{\bar{y}} &= \epsilon \delta(\bar{y}, \bar{u}, \bar{y}(t-\tau), \bar{u}(t-\tau)) \end{aligned} \qquad (18)$$
is slow-fast where $\bar{u} = \psi(\bar{y})$ is the (attracting) critical manifold. Finally, the output feedback strategy that we consider takes the following form
$$u = \alpha(y, \bar{u}, \bar{y}) \text{ with } \begin{cases} \dot{\bar{u}} = \beta(y, \bar{u}, \bar{y}) \\ \dot{\bar{y}} = \epsilon \delta(\bar{y}, \bar{u}, \bar{y}(t-\tau), \bar{u}(t-\tau)) \end{cases} \qquad (19)$$
Let us underline that the output y is naturally filtered by the dynamics to obtain the estimation $y^\star$ as
$$\hat{y}^\star = \lim_{t \to +\infty} \bar{y}(t) \qquad (20)$$
that provides a robustness w.r.t. to measurement noise. Of course, the price to pay is to have a slow convergence due to small ϵ.

4. EXTREMUM-SEEKING USING NUMERICAL OPTIMIZATION ALGORITHMS

We compare the continuous-time adaptation with classical numerical optimization algorithms that are difficult to express as dynamical systems such as golden-section iteration (which is applicable for single-parameter problems) and Newton iterations. These algorithms do not require any knowledge of the underlying system apart from the ability to evaluate the asymptotic output $\lim_{t \to \infty} y(t)$ for any given admissible inputs $(\bar{u}, \bar{y})$. Once the stabilizing feedback law $u(t) = \alpha(y(t), \bar{u}, \bar{y})$ is implemented, the problem is reduced to a classical numerical optimization problem: $\phi(h(x_{eq}(\bar{u}, \bar{y}))) \to \max$. In fact, if the feedback law is linear of the type
$$u(t) = \alpha(y(t), \bar{u}, \bar{y}) = \bar{u} + G[\bar{y} - y(t)] \qquad (21)$$
this is an optimization problem in the single parameter $\bar{v} = \bar{u} + G\bar{y}$:
$$\phi(h(x_{eq}(\bar{u} + G\bar{y}))) =: F(\bar{v}) \to \max \quad \text{w.r.t } \bar{v}. \qquad (22)$$
We test a combination of two classical optimization algorithms. We start with an initial admissible interval $[\bar{v}_{\text{low}}, \bar{v}_{\text{up}}]$. Then we use golden section search to narrow the initial interval down to a given fraction of its original size. Finally we apply a Newton iteration to solve $F'(\bar{v}) = 0$ in the remaining interval. Neither of these two methods requires knowledge about the structure of F. We only have to evaluate F and its derivatives at any desired point (where derivatives can be approximated by finite differences). For the evaluation of F and its derivatives in a given point $\bar{v}_0$ we use a naive *act-and-wait* approach [17]. First, one sets the parameter $\bar{v} = \bar{v}_0$, such that the feedback control law is $u(t) = \bar{v}_0 - Gy(t)$. Then one waits until the transients have settled (such that $x = \bar{x}_0 := x_{eq}(\bar{v}_0)$). Then one reads off $y = h(\bar{x}_0)$ and evaluates $\phi(y) = \varphi(y, \bar{v}_0 - Gy)$. For the golden section search we iterate from an initial interval $[\bar{v}_1, \bar{v}_3]$ by setting $\bar{v}_2 = \bar{v}_1 + (\bar{v}_3 - \bar{v}_1)(3 - \sqrt{5})/2$ and evaluating $F_1 = F(\bar{v}_1)$, $F_2 = F(\bar{v}_2)$ and $F_3 = F(\bar{v}_3)$. In this way we obtain an initial triplet $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and the corresponding objective function values (F_1, F_2, F_3). Then we proceed with the standard golden search to iteratively obtain new triplets $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ until $\bar{v}_3 - \bar{v}_1 < \texttt{tol}$. Correspondingly, for the Newton iteration with an initial guess $\bar{v}_{\text{old}}$ we evaluate F at the three points $(\bar{v}_{\text{old}} - h, \bar{v}_{\text{old}}, \bar{v}_{\text{old}} +$

