A Non–conservative Small-Gain Theorem for GAS Discrete–Time Systems

Roman Geiselhart *,[1] **Rob H. Gielen** ** **Mircea Lazar** **
Fabian R. Wirth *

* *Institute for Mathematics, University of Würzburg, Germany*
** *Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands*

Abstract: This paper makes use of the concept of a finite–time Lyapunov function to derive a non–conservative small-gain theorem for stability analysis of interconnected discrete–time nonlinear systems. Firstly, it is shown that the existence of a global finite–time Lyapunov function is equivalent to global asymptotic stability (GAS) of the overall interconnected system. Secondly, it is indicated that existence of Lyapunov–type functions for each subsystem, together with a small-gain condition implies GAS of the interconnected system. Thirdly, the main result of this paper establishes that GAS of the interconnected system always yields a set of Lyapunov–type functions that satisfy the small-gain condition for a rather general class of GAS nonlinear systems. A simple example demonstrates the non–conservatism of the proposed small-gain theorem.

Keywords: finite–time Lyapunov functions, non-conservative small-gain theorems, discrete–time nonlinear systems

1. INTRODUCTION

For large–scale systems it is in general quite difficult to prove global stability properties, such as global asymptotic stability (GAS). On the other hand, given a Lyapunov function for the system, then this implies GAS. Loosely speaking, a Lyapunov function is a positive definite function that is decreasing along solutions of the system, see Kellett and Teel (2005) for a detailed discussion in the discrete–time case. When we consider interconnected systems, the knowledge of Lyapunov functions for the subsystems yields GAS of the overall system provided a small-gain condition holds (see e.g. Zames (1966); Jiang et al. (1994, 1996); Jiang and Wang (2001); Dashkovskiy et al. (2010)). Especially for large–scale systems this approach is often used as it allows to split the system into several smaller subsystems for which Lyapunov functions can be derived. The small-gain idea is that the system is GAS if an associated comparison system is GAS (see e.g. Rüffer (2010)).

In this work we focus on small-gain results for discrete–time systems (see Jiang and Wang (2001, 2002); Jiang et al. (2004, 2008); Liu et al. (2012)). The small-gain approach (even in the continuous–time case) is quite conservative as it requires the subsystems to be GAS when considered decoupled from the other subsystems. To reduce this conservatism, we make use of the concept of a global finite–time Lyapunov function, which originates from Aeyels and Peuteman (1998), as a general concept for relaxing the Lyapunov conditions for time–varying systems. This concept essentially requires the Lyapunov function to decrease along the solutions of the system after a finite time instant, and not at every time instant. This relaxation was recently exploited in Gielen and Lazar (2012) to derive a non–conservative small-gain theorem for globally exponentially stable (GES) systems and linear gain functions.

Motivated by the results in Gielen and Lazar (2012), we provide a more general small-gain theorem, which involves nonlinear gain functions and applies to GAS systems. The proof of the theorem is based on a construction of a global finite–time Lyapunov function for the overall system. This construction requires the existence of a path which scales the Lyapunov-type functions of the subsystems and leads to an overall global finite–time Lyapunov function. Note that the existence of this path follows from the small-gain condition (see Dashkovskiy et al. (2010)) and we can compute a path using the algorithm in Geiselhart and Wirth (2012). We will show that the existence of a global finite–time Lyapunov function for a system is equivalent to the system being GAS.

We also state a converse of the relaxed small-gain theorem under which a GAS system can be considered as the interconnection of subsystems that admit suitable Lyapunov–type functions and satisfy a classical small-gain condition. The converse holds under a reasonable assumption, which allows for a general class of discrete–time systems, including GES systems, as considered in Gielen and Lazar (2012).

The outline of the paper is as follows. In Section 2 the required preliminaries are given. In Section 3 we state the problem and show the equivalence of GAS with the existence of a global finite–time Lyapunov function. The relaxed small-gain theorem and a converse of it are given in Section 4. Here we further discuss the additional assumption that is required to state the converse of the relaxed small-gain theorem. In Section 5 we give a nonlinear example and apply the proposed small-gain theorem.

[1] Corresponding author; roman.geiselhart@mathematik.uni-wuerzburg.de, r.h.gielen@hotmail.com, m.lazar@tue.nl and wirth@mathematik.uni-wuerzburg.de

2. PRELIMINARIES

By $\mathbb{N}$ we denote the natural numbers and we assume $0 \in \mathbb{N}$. Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers and $\mathbb{R}^N$ the vector space of real column vectors of length N; further $\mathbb{R}^N_+$ denotes the positive orthant. For any vector $v \in \mathbb{R}^N$ we denote by $[v]_i$ its ith component. Then $\mathbb{R}^N_+$ induces a partial order for vectors $v, w \in \mathbb{R}^N_+$. We denote $v \geq w :\iff [v]_i \geq [w]_i$ and $v > w :\iff [v]_i > [w]_i$, each for $i = 1, \dots, N$. We further denote $v \not\geq w :\iff$ there exists an index $i \in \{1, \dots, N\}$ such that $[v]_i < [w]_i$.

For $x_i \in \mathbb{R}^{n_i}, i = 1, \dots, N$ we use the notation $(x_1, \dots, x_N) := (x_1^\top, \dots, x_N^\top)^\top$. For $x \in \mathbb{R}^N$ we use any p-norm

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^N |[x]_i|^p\right)^{1/p} & \text{for } p \in [1, \infty) \\ \max_{i=1,\dots,N} |[x]_i| & \text{for } p = \infty \end{cases}.$$

To state the stability results we use standard *comparison functions*. We call a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a *function of class* $\mathcal{K}$ (denoted by $\alpha \in \mathcal{K}$), if it is strictly increasing, continuous, and satisfies $\alpha(0) = 0$. In particular, if $\alpha \in \mathcal{K}$ satisfies $\lim_{s\to\infty} \alpha(s) = \infty$, it is said to be of class $\mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a *function of class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if it is of class $\mathcal{K}$ in the first argument and $\lim_{t\to\infty} \beta(s, t) = 0$ for any fixed $s \in \mathbb{R}_+$.

