Finite-time Stabilization Using Implicit Lyapunov Function Technique ⋆

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Abstract: The Implicit Lyapunov Function (ILF) method for finite-time stability analysis is introduced. The control algorithm for finite-time stabilization of a chain of integrators is developed. The scheme of control parameters selection is presented by a Linear Matrix Inequality (LMI). The robustness of the finite-time control algorithm with respect to system uncertainties and disturbances is studied. The new high order sliding mode (HOSM) control is derived as a particular case of the developed finite-time control algorithm. The settling time estimate is obtained using ILF method. The algorithm of practical implementation of the ILF control scheme is discussed. The theoretical results are supported by numerical simulations.

1. INTRODUCTION

Frequently, the control practice needs a control and observation algorithms, which provide terminations of all transition processes in a finite time. Such problem statements usually appear in robotic systems, aerospace applications, underwater/surface vehicles control systems, etc. Finite-time control problems are subjects of intensive researches in the last years; e.g., see Haimo [1986], Bhat and Bernstein [2000], Orlov [2009]. Concerning the observation problems, a finite-time convergence of observed states to the real ones is always preferable, see Engel and Kreisselmeier [2002], Moulay and Perruquetti [2006], Perruquetti et al. [2008], Menard et al. [2010], Bejarano and Fridman [2010], Shen et al. [2011]. Sliding mode algorithms also ensure a finite-time convergence to the sliding surface, see Levant [2005a], Orlov [2005], Utkin et al. [2009], Polyakov and Poznyak [2009]; typically, the associated controllers have mechanical and electromechanical applications Bartolini et al. [2003], Ferrara and Giacomini [1998], Chernousko et al. [2008].

The finite-time control laws for a chain of integrators are developed in many papers (see, for example, Hong [2002], Levant [2005a], Bernuau et al. [2012]). Unfortunately, they are not supported by constructive algorithms of parameters tuning. The corresponding finite-time stability theorems usually guarantee only existence of appropriate control parameters.

The present paper elaborates the Implicit Lyapunov Function (ILF) method for the *finite-time stability* analysis. This method was initially used by Korobov [1979] for control synthesis problems and was called controllability function method. In order to be more precise we follow more actual terminology of Adamy and Flemming [2004]. The ILF method uses Lyapunov function defined in the implicit form by some algebraic equations. Stability analysis in this case does not require solving of this equation, since the Implicit function theorem (see, for example, Courant and John [2000]) helps to check all stability conditions analyzing the algebraic equation directly.

This paper addresses the problem of a finite-time stabilization of the chain of integrators. The developed finite-time ILF method is used in order to design a simple constructive control algorithm together with implicit Lyapunov function of closed-loop system. Finite-time stability conditions are formulated in the form of Linear Matrix Inequalities (LMI). This fact essentially simplifies the process of tuning of the control parameters. The ILF method allows us also to analyze robustness of the closed-loop system and to design *a high order sliding mode control algorithm* which rejects the bounded matched exogenous disturbances. The practical implementation of the ILF control algorithm requires development of special computational procedures. One of them is discussed in this paper.

1.1 Notations

Through the paper the following notations will be used:

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- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$, where $\mathbb{R}$ is the set of real number;
- for a differential equation numbered as (.), the time derivative of a function V along the solution of (.) is denoted by $\frac{dV}{dt}\big|_{(.)}$;
- $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^n$, i.e. $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$ for $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$;
- $\mathrm{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with the elements λ_i on the main diagonal;
- a continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is monotone increasing and $\sigma(0) = 0$;
- for a symmetric matrix $P = P^T$ the minimal and maximal eigenvalues are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively;
- if $P \in \mathbb{R}^{n\times n}$ then the inequality $P > 0$ ($P \geq 0$, $P < 0$, $P \leq 0$) means that P is symmetric and positive definite (positive semidefinite, negative definite, negative semidefinite).

2. PROBLEM STATEMENT

Consider a single input control system of the form

$$\dot{x} = Ax + bu + d(t, x), \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input,

$$A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$$

and the function $d(t, x) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ describes the system uncertainties and disturbances.

The system (1) in the disturbance free case describes a chain of integrators that is a basic model for demonstration of control algorithms. Typically, a well-developed control design scheme for a chain of integrators can be easily extended to the class of multi-input multi-output linear plants.