h) (where h is a small finite-difference deviation). Then we choose as the new point $\bar{v}_{\text{new}}$ the maximum point of the interpolating parabola. The golden section search is known to converge globally for unimodal functions with rate $(\sqrt{5}-1)/2$, whereas the Newton iteration converges quadratically close to a local maximum. In the presence of random disturbances in the evaluation of F we gradually increase the accuracy of the evaluation of F during the iteration by using the mean of y over a longer time interval after the transients have settled. Note that, when using act-and-wait and a discrete-time optimization algorithm, one has removed two of the two slower time scales present in [4]. They are replaced with the algorithmic iteration (the Newton iteration converges super-exponentially).

5. EXTREMUM SEEKING FOR THE CHEMOSTAT

Reference [31] has shown that there exists an output feedback that satisfies the assumptions A1, A2 and A4 for the model (1), with a simple saturated proportionate controller. Define first the saturation function as follows.

$$\text{sat}_{[D_{\min},D_{\max}]}(\xi) = \begin{vmatrix} D_{\max} & \text{if } \xi > D_{\max}, \\ \xi & \text{if } \xi \in [D_{\min},D_{\max}], \\ D_{\min} & \text{if } \xi < D_{\min}, \end{vmatrix} \quad (23)$$

For this saturation function [31] proved the following proposition.

Proposition 1. Assume that the reference values $(\bar{s},\bar{D})$ are in $[s_{\min},s_{\text{in}}] \times [D_{\min},D_{\max}]$, where the bounds satisfy

$$\begin{aligned} D_{\min} &< \mu(s) \quad \text{for all } s \in [s_{\min},s_{\text{in}}], \\ D_{\max} &> \mu(s) \quad \text{for all } s \in [0,s_{\text{in}}]. \end{aligned} \quad (24)$$

Then the output feedback

$$u(y,\bar{D},\bar{s}) = \text{sat}_{[D_{\min},D_{\max}]}\left(\bar{D} - G_1(y-\bar{s})\right) \quad (25)$$

with a gain G_1 satisfying

$$G_1 > \max\left(\max_{s \in [0,s_{\text{in}}]} -\mu'(s), \frac{\bar{D}-\mu(s_{\text{in}})}{s_{\text{in}}-\bar{s}}\right) \quad (26)$$

guarantees that the closed-loop dynamics (1), (25) has a stable equilibrium $(s_{\text{eq}},b_{\text{eq}}) \in [0,s_{\text{in}}) \times (0,\infty)$, which attracts all initial conditions $(s(0),b(0)) \in [0,s_{\text{in}}) \times (0,\infty)$.

See [31] for a proof. From system (1), one has at steady state

$$\mu(s_{\text{eq}}) = \bar{D} - G_1(s_{\text{eq}}-\bar{s}), \quad (27)$$

and consequently one has

$$s_{\text{eq}} = \bar{s} \Leftrightarrow \bar{D} = \mu(s_{\text{eq}}), \quad (28)$$

that is, Assumption A2 is fulfilled.

Note that in system (1) at steady state one has $b = s_{\text{in}} - s$. Thus, the objective function

$$\phi(s) = \mu(s)(s_{\text{in}}-s), \quad (29)$$

which is non-negative such that $\phi(0) = \phi(s_{in}) = 0$, is equivalent to the production rate $\mu(s)b$. For ϕ the assumptions A3 and A4 are fulfilled.

5.1 Implementation of continuous-time adaptation

For the continuous-time adaptation we can choose the same adaptive scheme for $\bar{u}$ for the step 1 described in Sec. 3.1 as was chosen in [31] for the identification of μ.

Proposition 2. For any $\bar{s} \in (0,s_{in})$ and numbers $D_{\min}$, $D_{\max}$ such that $0 < D_{\min} < \mu(\bar{s}) < D_{\max}$, the dynamical feedback law

$$\begin{aligned} u(y,\bar{D},\bar{s}) &= \text{sat}_{[D_{\min},D_{\max}]}\left(\bar{D} - G_1(y-\bar{s})\right) \\ \dot{\bar{D}} &= -G_2(y-\bar{s})(\bar{D}-D_{\min})(D_{\max}-\bar{D}) \end{aligned} \quad (30)$$

exponentially stabilizes the system (1) locally about $(\bar{s}, s_{in}-\bar{s})$, for any positive constants (G_1,G_2) such that $G_1 > -\mu'(\bar{s})$. Furthermore one has

$$\lim_{t \to +\infty} \bar{D}(t) = \mu(\bar{s})$$

See [31] for the proof.