For any two functions $\alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$ we write $\alpha_1 < \alpha_2$ if $\alpha_1(s) < \alpha_2(s)$ for all $s > 0$. A function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is called *positive definite*, if $\eta(0) = 0$ and $\eta(s) > 0$ for all $s > 0$. By id we denote the identity function $\mathrm{id}(s) = s$ for all $s \in \mathbb{R}_+$, and by 0 we denote the zero function $0(s) = 0$ for all $s \in \mathbb{R}_+$.

Definition 1. Let $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, for $i, j \in \{1, \dots, N\}$. Then we define the map $\Gamma_\oplus : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ by

$$\Gamma_\oplus(s) := \begin{pmatrix} \max\{\gamma_{11}([s]_1), \dots, \gamma_{1N}([s]_N)\} \\ \vdots \\ \max\{\gamma_{N1}([s]_1), \dots, \gamma_{NN}([s]_N)\} \end{pmatrix}. \quad (1)$$

We note that this map is continuous, satisfies $\Gamma_\oplus(0) = 0$ and is *monotone*, i.e., for all $s_1, s_2 \in \mathbb{R}^N_+$ with $s_1 \leq s_2$ we have $\Gamma_\oplus(s_1) \leq \Gamma_\oplus(s_2)$.

Definition 2. We say that the map $\Gamma_\oplus$ satisfies the *small-gain condition* if for all $s \in \mathbb{R}^N_+ \backslash \{0\}$

$$\Gamma_\oplus(s) \not\geq s. \quad (2)$$

Remark 3. The definition of the map $\Gamma_\oplus$ depends on taking the maximum of the functions γ_{ij}. Note that in other settings summation or so-called *monotone aggregation functions* (Dashkovskiy et al. (2010)) are used with, in general, other functions γ_{ij}.

A function $G : \mathbb{R}^N \to \mathbb{R}^N$ is called $\mathcal{K}$-*bounded*, if there exists a class $\mathcal{K}$-function ω, such that

$$\|G(x)\|_p \leq \omega(\|x\|_p)$$

holds for all $x \in \mathbb{R}^N$. Note that this implies that $G(0) = 0$, but does not necessarily imply continuity of $G(\cdot)$ except at $x = 0$.

3. PROBLEM STATEMENT

We consider N interconnected systems of the form

$$x_i(k+1) = g_i(x_1(k), \dots, x_N(k)) \in \mathbb{R}^{n_i}, \qquad k \in \mathbb{N} \quad (3)$$

with $x_i(0) \in \mathbb{R}^{n_i}$ and $g_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \to \mathbb{R}^{n_i}$ for $i \in \{1, \dots, N\}$. Let $n = \sum_{i=1}^N n_i$, $x = (x_1, \dots, x_N) \in \mathbb{R}^n$, and $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by $G = (g_1, \dots, g_N)$, then the overall system is given by

$$x(k+1) = G(x(k)), \qquad k \in \mathbb{N}. \quad (4)$$

Throughout the paper we will use the following assumption.

Assumption 4. The function $G(\cdot)$ in (4) is $\mathcal{K}$-bounded.

Note that Assumption 4 is rather mild, as it does not even require continuity of the map $G(\cdot)$ (except at $x = 0$). On the other hand, any continuous map $G : \mathbb{R}^n \to \mathbb{R}^n$ with $G(0) = 0$ is $\mathcal{K}$-bounded. Observe that existing results on small–gain theory typically assume continuity of the map $G(\cdot)$.

By $x(k, \xi) \in \mathbb{R}^n$ we denote the solution of system (4) at instance $k \in \mathbb{N}$ with initial condition $x(0) = \xi \in \mathbb{R}^n$.

Definition 5. The system (4) is called *globally asymptotically stable* (GAS) if there exists a $\mathcal{KL}$-function β such that for all $\xi \in \mathbb{R}^n$ and all $k \in \mathbb{N}$

$$\|x(k, \xi)\|_p \leq \beta(\|\xi\|_p, k).$$

As all norms on finite dimensional spaces are equivalent, the definition of GAS is of course independent of p.

Remark 6. It is known that if for any fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}_\infty$, then the system is even uniformly globally asymptotically stable (UGAS). Note that for time-invariant systems every continuous GAS system is UGAS, see (Jiang and Wang, 2002, Proposition 3.2).

Definition 7. A function $W : \mathbb{R}^n \to \mathbb{R}_+$ is a *global Lyapunov function* for system (4) if for some $p \in [1, \infty]$ it holds that

(i) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|_p) \leq W(\xi) \leq \alpha_2(\|\xi\|_p),$$

(ii) there exists a $\mathcal{K}$-function ρ satisfying $\rho < \mathrm{id}$ such that for all $\xi \in \mathbb{R}^n$

$$W(x(1, \xi)) \leq \rho(W(\xi)).$$

Remark 8. (i) In many prior works (e.g. Jiang and Wang (2002)) the definition of a Lyapunov function requires the existence of a positive definite function α_3 such that $W(x(1, \xi)) - W(\xi) \leq -\alpha_3(\|\xi\|_p)$ holds for all $\xi \in \mathbb{R}^n$. Let us briefly explain, that this requirement is equivalent to Definition 7. Note that by following similar steps as in (Lazar, 2006, Theorem 2.3.5) we conclude $0 \leq W(x(1, \xi)) \leq W(\xi) - \alpha_3(\|\xi\|_p) \leq (\mathrm{id} - \alpha_3 \circ \alpha_2^{-1})(W(\xi)) = \rho(W(\xi))$ with $\rho := (\mathrm{id} - \alpha_3 \circ \alpha_2^{-1})$ positive definite. We further have $0 \leq W(x(1, \xi)) \leq (\alpha_2 - \alpha_3)(\|\xi\|_p)$, so $\alpha_2 \geq \alpha_3$ and hence $\rho < \mathrm{id}$. Without loss of generality we can assume that $\rho \in \mathcal{K}$. On the other hand for given $\rho < \mathrm{id}$ we get $W(x(1, \xi)) - W(\xi) \leq -\alpha_3(\|\xi\|_p)$ for $\alpha_3 := (\mathrm{id} - \rho) \circ \alpha_1$.