Consideration of the control systems like (1) is additionally motivated by many mechanical and electromechanical applications, see, for example, Chernousko et al. [2008], Utkin et al. [2009].

The main aim of this paper is to propose a *constructive control* algorithm for finite-time stabilization of the system (1) and to study the robustness of this algorithm with respect to uncertainties and disturbances.

3. FINITE-TIME STABILITY AND IMPLICIT LYAPUNOV FUNCTION

Consider the system of the form

$$\dot{x} = f(t, x), \quad x(0) = x_0, \qquad (2)$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field. The case of discontinuous right-hand side of the system (2) is not excluded. In this case the solutions $x(t, x_0)$ of the system (2) are understood in the sense of Filippov [1988]. Namely, an absolutely continuous function $x(t, x_0)$ defined of the interval or a segment I is called the solution of the Cauchy problem associated to (2) if $x(0, x_0) = x_0$ and for almost all $t > 0$ it satisfies the following differential inclusion

$$\dot{x} \in K[f](t, x) := co \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} f(t, B(x, \varepsilon)\backslash N) \qquad (3)$$

where $co(M)$ defines the convex closure of the set M, $B(x, \varepsilon)$ is the ball with the center at $x \in \mathbb{R}^n$ and the radius ε, the equality $\mu(N) = 0$ means that the measure of the set $N \in \mathbb{R}^n$ is zero.

Assume that the origin is an equilibrium point of the system (2), i.e. $0 \in K[f](t, 0)$.

Definition 1. (Roxin [1966], Bhat and Bernstein [2000]). The origin of the system (2) is said to be ***finite-time stable*** if for a set $\mathcal{V} \subset \mathbb{R}^n$:

(1) **Finite-time attractivity**: there exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_+$, such that for all $x_0 \in \mathcal{V} \setminus \{0\}$, $x(t, x_0)$ is defined on $[0, T(x_0))$ and $\lim_{t\to T(x_0)} x(t, x_0) = 0$.
(2) **Lyapunov stability**: there exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$, $\|x(t, x_0)\| \leq \delta(\|x_0\|)$.

The function $T(\cdot)$ from Definition 1 is called the *settling-time function* of the system (2).

If in Definition 1 the set $\mathcal{V}$ coincides with $\mathbb{R}^n$ then the origin of the system (2) is said to be *globally finite time stable.*

Notice that:

- Finite-time stability is obtained through an "infinite eigenvalue assignation" for the system at the origin.
- There exists a function called the *settling time* that performs the time for a solution to reach the equilibrium. The function depends on the initial condition of a solution.
- The right hand side of the ordinary differential equation *cannot be locally Lipschitz at the origin.*

The next theorem presents the ILF method (Korobov [1979], Adamy and Flemming [2004]) for finite-time stability analysis.

Theorem 2. If there exists a continuous function

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$
$$(V, x) \mapsto Q(V, x)$$

such that

C1) $Q(V, x)$ is continuously differentiable for $\forall x \in \mathbb{R}^n\backslash\{0\}$ and $\forall V \in \mathbb{R}_+$;

C2) for any $x \in \mathbb{R}^n\backslash\{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+$:

$$Q(V^-, x) < 0 < Q(V^+, x);$$

C3) $\lim_{\substack{x\to 0\\(V,x)\in\Omega}} V = 0^+, \quad \lim_{\substack{V\to 0^+\\(V,x)\in\Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\|\to\infty\\(V,x)\in\Omega}} V = +\infty,$

where $\Omega = \left\{(V, x) \in \mathbb{R}^{n+1} : Q(V, x) = 0\right\}$;

C4) the inequality $-\infty < \frac{\partial Q(V,x)}{\partial V} < 0$ holds for $\forall V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n\backslash\{0\}$;

C5) for $\forall t \in \mathbb{R}_+$ and $\forall(V, x) \in \Omega$;

$$\sup_{\forall y\in K[f](t,x)} \frac{\partial Q(V, x)}{\partial x} y \leq cV^{1-\mu} \frac{\partial Q(V, x)}{\partial V}$$

where $c > 0$ and $0 < \mu \leq 1$ are some constants, then the origin of system (2) is globally finite time stable with the following settling time estimate

$$T(x_0) \leq \frac{V_0^\mu}{c\mu},$$

where $V_0 \in \mathbb{R}_+ : Q(V_0, x_0) = 0$.