Our continuous-time extremum seeking output feedback for the chemostat model (1) is thus given by the following equations

$$u(y,\bar{D},\bar{s}) = \text{sat}_{[D_{\min},D_{\max}]}\left(\bar{D} - G_1(y-\bar{s})\right) \quad (31)$$

$$\dot{\bar{D}} = -G_2(y-\bar{s})(\bar{D}-D_{\min})(D_{\max}-\bar{D}) \quad (32)$$

$$\frac{1}{\epsilon}\dot{\bar{s}} = \text{sgn}\big[(\bar{D}(s_{in}-\bar{s})- \quad (33)$$

$$\bar{D}(t-\tau)(s_{in}-\bar{s}(t-\tau)))(\bar{s}-\bar{s}(t-\tau))\big]$$

5.2 Implementation of discrete-time optimization

The discrete-time optimization uses only the saturated feedback, combining the parameters $\bar{D}$ and $\bar{S}$ into a single parameter $\bar{v} = \bar{D} + G_1\bar{s}$, such that

$$u(y,\bar{v}) = \text{sat}_{[D_{\min},D_{\max}]}(\bar{v} - G_1 y). \quad (34)$$

The initial bracketing interval $[\bar{v}_1,\bar{v}_2]$ for $\bar{v}$ is chosen such the line $\ell(\bar{v}) = \{(\bar{D},\bar{s}) : \bar{D} + G_1\bar{s} = \bar{v}\}$ intersects the domain of admissible reference values $[s_{\min},s_{\text{in}}] \times [D_{\min},D_{\max}]$ for all $\bar{v} \in [\bar{v}_1,\bar{v}_2]$. Thus, Proposition 1 ensures the existence of a unique equilibrium $s_{\text{eq}}(\bar{v})$ for all $\bar{v} \in [\bar{v}_1,\bar{v}_2]$. This implies that we can proceed with the golden section search for the objective functional

$$F(\bar{v}) = u(s_{\text{eq}}(\bar{v}),\bar{v})[s_{\text{in}} - s_{\text{eq}}(\bar{v})], \quad (35)$$

where u, as defined in (34), is the asymptotic value of the input u. Once, the golden section search has shrunk the bracketing interval to a certain size, we apply a single step of the Newton iteration for the objective functional F.

6. NUMERICAL SIMULATIONS

The output feedback law (31)–(33) has been simulated for a Haldane function

$$\mu(s) = \frac{\mu_{\max}s}{K + s + s^2/K_i} \quad (36)$$

and values of the parameters: $\mu_{\max} = 1$, $K = 1$, $K_i = 0.1$, $S_{in} = 1$, $D_{\min} = 0$ and $D_{\max} = 1$. Note however, that in our simulations we make no assumptions about the type of μ apart from those required in the assumptions of propositions 1 and 2 (leading to the validity of assumptions A1–A4). For the control laws in (31)–(33), we have chosen $G_1 = 1$, $G_2 = 1$ and $\epsilon = 1e^{-3}$. To simulate measurement noise, we have considered that the output is corrupted in the following way

$$y(t) = s(t)(1+b(t)) \quad (37)$$

where $b(\cdot)$ is a random square signal of frequency ω and amplitude a, whose values have been chosen with $\omega = 0.2$ and $a = 0.05$. The random function b mimics disturbances

of the measurements. The low frequency ω prevents the disturbances from averaging out too quickly (to get an averaging effect one has to wait for a time $t_{\text{wait}} \gg 1/\omega$).
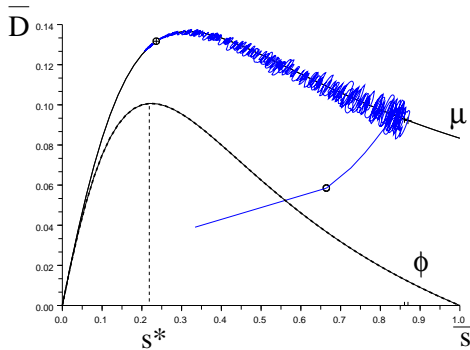
6.1 Continuous-time adaptation

Fig. 1. Trajectory of (1) with feedback laws (31)–(33) in the $(\bar{s}, \bar{D})$ plane.