(ii) For the case $\alpha_1(s) := as^\lambda, \alpha_2(s) := bs^\lambda, \alpha_3(s) := cs^\lambda$ for some $a, b, c, \lambda > 0$ we obtain $W(x(1, \xi)) \leq \rho W(\xi)$ with $\rho := (1 - \frac{c}{b}) \in [0, 1)$, see (Lazar, 2006, Theorem 2.3.5).

We will now relax the assumptions on the global Lyapunov function given in Definition 7.

Definition 9. A function $W : \mathbb{R}^n \to \mathbb{R}_+$ is called a *global finite–time Lyapunov function* for system (4) if for some $p \in [1, \infty]$ it holds that

(i) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|_p) \leq W(\xi) \leq \alpha_2(\|\xi\|_p),$$

(ii) there exists an $M \in \mathbb{N}$, $M \geq 1$, and a $\mathcal{K}$-function ρ satisfying $\rho < \mathrm{id}$ such that for all $\xi \in \mathbb{R}^n$

$$W(x(M,\xi)) \leq \rho(W(\xi)).$$

Remark 10. Here the function W is required to decrease after M steps, whereas a classical Lyapunov function is required to decrease at each step. In particular, any global Lyapunov function is a particular global finite–time Lyapunov function.

The next results show that the existence of a global finite–time Lyapunov function is sufficient and necessary to conclude GAS of the underlying system class at least if $G(\cdot)$ in (4) is continuous.

Proposition 11. If system (4) is GAS then there exists a global finite–time Lyapunov function.

Remark 12. Any global Lyapunov function as defined in Definition 7 is a global finite–time Lyapunov function with $M = 1$. So with (Jiang and Wang, 2002, Theorem 1) GAS together with a continuous dynamic map $G(\cdot)$ implies the existence of a global finite–time Lyapunov function. Note that in Jiang and Wang (2002) the authors take the Euclidian norm and define stability with respect to a set $\mathcal{A}$. Since the proof does not change by taking any arbitrary p-norm this results holds with $\mathcal{A} = \{0\}$. See also Nešić et al. (1999) for the case that $G(\cdot)$ is discontinuous.

We note that that Proposition 11 also holds true for any $\mathcal{K}$-bounded function G, which is not proved here due to space limitations.

Theorem 13. The existence of a global finite–time Lyapunov function for (4) implies that system (4) is GAS.

Proof. Assume that there exists a global finite–time Lyapunov function W as defined in Definition 9. First note that from the standing Assumption 4 we conclude that for any $j \in \mathbb{N}$

$$\|x(j,\xi)\|_p \leq \omega^j(\|\xi\|_p). \tag{5}$$

Then for any $k = lM + j$, $l \in \mathbb{N}$, $j \in \{0,\dots,M-1\}$ we have

$$\begin{aligned}
\|x(k,\xi)\|_p &\leq \alpha_1^{-1}(W(x(k,\xi))) \\
&\leq \alpha_1^{-1}(W(x(lM, x(j,\xi)))) \\
&\leq \alpha_1^{-1} \circ \rho^l(W(x(j,\xi))) \\
&\leq \alpha_1^{-1} \circ \rho^l \circ \alpha_2(\|x(j,\xi)\|_p) \\
&\overset{(5)}{\leq} \alpha_1^{-1} \circ \rho^l \circ \alpha_2 \circ \omega^j(\|\xi\|_p) \\
&\leq \max_{i \in \{0,\dots,M-1\}} \alpha_1^{-1} \circ \rho^l \circ \alpha_2 \circ \omega^i(\|\xi\|_p) \\
&\leq \max_{i \in \{0,\dots,M-1\}} \alpha_1^{-1} \circ \rho^{\frac{k}{M}-1} \circ \alpha_2 \circ \omega^i(\|\xi\|_p) \\
&=: \beta(\|\xi\|_p, k).
\end{aligned}$$

It is easy to see that β is a $\mathcal{KL}$-function. Then by Definition 5 system (4) is GAS. □

4. MAIN RESULTS

We will now relax the assumption of standard small-gain theorems, that the subsystems have to admit Lyapunov functions. Here we only assume that there exist Lyapunov–type functions for the subsystems. The first result, Theorem 14, states the sufficiency to conclude GAS. On the other hand, we state a converse of this result in Theorem 17. This shows the non–conservativeness of the proposed small-gain theorem.

Theorem 14. The system (4) is GAS if there exists an $M \in \mathbb{N}$, $M \geq 1$, functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, and $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ such that the following conditions hold.

(i) For all $i \in \{1,\dots,N\}$ there exist $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$ such that for all $\xi_i \in \mathbb{R}^{n_i}$ it holds

$$\alpha_{1i}(\|\xi_i\|_p) \leq V_i(\xi_i) \leq \alpha_{2i}(\|\xi_i\|_p). \tag{6}$$

(ii) For all $\xi \in \mathbb{R}^n$ it holds

$$\begin{bmatrix} V_1(x_1(M,\xi)) \\ \vdots \\ V_N(x_N(M,\xi)) \end{bmatrix} \leq \Gamma_\oplus \left(\begin{bmatrix} V_1(\xi_1) \\ \vdots \\ V_N(\xi_N) \end{bmatrix} \right). \tag{7}$$

(iii) The map $\Gamma_\oplus$ from (1) satisfies the small-gain condition (2).