Proofs of all propositions are skipped.

4. CONTROL DESIGN

Introduce the function

$$Q(V, x) := x^T D_\mu(V^{-1}) P D_\mu(V^{-1}) x - 1, \qquad (4)$$

where $D_\mu(\lambda)$ is the dilation matrix of the form

$$D_\mu(\lambda) = \text{diag}\{\lambda^{1+(n-i)\mu}\}_{i=1}^n, \quad 0 < \mu \leq 1$$

and $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, i.e. $P = P^T > 0$. Similar equation for Implicit Lyapunov Function was considered in Korobov [1979], Adamy and Flemming [2004]. For $\mu = 0$ the equation (4) admits the explicit solution $V = \sqrt{x^T P x}$.

Denote $H_\mu := \text{diag}\{-1 - (n-i)\mu\}_{i=1}^n$.

Theorem 3. (**Disturbance-free case**). If $d(t, x) \equiv 0$ and the system of matrix inequalities:

$$\begin{cases} AX + XA^T + by + y^T b^T + \alpha X \leq 0, \\ -\gamma X \leq XH_\mu + H_\mu X < 0, \quad X > 0, \end{cases} \qquad (5)$$

is feasible for some $\alpha, \gamma \in \mathbb{R}_+$, $\mu \in (0, 1]$ and $X \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^{1\times n}$ then the control of the form

$$u(V, x) = V^{1-\mu} k D_\mu(V^{-1}) x, \qquad (6)$$

where $k := yX^{-1}$,

$$V \in \mathbb{R}_+ \; : \; Q(V, x) = 0$$

and $Q(V, x)$ is defined by (4) with $P := X^{-1}$, stabilizes the system (1) in a finite time and the settling time function is bounded as follows

$$T(x_0) \leq \frac{\gamma V_0^\mu}{\alpha\mu}, \qquad (7)$$

where $V_0 \in \mathbb{R}_+ : Q(V_0, x_0) = 0$.

Let us make some remarks about presented control scheme:

- A practical implementation of the control (6) requires solving of the equation $Q(V, x) = 0$ in order to obtain $V(x)$. In some cases (for example, $n = 2, \mu = 1$), the function $V(x)$ can be found analytically. However, the obtained analytical representation will be very cumbersome. If a digital control device has enough memory then the function V can also be calculated off-line on some grid, approximated (for example, by local splines) and stored in the controller for on-line usage. In other cases the equation can be numerically solved on-line using rather simple procedures (see, the next section for details).
- The advantage of the control design scheme presented in Theorem 3 is related to simplicity of the control parameters tuning. It is based on LMI technique, which is very well developed for linear systems. For any fixed $\alpha, \mu, \gamma \in \mathbb{R}_+$ the system of matrix inequalities (5) becomes a feasible LMI at least for sufficiently small $\mu \in (0, 1]$ and sufficiently large $\gamma \in \mathbb{R}_+$. Indeed, controllability of the pair $\{A, b\}$ implies feasibility of the first LMI from the system (5) (see, for example, Boyd et al. [1994]). On the other hand, the following representation $H_\mu X + XH_\mu = -2X - \mu(FX + XF)$, where $F = \text{diag}\{n-1, n-2, ..., 1, 0\}$, holds, so for $0 < \mu < \frac{2\lambda\min(X)}{|\lambda_{\min}(FX+XF)|}$ we always have $H_\mu X + XH_\mu < 0$.
- Parameters γ and α introduced in (5) are tuning parameters for the upperbound of the settling time of the closed-loop system (see, the estimate (7)).
- For $\mu = 1$ the control (6) is globally bounded, since $x^T D_\mu(V^{-1}) P D_\mu(V^{-1}) x = 1 \Rightarrow \|D_\mu(V^{-1})x\|_2^2 \leq \frac{1}{\lambda_{\min}(P)}$ and we have
$$|u(x)| \leq \|k\| \cdot \|D_\mu(V^{-1})x\| \leq \frac{\|k\|}{\sqrt{\lambda_{\min}(P)}}.$$
Moreover, in this case the condition $|u(x)| \leq u_0$, where $u_0 \in \mathbb{R}_+$, can be rewritten into LMI form:
$$\begin{pmatrix} X & y^T \\ y & u_0^2 \end{pmatrix} \geq 0. \qquad (8)$$
Indeed, since
$$\frac{1}{u_0^2} u^2(x) = \frac{1}{u_0^2} x^T D_\mu(V^{-1}) k^T k D_\mu(V^{-1}) x \leq 1 =$$
$$x^T D_\mu(V^{-1}) P D_\mu(V^{-1}) x,$$
then the matrix inequality $\frac{1}{u_0^2} k^T k \leq P$ implies $|u(x)| \leq u_0$ and using Schur complement in the new variables $y = kP^{-1}$ and $X = P^{-1}$ we obtain (8).
- For $\mu \in (0, 1)$ the control of the form (6) is *continuous* (but not locally Lipschitz at the origin) function of the state x. If $\mu = 1$ then the control function $u(x)$ is *continuous outside the origin* and globally *bounded* for all $x \in \mathbb{R}^n$. If $\mu \to 0$ then the control becomes a linear feedback $u(x) = kx$.