Figure 1 shows how the trajectory of the variables $(\bar{s}, \bar{D})$ converges to the graph of the function $\mu(\cdot)$ i nthe $(\bar{s}, \bar{D})$-plane initially up to disturbances due to the measurement noise. This convergence toward $\mu(\cdot)$ is effected by the control laws (31) and (32) on the fast timescale. Once the feedback laws (31) and (32) have achieved quasi-equilibrium for fixed $\bar{s}$ the dynamics of (33) maximizes the functional ϕ on the slow timescale. Timescale separation $\epsilon = 10^{-3}$ is chosen for our simulations. In Figure 1 one can also see that the adaptation of $\bar{s}$ via (33) forces the state (s, b) to converge near the equilibrium $(s^\star, b^\star)$ (indicated as intersection between dotted line and graph $\mu(\cdot)$, where $s^\star$ is maximizing the function $\phi(\cdot)$. We note that the adaptation has converged already after $t = 3500$.

Remark If we choose the timescales not sufficiently different (say, $\epsilon = 10^{-2}$ in our example) we observe that the chemostat (1) with feedback laws (31)–(33) becomes dynamically unstable in in its fixed point $(s, b, \bar{s}, \bar{D}) = (s^\star, b^\star, s^\star, \mu(s^\star))$.

6.2 Discrete-time algorithms

For the discrete-time optimization algorithm we chose an initial bracketing interval $[\bar{v}_1, \bar{v}_2] = [0.04, 1]$ (gain $G_1 = 1$ in feedback law (34) identical to the continuous-time adaptation). During the golden section search (until $t \approx 700$ in Figure 2(a)) we checked every $t_{\text{inc}} = 25$ time units if the transients have settled (criterion is a decrease of the standard deviation compared to the previous time interval). Once the output is accepted as stationary we used the mean of output and input over the last time interval to calculate the objective F. A larger t_{inc} inceases the accuracy of the evaluation of F in the presence of disturbances due to averaging. We observe that after 7 evaluations of F the golden section search reduces the bracketing interval to length 0.2 and the estimated optimal value of s is already close to $s^\star$ ($t \approx 700$ in Figure 2(a)). The Newton step then only confirms the optimality of the final estimate of $s^\star$ (up to disturbance level). For the Newton iteration we chose t_{inc} larger ($t_{\text{inc}} = 100$) to increase the accuracy of F evaluations and use the points
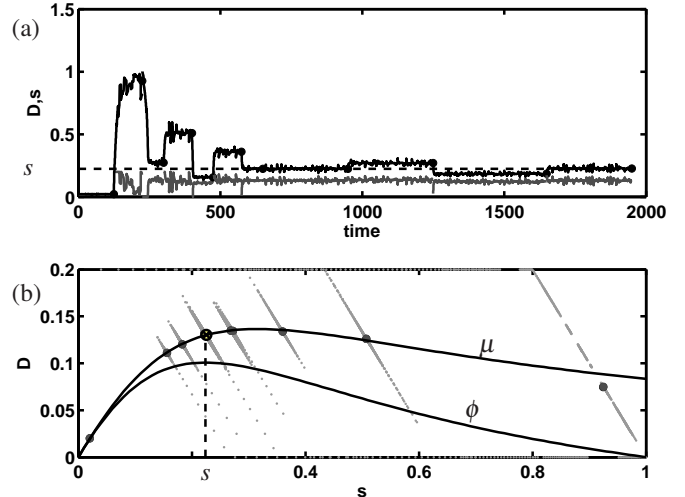

Fig. 2. Time profiles (a) and input-output plane (b) for time-discrete optimization. The grey time profile in (a) is $u(t) = D(t)$ and the black profile is $s(t)$. Full circles in (a) and (b) indicate inputs and outputs where the objective F was evaluated.