Proof. Assume that we have V_i and γ_{ij} satisfying the hypothesis of the theorem. Then from Theorem 14-(iii) and (Dashkovskiy et al., 2010, Theorem 5.2-(iii)) it follows that there exists an Ω-path $\sigma \in \mathcal{K}_\infty^N$, i.e. a function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+^N$, for which any component function $\sigma_i \in \mathcal{K}_\infty$, $i \in \{1,\dots,N\}$, and that satisfies

$$\Gamma_\oplus(\sigma(r)) < \sigma(r) \tag{8}$$

for all $r > 0$. Let in the following $i, j, j' \in \{1,\dots,N\}$. Define

$$W(\xi) := \max_i \sigma_i^{-1}(V_i(\xi_i)). \tag{9}$$

We will show that W is a global finite–time Lyapunov function for the overall system (4). To show this note that 14-(i) implies $W(\xi) \geq \max_i \sigma_i^{-1}(\alpha_{1i}(\|\xi_i\|_p)) \geq \alpha_1(\|\xi\|_p)$ with $\alpha_1 := \min_j \sigma_j^{-1} \circ \alpha_{1j} \circ \frac{1}{N^{1/p}} \mathrm{id} \in \mathcal{K}_\infty$. On the other hand we have $W(\xi) \leq \max_i \sigma_i^{-1}(\alpha_{2i}(\|\xi_i\|_p)) \leq \alpha_2(\|\xi\|_p)$ with $\alpha_2 := \max_i(\sigma_i^{-1} \circ \alpha_{2i}) \in \mathcal{K}_\infty$, which shows the properness (6) of W. To show the decay of W note that (8) implies

$$\max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j(r) < r, \qquad \forall r > 0.$$

Define $\rho := \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j$, then $\rho < \mathrm{id}$ and we have

$$\begin{aligned}
W(x(M,\xi)) &= \max_i \sigma_i^{-1}(V_i(x_i(M,\xi))) \\
&\overset{(7)}{\leq} \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij}(V_j(\xi_j)) \\
&= \max_{i,j} \sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j \circ \sigma_j^{-1}(V_j(\xi_j)) \\
&\leq \max_{i,j,j'} \left(\sigma_i^{-1} \circ \gamma_{ij} \circ \sigma_j\right) \circ \left(\sigma_{j'}^{-1}(V_{j'}(\xi_{j'}))\right) \\
&= \rho(W(\xi)).
\end{aligned}$$

This shows that W is a global finite–time Lyapunov function. Then from Theorem 13 we conclude that system (4) is GAS. □

Remark 15. In the case that $M = 1$, the proof shows that W defined in (9) is a global Lyapunov function. For $M > 1$ this is, in general, false.

The converse of Theorem 14, i.e. the existence of particular functions V_i and γ_{ij}, holds under an appropriate assumption, which we will state now, and which we will discuss in the remainder of this section.

Assumption 16. System (4) admits a global Lyapunov function W that satisfies for some $M \in \mathbb{N}$, $M \geq 1$ and all $s > 0$

$$\rho^M(s) < \alpha_1 \circ \tfrac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}(s), \tag{10}$$

where α_1, α_2, ρ are related to W as defined in Definition 7.

Note that p comes from the norm $\|\cdot\|_p$ and for the case $p = \infty$ we define $\frac{1}{N^{1/\infty}} = 1$. Under this assumption we prove the converse of Theorem 14.

Theorem 17. If the system (4) is GAS and satisfies Assumption 16, then there exist functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ such that the following holds.

(i) For all $i \in \{1, \dots, N\}$ there exist $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$ such that for all $\xi_i \in \mathbb{R}^{n_i}$, (6) holds.
(ii) For all $\xi \in \mathbb{R}^n$ and each $M \in \mathbb{N}$, $M \geq 1$ satisfying (10) it holds (7).
(iii) The map $\Gamma_\oplus$ from (1) satisfies the small-gain condition (2).

Proof. Since the system (4) is GAS there exists a global Lyapunov function W as defined in Definition 7. From Definition 7-(ii) we get by iteration

$$W(x(k,\xi)) \leq \rho^k(W(\xi)). \tag{11}$$

Take any $\eta \in \mathcal{K}_\infty$ and define

$$V_i(\xi_i) := \eta(\|\xi_i\|_p) \tag{12}$$

for $i \in \{1, \dots, N\}$. Then Theorem 17-(i) holds for $\alpha_{1i} = \alpha_{2i} = \eta$ for all $i \in \{1, \dots, N\}$. Let $M \in \mathbb{N}$ satisfy (10). Then

$$\begin{aligned} V_i(x_i(M,\xi)) &= \eta(\|x_i(M,\xi)\|_p) \\ &\leq \eta(\|x(M,\xi)\|_p) \\ &\leq \eta \circ \alpha_1^{-1}(W(x(M,\xi))) \\ &\overset{(11)}{\leq} \eta \circ \alpha_1^{-1} \circ \rho^M(W(\xi)) \\ &\leq \eta \circ \alpha_1^{-1} \circ \rho^M \circ \alpha_2(\|\xi\|_p) \\ &\overset{(13)}{\leq} \max_j \eta \circ \alpha_1^{-1} \circ \rho^M \circ \alpha_2(N^{1/p}\|\xi_j\|_p) \\ &\leq \max_j \eta \circ \alpha_1^{-1} \circ \rho^M \circ \alpha_2 \circ N^{1/p}\mathrm{id} \circ \eta^{-1}(V_j(\xi_j)), \end{aligned}$$

where we used

$$\|\xi\|_p \leq \max_j N^{1/p}\|\xi_j\|_p. \tag{13}$$

By (10) we obtain

$$\gamma := \eta \circ \underbrace{\alpha_1^{-1} \circ \rho^M \circ \alpha_2 \circ N^{1/p}\,\mathrm{id}}_{<\mathrm{id}} \circ \eta^{-1} < \mathrm{id}.$$