Theorem 4. (**Disturbed case**). If

1) the system of matrix inequalities

$$\begin{cases} AX + XA^T + by + y^T b^T + \alpha X + \beta I_n \leq 0, \\ -\gamma X \leq XH_\mu + H_\mu X < 0, \quad X > 0, \end{cases} \qquad (9)$$

is feasible for some $\mu \in (0, 1]$, $\alpha, \beta, \gamma \in \mathbb{R}_+ : \alpha > \beta$, $X \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^{1\times n}$;

2) the control $u(V, x)$ has the form (6) with $P := X^{-1}$ and $k = yX^{-1}$;

3) the disturbance function $d(t, x)$ satisfy

$$d^T(t, x) D_\mu^2(V^{-1}) d(t, x) \leq \beta^2 V^{-2\mu} \qquad (10)$$

where $V \in \mathbb{R}_+ \; : \; Q(V, x) = 0$.

Then the closed-loop system is globally finite-time stable and the settling time function estimate has the form

$$T(x_0) \leq \frac{\gamma V_0^\mu}{\mu(\alpha - \beta)},$$

where $V_0 \in \mathbb{R}_+ : Q(V_0, x_0) = 0$.

The system of matrix inequalities (9) is also feasible for any $\alpha, \beta \in \mathbb{R}_+$, at least for small $\mu \in (0, 1]$ and large $\gamma \in \mathbb{R}_+$. If β tends to zero, the inequality (10) gives $d(t, x) \equiv 0$, Theorem 4 becomes the same as Theorem 3.

Remark 5. The implicit restriction (10) for the system disturbances becomes an explicit form when $\mu = 1$ and the matching condition holds: $d_i(t,x) = 0$, $i = 1, ..., n-1$. Fulfilling the matching condition is rather a natural assumption for the system (1), which describes a chain of integrators. In this case the condition (10) becomes

$$|d_n(t,x)| \leq \beta.$$

So, the finite-time control (6) designed for $\mu = 1$ and $\alpha > \beta$ rejects the bounded matched system disturbances showing the typical property of **the high order sliding mode control (HOSM)**. The HOSM version of the control (6) has the unique discontinuity point $x = 0$ similarly to quasi-continuous HOSM control, see Levant [2005b].

Some obvious remarks can be done for local modifications of the presented theorem. If the restriction (10) holds only for $0 < V \leq V_{\max} : Q(V,x) = 0$ then Theorem 4 provides only local finite time stability with the attraction domain $\Omega := \{x \in \mathbb{R}^n : x^T D_\mu(V_{\max}^{-1}) P D_\mu(V_{\max}^{-1}) x \leq 1\}$. Otherwise, if (10) holds only for $V \geq V_{\min} : Q(V,x) = 0$ then the set $\{x \in \mathbb{R}^n : x^T D_\mu(V_{\min}^{-1}) P D_\mu(V_{\min}^{-1}) x \leq 1\}$ is a finite-time stable invariant set of the system (1), (6).

5. PRACTICAL IMPLEMENTATION OF CONTROL

5.1 Some supporting facts

In Adamy and Flemming [2004], an implicit control scheme similar to (6) is called a soft variable structure control. Indeed, for each given V the closed-loop control system has a linear form. If V is some switching parameter we have linear switching (variable structure) system. We will follow this interpretation in discussion of practical implementation of the ILF controls.