$(\bar{v}-h, \bar{v}, \bar{v}+h)$ with $h = 0.05$ to evaluate the approximating parabola for F in $\bar{v}$. Figure 2(b) shows the trace (grey dots) of outputs s and inputs u (equaling dilution rate D) in the (s, D)-plane. The gaps in the trace are due to finite sampling time. The full circles show the points where the transients were accepted as settled.

The speed and accuracy of the discrete-time optimization is limited by the choice of t_{inc}, which in turn has to be determined sufficiently large to enable an averaging effect for the disturbances.

7. CONCLUSION

We have proposed two extremum-seeking schemes: one approach uses dynamical output feedback that is based on a continuation method. Its closed-loop dynamics possesses two time scales: the timescale of the original system with stabilizing feedback and a slow timescale for the adaptation scheme. As an alternative we test classical optimization algorithms, applied to the steady-state outputs of the feedback-stabilized system, which have a continuous timescale (which is exponentially convergent) and a discrete timescale (which is exponentially or super-exponentially convergent). This is in contrast to extremum seeking techniques that use dither signals, which need three timescales for nonlinear systems (the time scale of the stabilized system is the fastest).

REFERENCES

[1] E. L. Allgower and K. Georg *Introduction to Numerical Continuation Methods*, Society for Industrial and Applied Mathematics, 2003.

[2] R. Antonelli, J. Harmand and J. P. Steyer and A. Astolfi. *Set point regulation of an anaerobic digestion process with bounded output feedback*, IEEE transactions on Control Systems Technology, vol. 11, pp. 495–504, 2003.

[3] K. Ariyur and M. Kristic. *Real-Time Optimization by Extremum-Seeking Control.* Wiley, 2003.

[4] K. Ariyur and M. Krstic, *Slope seeking: A generalization of extremum seeking.* International Journal of Adaptive Control and Signal Processing, vol. 18, pp. 1-22, 2004.

[5] K. Astrom and B. Wittenmark. *Adaptive Control,* Prentice-Hall, 2nd Edition, 1994.

[6] E. Busvelle and J.-P. Gauthier, *On determining unknown functions in differential systems, with an application to biological reactors* ESAIM Control, Optimisation and Calculs of Variations, Vol. 9, pp. 509–522, 2003.

[7] G. Bastin and D. Dochain, *On-Line Estimation and Adaptive Control of Bioreactors,* Elsevier, Amsterdam, 1990.

[8] G. Bastin and J.F. Van Impe, *Nonlinear and adaptive control in biotechnology : A tutorial,* European Journal of Control, Vol. 1(1), pp. 37–53, 1995.

[9] G. D'ans, D. Gottlieb and P. Kokotovic. *Optimal control of bacterial growth.* Automatica, Vol. 8(6), pp. 729–736, 1972.

[10] D. Dochain and G. Bastin. *Adaptive identification and control algorithms for non linear bacterial growth systems.* Automatica, vol 20 (5), pp. 621–634, 1984.

[11] D. Dochain and G. Bastin. *On-line estimation of microbial growth rates.* Automatica, Vol. 22 (6), pp. 705–711, 1986.

[12] D. Dochain, M. Perrier and M. Guay. *Extremum Seeking Control and its Application to Process and Reaction Systems: A Survey.* Mathematics and Computers in Simulation, Vol. 16 (6), pp. 535–553, 2010.

[13] G. François, B. Srinivasan and D. Bonvin. *Comparison of Six Implicit Real-Time Optimization Schemes.* Journal Europen des Systèmes Automatisés, Vol. 46 (2-3) pp. 291–305, 2012.

[14] J.P. Gauthier, H. Hammouri and S. Othman, *A Simple Observer for Nonlinear Systems. Applications to Bioreactors.* IEEE Trans. Autom. Control, Vol. 37(6) , pp. 875–880, 1992.

[15] M. Guay, D. Dochain and M. Perrier. *Adaptive extremum seeking control of stirred tank bioreactors.* Automatica, 40 (5), pp. 881–888, 2004.

[16] A. Holmberg and J. Ranta, *Procedures for parameter and state estimation of microbial growth process models.* Automatica, Vol. 18, pp. 181–193, 1982.