Let $\gamma_{ij} := \gamma$, for $i, j \in \{1, \dots, N\}$, then $V_i(x_i(M,\xi)) < \max_j \gamma_{ij}(V_j(\xi_j))$ holds for all $i \in \{1, \dots, N\}$, which shows Theorem 17-(ii) and from $\gamma_{ij} < \mathrm{id}$ we conclude that Theorem 17-(iii) holds. This concludes the proof. □

Remark 18. The number $M \in \mathbb{N}$ in Theorem 14 and 17 depends on the systems dynamics (4) and, of course, on the functions V_i and γ_{ij}. General small-gain theorems as e.g. Jiang et al. (2004, 2008); Liu et al. (2012) are similar to Theorem 14, where typically condition (7) is assumed to be satisfied for $M = 1$. It is known that this approach is conservative, and hence, might fail. The purpose of Theorem 17 is to reduce the conservativeness of current small-gain theory. This goal is attained, however, at the price of finding a suitable M, which might be a difficult problem in itself. Nevertheless, since the only constraint on M is that it is large enough, the developed results hold the promise of delivering applicable ISS conditions, as also demonstrated by the example provided in Section 5.

We will now consider different settings under which Assumption 16 is satisfied. The proofs can be found in the appendix. Let us start with the case where (10) in Assumption 16 does not have to be satisfied for all $s > 0$.

Theorem 19. Let system (4) be GAS and assume that there exists a global Lyapunov function W for system (4) with α_1, α_2, ρ satisfying Definition 7. Then the following holds.

(i) For any compact set $\mathbb{Y} \subset \mathbb{R}^N$ with $0 \notin \mathbb{Y}$, there exists an $M \in \mathbb{N}$, $M \geq 1$ such that (10) holds for any $\|s\|_p$ with $s \in \mathbb{Y}$.
(ii) If, additionally, α_1, α_2, ρ are continuously differentiable and satisfy $\rho'(0) < 1$ and $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$, then for any compact set $\mathbb{Y} \subset \mathbb{R}^N$ with $0 \in \mathbb{Y}$, there exists an $M \in \mathbb{N}$, $M \geq 1$ such that (10) holds for any $\|s\|_p$ with $s \in \mathbb{Y}$.

Note that it is not restriction to assume α_1, α_2, ρ to be continuously differentiable (see Malisoff and Mazenc (2005)).

Theorems 17 and 19-(i) imply, that under the assumption that system (4) admits a global Lyapunov function W, there exists an $M \in \mathbb{N}$, $M \geq 1$, Lyapunov–type functions $V_i : \mathbb{Y}_i \subset \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ satisfying the conditions of Theorem 17-(i)-(ii) for $\xi_i \in \mathbb{Y}_i$, and Theorem 17-(iii) for all $s > 0$, where $s \in \{\|y\|_p : y \in \mathbb{Y} = \mathbb{Y}_1 \times \ldots \times \mathbb{Y}_N\}$. This may be used to show that the construction (12) can at least be used to obtain a finite–time Lyapunov function (similar to (9)) that can guarantee semi-global practical asymptotic stability of system (4).

We now briefly explain why the assumption on the derivatives in Theorem 19-(ii) is reasonable. Assume that system (4) admits a global Lyapunov function W for which the bounds in Theorem 19-(ii) are satisfied. If we fix $\eta \in \mathcal{K}_\infty$ and V_i given by (12) then for any $M \in \mathbb{N}$ large enough we have that Theorem 17-(iii) holds for all $s \in [0, \bar{s}]$, $\bar{s} := \max_{y \in \mathbb{Y}} \|y\|_p$. Again, this can now be used to obtain a finite–time Lyapunov function that guarantees semi-global asymptotic stability of system (4).

Note that the bounds on the derivatives are satisfied if the equilibrium point 0 is locally exponentially convergent (i.e. there exists a local Lyapunov function with exponential bounds), see also Theorem 20–(i).

Theorem 19 only considers the case, where (10) is satisfied on a compact subset of $\mathbb{R}_+$. The next theorem states particular cases, where (10) is globally satisfied so that Assumption 16 holds globally.

Theorem 20. Let system (4) be GAS and assume that there exists a global Lyapunov function W for system (4) with α_1, α_2, ρ satisfying Definition 7. If one of the following conditions holds, then Assumption 16 is globally satisfied.

(i) $\alpha_1(s) = as^\lambda, \alpha_2(s) = bs^\lambda, \alpha_3(s) = cs^\lambda$ holds for some $a, b, c, \lambda > 0$.
(ii) $\rho'(0) < 1$, $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$ and $\rho \in \mathcal{K} \backslash \mathcal{K}_\infty$.
(iii) $\rho'(0) < 1$, $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$, $\liminf_{s\to\infty} (\alpha_1)'(s) \in (0, \infty)$, $\liminf_{s\to\infty} \left(\alpha_2^{-1}\right)'(s) \in (0, \infty)$ and $\limsup_{s\to\infty} \rho'(s) \in (0, 1)$.