Corollary 6. Let the control $u(V,x)$ be defined according to Theorem 3 and $d(t,x) \equiv 0$ then the control $u_0(x) = u(V_0, x)$ is the linear stabilizing feedback control for the system (1) for any given $V_0 \in \mathbb{R}_+$.

The next corollary will help us to analyze the discrete-time version of the developed control schemes.

Corollary 7. Let $\{t_i\}_{i=0}^{\infty}$ be an arbitrary strictly increasing sequence of time instants, $0 = t_0 < t_1 < t_2 < ....$ Let the function $u(V,x)$ is defined according to Theorem 3 and $d(t,x) \equiv 0$ then the origin of the system (1) with the switching control

$$u(x) := u(V_i, x) \quad \text{for} \quad t \in [t_i, t_{i+1}), \tag{11}$$

where $V_i > 0 : Q(V_i, x(t_i)) = 0$, is asymptotically stable.

The corollary shows that the sampled-time realization of the developed "implicit" control scheme keeps the stability property of the closed-loop system (1) independently on the sampling period. Between two switching instants the system is *linear*, so it can be studied using the standard schemes for discretization of linear systems (see, for example, Dorf and Bishop [2008]).

5.2 On Digital Implementation

The considerations presented below give just general remarks on a possible implementation of the developed control scheme. A detailed study of sampled-time and discrete-time versions of the presented control algorithms goes beyond the scope of this paper providing the subject for a future research.

To realize the control algorithm (6) in practice we need to know $V(x)$. In some cases the function $V(x)$ can be calculated analytically. For example, if $n = 2$ and $\mu = 1$ the equation $Q(V,x) = 0$ can be represented as

$$V^4 - p_{22}x_2^2V^2 - 2p_{12}x_1x_2V - p_{11}x_1^2 = 0, \; P := \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \tag{12}$$

where P satisfies the LMI system (5). The equation (12) is the 4th order polynomial and using the Ferrari formulas it can be found an analytical representation of $V(x_1, x_2)$, but this representation will be very cumbersome.

The proposed control scheme can be implemented in digital control devices that allow us to calculate the value of V at some point x by means of resolving the equation $Q(V,x) = 0$ numerically and on-line. Fortunately, for practical reasons rather simple numerical procedures can be utilized in order to find the solution of the equation $Q(V,x) = 0$ for any given $x \in \mathbb{R}^n \backslash \{0\}$.

Since $Q(V,x)$ satisfy the properties *C1*)-*C4*) of Theorem 2, then for each fixed $\overline{x} \in \mathbb{R}^n \backslash \{0\}$ the scalar function $\overline{Q}(V) := Q(V, \overline{x})$ is monotone decreasing and has the unique zero on the interval $(0, +\infty)$. So, in practice to resolve the scalar equation $\overline{Q}(V) = 0$ we may use, for example, the bisection method.

Taking into account the results presented in Corollaries 6 and 7 the following scheme of digital implementation can be presented.

Let the control $u(V,x)$ of the form (6) be realized in a digital device and the parameter V may change its value at some time instants: $t_0 = 0, t_i > 0, i = 1, 2, ...$ Denote $V_i := V(t_i)$ and $x_i := x(t_i)$. On the time interval $[t_i, t_{i+1})$ the control has the form $u(V_i, x)$, which is a linear stabilizing feedback for any $V_i \in \mathbb{R}_+$ (see, Corollary 6).

The simplest scheme for selection of the switching control parameter V_i is described by the following algorithm. If

```
INITIALIZATION:
                 V_0 = 1;  a = V_min;  b = 1;
STEP :
  If x_i^T D_μ(b^{-1}) P D_μ(b^{-1}) x_i > 1 then
          a = b;  b = 2b;
  elseif x_i^T D_μ(a^{-1}) P D_μ(a^{-1}) x_i < 1 then
          b = a;  a = max{a/2, V_min};
  else
      c = (a+b)/2;
      If x_i^T D_μ(c^{-1}) P D_μ(c^{-1}) x_i < 1 then b = c;
      else a = max{V_min, c};
      endif;
  endif;
  V_i = b;
```

$x_i \in \mathbb{R}^n$ is some given vector and `STEP` of this algorithm is applied recurrently many times to the same x_i then Algorithm 5.2 realizes:

1) a localization of the unique positive root of the equation $Q(V, x_i) = 0$;

2) improvement of the obtained localization by means of the bisection method.