[17] T. Insperger. *Act-and-wait concept for continuous-time control systems with feedback delay.* IEEE Transactions on Control Systems Technology, 14 (5), pp. 974–977, 2006.

[18] F. Jadot, G. Bastin and F. Viel, *Robust global stabilization of stirred tank reactors with saturated output-feedback,* Eur. J. Control, Vol5, pp. 361–371, 1999.

[19] M. Krstic, *Performance improvement and limitations in extremum seeking control.* Systems & Control Letters, vol. 39, pp. 313-326, 2000.

[20] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design.* Wiley, 1995

[21] M. Krstic and H. H. Wang, *Stability of extremum seeking feedback for general dynamic systems.* Automatica, vol. 36, pp. 595-601, 2000.

[22] R. O'Malley, *Singular perturbation methods for ordinary differential equations,* Springer-Verlag, New York, 1991.

[23] M. Mansour and J. Ellis. *Comparison of methods for estimating real process derivatives in on-line optimization.* Applied Mathematical Modelling, Vol. 27 (4), pp. 275–291, 2003.

[24] N. Marcos, M. Guay, T. Zhang and D. Dochain. *Adaptive extremum-seeking control of a continuous stirred tank bioreactor with Haldane's kinetics.* Journal of Process Control, Vol. 14 (3), pp. 317–328, 2004.

[25] N. Marcos, M. Guay and D. Dochain. *Output feedback adaptive extremum seeking control of a continuous stirred tank bioreactor with Monod's kinetics.* Journal of Process Control, Vol. 14 (7), pp. 807–818, 2004.

[26] M. Perrier, S. Feyo de Azevedo, E. Ferreira and D. Dochain. *Tuning of observer-based estimators: theory and application to the on-line estimation of kinetic parameters* Control Engineering Practice, vol. 8 (4), pp. 377–388, 2000.

[27] A. Rapaport and D. Dochain. *Interval observers for biochemical processes with uncertain kinetics.* Mathematical Biosciences, 193 (2), pp. 235–253, 2004.

[28] A. Rapaport and J. Harmand. *Robust regulation of a class of partially observed nonlinear continuous bioreactors,* J. Process Control, vol. 12, pp. 291–302, 2002.

[29] B. Satishkumar and M. Chidambaram. *Control of unstable bioractor using fuzzy-tuned PI controller,* Bioproc. Eng. Vol. 20, pp. 127–132, 1999.

[30] A. Schaum, J. Alvarez and T. Lopez-Arenas. *Saturated PI control of continuous bioreactors with Haldane kinetics* Chem. Eng. Science, Vol. 68, pp. 520–529, 2012.

[31] J. Sieber, A. Rapaport, S. Rodrigues and M. Desroches. *A method for the reconstruction of unknown non-monotonic growth function in the chemostat.* Bioprocess and Biosystems Engineering, 2013 (to appear).

[32] I.Y. Smets, J.E. Claes, E.J. November, G. Bastin and J.F. Van Impe, *Optimal adaptive control of (bio)chemical reactors: past, present and future.* Journal of Process Control, Vol. 14(7), pp. 795–805, 2004.

[33] Y. Tan, D. Nesic and I.M.Y. Mareels. *On the dither choice in extremum seeking control.* Automatica, Vol. 44(5), pp. 1446–1450, 2008.

[34] Y. Tan, D. Nesic, I.M.Y. Mareels and A. Astolfi, *Global extremum seeking in the presence of local extrema.* Automatica, Vol. 45(1), pp. 245-251, 2009.

[35] K. Versyck, J. Claes and J. Van Impe. *Optimal experimental design for practical identification of unstructured growth models.* Mathematics and computers in simulation, Vol. 46 (5-6), pp. 621–629, 1998.

[36] H.-H. Wang, M. Krstic, and G. Bastin, *Optimizing bioreactors by extremum seeking,* International Journal of Adaptive Control and Signal Processing, vol. 13, pp. 651-669, 1999.

[37] T. Zhang, M. Guay and D. Dochain. *Adaptive extremum seeking control of continuous stirred tank bioreactors.* AIChE J., Vol. 49(1), pp. 113–123, 2003.