Remark 21. The assumption on the functions $\alpha_1, \alpha_2, \alpha_3$ in Theorem 20-(i) implies that system (4) is even globally exponentially stable (GES). Combining Theorems 14, 17 and 19-(i) we partially recover (Gielen and Lazar, 2012, Theorem 4). Additionally, the functions γ_{ij} in Theorem 17 can be chosen to be linear.

Remark 22. The bound on the derivative in Theorem 19-(ii) ensures that system (4) is semi-globally asymptotically stable instead of semi-globally practically asymptotically stable. To ensure global stability, Theorem 20-(ii)-(iii) proposes two conditions.
(i) Theorem 20-(ii) indicates that there exists a compact set U containing 0 such that the system dynamic maps any point $\xi \in \mathbb{R}^n$ in one step into U.

(ii) Theorem 20-(iii) considers the case where for large $s > 0$, α_1 and α_2 are bounded from below and above by affine functions, and ρ is bounded from above by an affine function with slope less than one.

5. ILLUSTRATIVE EXAMPLE

Consider the nonlinear system

$$\begin{aligned} x_1(k+1) &= x_1(k) - 0.3x_2(k) \\ x_2(k+1) &= x_1(k) + 0.3\frac{x_2^2(k)}{1+x_2^2(k)}. \end{aligned} \tag{14}$$

We want to show that this system is GAS. Since in practice finding a suitable global Lyapunov function is quite hard, we are trying to find a suitable global finite–time Lyapunov function. Therefore, we want to split the system into two subsystems. Note that the first subsystem decoupled from the second subsystem is globally stable, but not GAS. So we cannot find a Lyapunov function for this subsystem. At this point standard small-gain theorems would fail, as they consider the subsystems to be at least GAS when decoupled from the other subsystems.

If we assume that Assumption 16 is satisfied, then we note from the proof of Theorem 17 that we can take any $\mathcal{K}_\infty$-function η and define $V_i(\xi_i) := \eta(|\xi_i|)$. Then we can find, in a straightforward manner, an $M \in \mathbb{N}$, $M \geq 1$ and functions $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ such that Theorem 17–(i)-(iii) is satisfied. By implication, this leads to a global finite–time Lyapunov function for the overall system, which implies GAS.

So let us start with $V_i(\xi_i) := |\xi_i|$, $i = 1, 2$. Then we compute for all $\xi \in \mathbb{R}^2$

$$V_1(x_1(1,\xi)) = |\xi_1 - 0.3\xi_2| \leq \max\{2V_1(\xi_1), 0.6V_2(\xi_2)\},$$
$$V_2(x_2(1,\xi)) = |\xi_1 + 0.3\tfrac{\xi_2^2}{1+\xi_2^2}| \leq \max\Big\{2V_1(\xi_1), 0.6\tfrac{V_2^2(\xi_2)}{1+V_2^2(\xi_2)}\Big\}.$$

Since $\gamma_{11}(s) = 2s$, the small-gain condition is violated and we cannot conclude stability. This is also clear from the above observation that the first subsystem decoupled from the second subsystem is not GAS. Computing solutions $x(k,\xi)$ we see that for $k = 3$ we obtain (15).

Note that for all $x \in \mathbb{R}$, we have $\frac{x^2}{1+x^2} \leq \frac{|x|}{2}$. Then

$$\begin{aligned} V_1(x_1(3,\xi)) &\leq 0.4|\xi_1| + 0.21|\xi_2| + \tfrac{0.09}{2}|\xi_2| + \\ &\qquad \tfrac{0.09}{2}\left(|\xi_1| + \tfrac{0.3}{2}|\xi_2|\right) \\ &= \max\{0.89V_1(\xi_1), 0.5235V_2(\xi_2)\}, \\ V_2(x_2(3,\xi)) &\leq 0.7|\xi_1| + 0.3|\xi_2| + \tfrac{0.09}{2}|\xi_2| + \\ &\qquad \tfrac{0.3}{2}\left(|\xi_1| + 0.3|\xi_2| + \tfrac{0.3}{2}(|\xi_1| + \tfrac{0.3}{2}|\xi_2|)\right) \\ &= \max\{1.745V_1(\xi_1), 0.78675V_2(\xi_2)\}. \end{aligned}$$

From that we derive the linear functions

$$\begin{aligned} \gamma_{11}(s) &= 0.89s, & \gamma_{21}(s) &= 1.745s, \\ \gamma_{12}(s) &= 0.5235s, & \gamma_{22}(s) &= 0.78675s. \end{aligned}$$

Since $\gamma_{11} < \mathrm{id}$, $\gamma_{22} < \mathrm{id}$ and $\gamma_{12} \circ \gamma_{21} < \mathrm{id}$, we conclude from the cycle condition (Dashkovskiy et al., 2007, Sec. 4.3) that the small-gain condition (2) is satisfied. From Theorem 14 we can now conclude GAS of the overall system (14).

Remark 23. Theorem 14 proves the GAS property for the interconnected system (4) by constructing a finite–time global Lyapunov function. Note that this procedure is straightforward as it can be seen by means of this example. To do this we use the method proposed in Geiselhart and Wirth (2012) to compute an Ω-path $\sigma(r) := (0.5r, 0.9r)$ that satisfies

$$\Gamma_\oplus(\sigma(r)) = \begin{pmatrix} 0.47115r \\ 0.8725r \end{pmatrix} < \begin{pmatrix} 0.5r \\ 0.9r \end{pmatrix} = \sigma(r)$$

for all $r > 0$. From the proof of Theorem 14 we can now conclude that $W(\xi) := \max_i \sigma_i^{-1}(V_i(\xi_i)) = \max\{2|\xi_1|, \frac{10}{9}|\xi_2|\}$ is a global finite–time Lyapunov function for the overall system (14).