Such an application of Algorithm 5.2 allows us to calculate $V(x_i)$ with high precision but it requests a high computational capability of a control device. So, it is more reasonable to realize STEP of Algorithm 5.2 *just once in each sampled time instant*. In this case $x_i = x(t_i) \neq x(t_{i+1}) = x_{i+1}$ and some additional justification of functionality of this algorithm has to be presented. Denote

$$\Pi(V_i) := \{x \in \mathbb{R}^n : x^T D_\mu(V_i^{-1}) P D_\mu(V_i^{-1}) x \leq 1\}. \quad (13)$$

Considerations presented in the proofs of Corollaries 6 and 7 imply that for any $V_i \in \mathbb{R}_+$ if $x(t_i) \in \Pi(V_i)$ then $x(t) \in \Pi(V_i)$ for $t \in [t_i, t_{i+1}]$, where $x(t)$ is the solution of the system (1) with the control $u(V_i, x)$ of the form (6). Monotonicity condition $\frac{\partial Q(V,x)}{\partial V} < 0$ implies that $\Pi(V') \subset \Pi(V'')$ for $V' < V''$. So, in order to guarantee stability of the sampled time realization of the developed control, on each step of Algorithm 5.2 the *upper estimate* of the root is selected for generation of the linear feedback for the next sampled interval. For *disturbance free case* such selection ensures that at the next sampling time instant we will have $V(x_{i+1}) \in (0, V_i)$, i.e. Algorithm 5.2 does not lose the localization of the root of the equation $Q(V, x_{i+1}) = 0$ between two sampled time instants. This means that if the root is localized on some step $i^* \in \{0, 1, 2, ...\}$ then the sequence $\{V_i\}_{i=i^*}^{\infty}$ generated by Algorithm 5.2 for later time instants is non-increasing, i.e. $V_{i+1} \leq V_i$ for all $i \geq i^*$. Moreover, the procedure of the improvement of the root localization (i.e. bisection method) will operate until $x(t_i) \notin \Pi(V_{\min})$. This guarantees convergence of of all trajectories of the system (1) with the presented switching version of the control (6) to the ellipsoid $\Pi(V_{\min})$.

The parameter $V_{\min}$ defines lower possible value of V and the corresponding "minimal" attractive and invariant set $\Pi(V_{min})$ for the closed-loop system. This parameter cannot be selected arbitrary small due to finite numerical precision of digital devices.

Remark 8. An additional advantage of the developed control scheme is related to possible reduction of the chattering effect for the high order sliding mode control case ($\mu = 1$) of the developed algorithm. The HOSM control of the form (6) has the unique discontinuity point $x = 0$. If we modify the finite time control (6) redefining $u(x) := kD_\mu(V_{\min}^{-1})x$, $\forall x \in \Pi(V_{\min})$ for some $V_{\min} \in \mathbb{R}_+$, then we will have the linear control law inside the invariant finite time attractive ellipsoid $\Pi(V_{\min})$. Such a modification of control obviously follows the classical ideas of the chattering reduction technique based on smoothing of the discontinuous sliding mode control around the switching manifold, see Utkin et al. [2009]. As we can see required smoothing is properly realized by Algorithm 5.2.

The numerical simulations presented below have been done using Algorithm 5.2, which demonstrate the effectiveness of the presented scheme of the ILF control implementation even for the disturbed case.

6. NUMERICAL EXAMPLES

6.1 Finite time stabilization

Consider the system (1) for $n = 3$ in the disturbance free case ($d(t, x) \equiv 0$).

Define the finite time control u in the form (6) with the parameter $\mu = 1/2$, where the matrix $P \in \mathbb{R}^{3\times3}, P > 0$ and the vector $k \in \mathbb{R}^{1\times3}$ are obtained from the LMI (5) with parameters $\alpha = 1$, $\gamma = 4.5$:

$$P = \begin{pmatrix} 8.5388 & 9.5299 & 4.0408 \\ 9.5299 & 16.3809 & 6.9417 \\ 4.0408 & 6.9417 & 5.7131 \end{pmatrix},$$

$$k = (-1.4059 \quad -3.0735 \quad -2.0150).$$

The numerical solving of ODE for the closed-loop system have been done by the Euler method with the fixed step size $h = 0.01$ and the discrete time application of the finite time ILF control is realized by the scheme presented in Algorithm 5.2 for $V_{\min} = 0.01$. The simulation results are shown on Fig. 1.