6. CONCLUSIONS

The existence of global finite–time Lyapunov functions, which decrease after a finite time, as a relaxation of common global Lyapunov functions, were shown to be equivalent to the system being GAS. This fact was then used to provide a non–conservative general small-gain theorem that generalizes recent small-gain theorems. A converse theorem, that states the existence of suitable Lyapunov–type and gain functions, is stated and the used assumption is discussed and shown to hold for a rather general class of systems. An example demonstrated the straightforward application and emphasized the non–conservativeness of the proposed approach.

REFERENCES

Aeyels, D. and Peuteman, J. (1998). A new asymptotic stability criterion for nonlinear time-variant differential equations. *IEEE Trans. Autom. Control*, 43(7), 968–971.

Dashkovskiy, S.N., Rüffer, B.S., and Wirth, F.R. (2007). An ISS small gain theorem for general networks. *Math. Control Signals Systems*, 19, 93–122.

Dashkovskiy, S.N., Rüffer, B.S., and Wirth, F.R. (2010). Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J. Control Optim.*, 48, 4089–4118.

Geiselhart, R. and Wirth, F. (2012). Numerical construction of LISS Lyapunov functions under a small-gain condition. *Math. Control Signals Systems*, 24, 3–32.

Gielen, R.H. and Lazar, M. (2012). Non-conservative dissipativity and small-gain conditions for stability analysis of interconnected systems. In *Proc. 51th IEEE Conf. Decis. Control*, 4187–4192. Maui, HI.

Jiang, Z.P., Teel, A., and Praly, L. (1994). Small-gain theorem for ISS systems and applications. *Math. Control Signals Syst.*, 7(2), 95–120.

Jiang, Z.P., Lin, Y., and Wang, Y. (2004). Nonlinear small-gain theorems for discrete-time feedback systems and applications. *Automatica*, 40(12), 2129–2136.

Jiang, Z.P., Lin, Y., and Wang, Y. (2008). Nonlinear small-gain theorems for discrete-time large-scale systems. In *In Proc. 27th Chinese Control Conf.* Kumming, Yunnan, China.

Jiang, Z.P., Mareels, I.M., and Wang, Y. (1996). A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8), 1211–1215.

Jiang, Z.P. and Wang, Y. (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37(6), 857–869.

Jiang, Z.P. and Wang, Y. (2002). A converse Lyapunov theorem for discrete-time systems with disturbances. *Systems Control Lett.*, 45(1), 49–58.

Kellett, C.M. and Teel, A.R. (2005). On the robustness of $\mathcal{KL}$-stability for difference inclusions: Smooth discrete-time

$$x(3,\xi)=\begin{pmatrix} 0.4\xi_1-0.21\xi_2-0.09\frac{\xi_2^2}{1+\xi_2^2}-0.09\frac{(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}{1+(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2} \\ 0.7\xi_1-0.3\xi_2-0.09\frac{\xi_2^2}{1+\xi_2^2}+0.3\frac{\left(\xi_1-0.3\xi_2+0.3\frac{(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}{1+(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}\right)^2}{1+\left(\xi_1-0.3\xi_2+0.3\frac{(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}{1+(\xi_1+0.3\frac{\xi_2^2}{1+\xi_2^2})^2}\right)^2} \end{pmatrix}. \tag{15}$$

Lyapunov functions. *SIAM J. Control Optim.*, 44(3), 777–800.

Lazar, M. (2006). *Model Predictive Control of Hybrid Systems: Stability and Robustness*. Ph.D. thesis, Eindhoven University of Technology.

Liu, T., Hill, D.J., and Jiang, Z.P. (2012). Lyapunov formulation of the large-scale, ISS cyclic-small-gain theorem: The discrete-time case. *Syst. Control Lett.*, 61(1), 266–272.

Malisoff, M. and Mazenc, F. (2005). Further construction of strict Lyapunov functions for time-varying systems. In *Proc. 2005 IEEE American Control Conf.*, volume 3, 1889–1894.

Nešić, D., Teel, A.R., and Kokotović, P.V. (1999). Sufficient conditions for the stabilization of sampled-data nonlinear systems via discrete-time approximations. *Systems Control Lett.*, 38(4–5), 259–270.

Rüffer, B.S. (2010). Small-gain conditions and the comparison principle. *IEEE Trans. Autom. Control*, 55(7), 1732–1736.

Zames, G. (1966). On input-output stability of time-varying nonlinear feedback systems I. conditions derived using concepts of loop gain conicity and positivity. *IEEE Trans. Autom. Control*, 11, 228–238.

Appendix A. PROOF OF THEOREM 19

(i) Since $\rho < \mathrm{id}$, there exists for any $t, \tau > 0$ a minimal $M \in \mathbb{N}$ such that $\rho^M(t) < \tau$. Define $\eta := \alpha_1 \circ \frac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}$ and for any $s \in \mathbb{Y}$

$$M(s) := \min\{M \in \mathbb{N} : \rho^M(\|s\|_p) < \eta(\|s\|_p)\}.$$

Let $\bar{M} := \sup\{M(s) : s \in \mathbb{Y}\}$. We have to show that $\bar{M} < \infty$. So assume to the contrary that $\bar{M} = \infty$. Then there exists a sequence $\{s_l\}_{l\in\mathbb{N}} \subset \mathbb{Y}$ such that $\{M(s_l)\}_{l\in\mathbb{N}} \to \infty$ for $l \to \infty$. Since $\mathbb{Y} \subset \mathbb{R}^n$ is compact, we can without loss of generality assume that the sequence $\{s_l\}_{l\in\mathbb{N}}$ is convergent to a point $S \in \mathbb{Y}$, else take a convergent subsequence. But this means that in any open neighborhood U around S there exist infinitely many $s_i \in U$ with $M(s_i)$ pairwise distinct. On the other hand $M(S)$ is well defined, and, by continuity, there exists an open neighborhood $\tilde{U}$ around S with $M(\tilde{s}) \le M(S)$ for all $\tilde{s} \in \tilde{U}$. But this contradicts the unboundedness of the sequence $\{M(s_i)\}$, where $\tilde{s}_i \in U \subset \tilde{U}$. So $\bar{M} < \infty$.

(ii) From the proof of part (i) of the theorem, it remains to show that there exists an $\varepsilon > 0$ and an $M \in \mathbb{N}$ such that for all $s \in \mathbb{Y}$ with $\|s\|_p \in [0, \varepsilon)$ it holds $\rho^M(\|s\|_p) < \eta(\|s\|_p)$. Since $\rho'(0) < 1$ there exists a $c_1 < 1$ such that $\rho(t) < ct$ for all $t \in [0, \varepsilon)$ and $\varepsilon > 0$ sufficiently small, and since $(\alpha_1 \circ \alpha_2^{-1})'(0) > 0$ there exists a $c_2 > 0$ such that $\alpha_1 \circ \frac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}(t) > c_2 t$ for all $t \in [0, \varepsilon)$. Pick any $M \in \mathbb{N}$ such that $c_1^M < c_2$ then

$$\rho^M(t) < c_1^M t < c_2 t < \alpha_1 \circ \tfrac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}(t)$$

for all $t \in [0, \varepsilon)$, which concludes the proof. □

Appendix B. PROOF OF THEOREM 20

(i) From Remark 8-(ii) we obtain $\rho := (1 - \frac{c}{b}) \in [0, 1)$. Then (10) is equivalent to $\rho^M < \frac{a}{N^{1/p} b}$, and there always exists an $M \in \mathbb{N}$ such that this condition holds.

(ii) From $\rho \in \mathcal{K}\backslash\mathcal{K}_\infty$ we conclude that there exists a $C > 0$ such that $\rho(s) \le C$ for all $s \in \mathbb{R}_+$. Take $v := \alpha_1^{-1}(C) \in \mathbb{R}_+$. From Theorem 19–(ii) there exists an $M \in \mathbb{N}$ such that (10) holds for all $s \in [0, \alpha_2(N^{1/p} v)]$. Then for all $s > \alpha_2(N^{1/p} v)$,

$$\rho^M(s) < \rho(s) \le C = \alpha_1\left(\tfrac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1} \circ \alpha_2 \circ N^{1/p} \mathrm{id}\right)(v) < \alpha_1 \circ \tfrac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}(s).$$

This shows that (10) holds for all $s \in \mathbb{R}_+$.

(iii) Define $c_1 := \liminf_{s\to\infty} (\alpha_1)'(s) > 0$, $c_2 := \liminf_{s\to\infty} \left(\alpha_2^{-1}\right)'(s) > 0$ and $c_3 := \limsup_{s\to\infty} \rho'(s) \in (0, 1)$. By assumption there exists a $\hat{s}_1 > 0$ suitably large and constants $K_1, K_2, K_3 \in \mathbb{R}$ such that for all $s \ge \hat{s}$ we have

$$\alpha_1(s) > c_1 s + K_1 \tag{B.1}$$

$$\alpha_2^{-1}(s) > c_2 s + K_2 \tag{B.2}$$

$$\rho(s) < c_3 s + K_3. \tag{B.3}$$

From equations (B.1) and (B.2) we conclude for $s \ge \hat{s}_1$, and $\hat{s}_1$ suitably large

$$\alpha_1 \circ \frac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}(s) > \frac{c_1 c_2}{N^{1/p}} s + \left(\frac{c_1 K_2}{N^{1/p}} + K_1\right). \tag{B.4}$$

From $\rho < \mathrm{id}$ and (B.3) we conclude for $s \ge \hat{s}_1$ and $\hat{s}_1$ suitably large that

$$\rho^k(s) < c_3^k s + K_3 \sum_{j=0}^{k-1} c_3^j < c_3^k s + K_3 \sum_{j=0}^{\infty} c_3^j < c_3^k s + \frac{K_3}{1 - c_3} \tag{B.5}$$

by evaluating the geometric series. Take $M_1 \in \mathbb{N}$ such that $c_3^{M_1} < \frac{c_1 c_2}{N^{1/p}}$ and define $\hat{s}_2 := \frac{K_3/(1-c_3) - K_2 c_1/N^{1/p} - K_1}{c_1 c_2/N^{1/p} - c_3^{M_1}}$. Then this implies for all $s > \hat{s}_2$

$$c_3^{M_1} s + \frac{K_3}{1 - c_3} < \frac{c_1 c_2}{N^{1/p}} s + \left(\frac{c_1 K_2}{N^{1/p}} + K_1\right). \tag{B.6}$$

Altogether we conclude for all $s > \hat{s} := \max\{\hat{s}_1, \hat{s}_2\}$

$$\rho^{M_1}(s) \overset{\text{(B.5)}}{<} c_3^{M_1} s + \frac{K_3}{1 - c_3} \overset{\text{(B.6)}}{<} \frac{c_1 c_2}{N^{1/p}} s + \frac{c_1}{N^{1/p}} K_2 + K_1 \overset{\text{(B.4)}}{<} \alpha_1 \circ \frac{1}{N^{1/p}} \mathrm{id} \circ \alpha_2^{-1}(s). \tag{B.7}$$

From Theorem 19–(ii) we conclude that there exists an $M_2 \in \mathbb{N}$ such that (10) holds for all $s \in [0, \hat{s}]$. Take $M := \max\{M_1, M_2\}$, then (10) holds on $[0, \hat{s}]$, since $\rho^M \le \rho^{M_2}$. And (10) holds for all $s \ge \hat{s}$ since $\rho^M \le \rho^{M_1}$ and (B.7). □