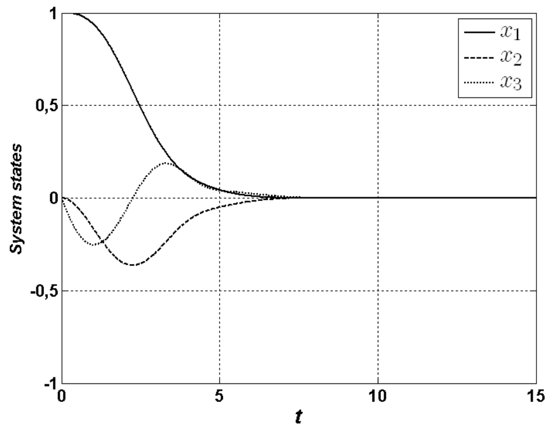

Fig. 1. The simulation results for the finite-time ILF control.

6.2 HOSM ILF control

Consider the system (1) for $n = 3$ in the case of matched disturbances, i.e. $d_1(t, x) = 0$, $d_2(t, x) = 0$ and $|d_3(t, x)| \leq \beta = 0.3$. For simulations we take $d_3(t, x) = 0.3 \sin(t)$ and $x(0) = (1, 0, 0)^T$.

The numerical solving of ODE for the closed-loop system has been done by the Euler method with a fixed step size $h \in \mathbb{R}_+$. In order to show the effectiveness of the developed control scheme with respect to the chattering reduction, we select the quite large (at least for sliding mode control application) step size $h = 0.1$.

The Fig. 2 shows the simulation results for the closed-loop system with the HOSM ILF control (6) ($\mu = 1$) that is restricted by $|u(x)| \leq 1$ (see, (8)) and applied by the scheme presented in Algorithm 5.2 for $V_{\min} = 0.1$.

The parameters of the control (6) were selected solving the LMI system (5),(8) for $\mu = 1$ and $\alpha = 0.6$, $\beta = 0.3$, $\gamma = 7.5$, $u_0 = 1$:

$$P = \begin{pmatrix} 0.2060 & 0.6062 & 0.6203 \\ 0.6062 & 2.1613 & 2.2972 \\ 0.6203 & 2.2972 & 3.2538 \end{pmatrix},$$

$$k = (-0.2946 \quad -1.2280 \quad -1.7648).$$

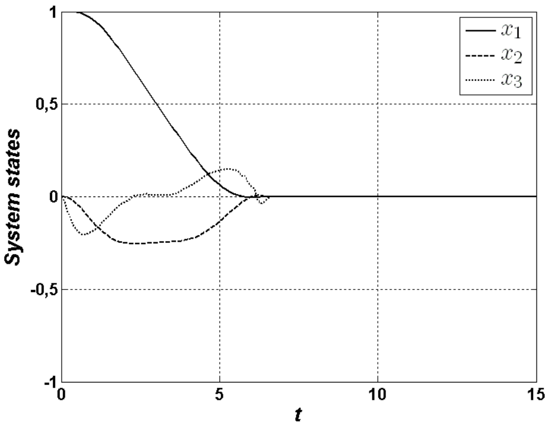

Fig. 2. The simulation results for the HOSM ILF control.

7. CONCLUSIONS

In the paper a new finite-time control algorithm is presented. It has the following **advantages**:

- The control algorithm is constructive. The scheme for parameters tuning has a simple LMI representation.
- For $\mu = 1$ the control law (6) realizes a high order sliding mode control rejecting matched bounded uncertainties and disturbances.
- The digital implementation of the developed control scheme admits the effective chattering reduction by means of tuning the parameter $V_{\min}$.

and the following **disadvantages**:

- the algorithm is applicable only for digital controllers;
- the practical realization of the developed finite-time control scheme asks for additional computational power of the digital control device, which is required for on-line computation of the ILF value at the current state.

The presented approach to finite-time controller design opens a lot of topics for future research. For example, optimal tuning of the control parameters using SDP programming technique, development of ILF finite-time observers, finite-time ILF control for nonlinear and MIMO systems, etc.

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