Nonlinear Observer Normal Forms for Some Predator-Prey Models

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Abstract: This paper considers the nonlinear observer normal forms and their application in an ecological Predator-Prey system. These forms allow for the design of robust observers for Predator-Prey models where full measurement is not available. Thus, from a measured population of one specie (prey or predator), one can estimate the population that is not directly measured.

1. INTRODUCTION

Observer design for nonlinear systems is not an easy task and although research in this area has intensified in recent years, many problems remain. In 1983, two important and distinct contributions were made to this field. These were based on the notion of high-gain observer design (Gauthier et al. [1992]), and the so-called nonlinear observer normal forms Krener and Isidori [1983]. The latter consists of transforming a single output nonlinear dynamical system into an observable linear part and a non-linear term involving the output. It turns out that this particular form allows for adaption of the linear Leunberger observer approach for this particular class of nonlinear systems.

The case of multiple output dynamical system was then treated in Krener and Respondek [1985] and Xia and Gao [1989]. Since then, many geometrical algorithms based on the above body of work have been published, e.g. see (Boutat et al. [2009], Lynch and Bortoff [2001]). Others methods called direct transformations can be found in Lopez et al. [1999], Glumineau et al. [1996] and references therein. The concept of nonlinear observer normal forms depending on the output was addressed in Respondek et al. [2004], Krener and Respondek [1985], Zheng et al. [2007], Wang and Lynch [2010]. The most recent concept is the extended nonlinear observer normal forms introduced in Jouan [2003] and developed in Noh et al. [2004], Back et al. [2006], Boutat [2007], Boutat and Busawon [2011].

The above observer design (nonlinear observer forms) methodology has found applications in electrical/electronic and robotic fields, however, we are not aware of any study considering the applicability of the approach to the field of ecosystems modelled as a Lotka-Volterra system (see Volterra [1931, 1928]). This model, which also is called Predator-prey model, was developed by Volterra [1928] to study the interaction of competing or cooperative species. The general model called Kolmogorov's predator-prey model, that undergoes n populations (more precisely the non-dimensional population density) x_i for $i = 1 : n$ in competition or in cooperation, and can be described as follows

$$\dot{x}_i = x_i f_i(x) \tag{1}$$

where $x = (x_1, ..., x_n)^T$, functions f_i are the per capita growth rate of the species. A population i is a predator for a prey population $j \neq i$ if and only if the following inequalities hold

$$\frac{\partial f_j}{\partial x_i} > 0 \text{ and } \frac{\partial f_i}{\partial x_j} < 0$$

In addition, if these condition are fulfilled we say that the populations i and j are in competition. If the above partial differential inequalities are positive, it is said that the populations i and j are in cooperation.

To our knowledge, research that relate to the synthesis of observers for predator prey models are all based on the Leunberger's observer applied to the linear part using the Lipschitz structure of those models (López et al. [2007a], Vaidyanathan [2010], López et al. [2007b], Varga et al. [2010] and references therein), works using high order polynomial observer Mata-Machuca et al. [2010], or interval observer Bernard et al. [1998], Rapaport and Harmand [2002].

In this work, we propose to develop nonlinear observer normal forms for some prey-predator models. Our goal is not to present a general model that responds all situations, but rather, to show that these methods work very well and can open a new field of applications.

Along this work we assume that we know the parameters of the systems to be studied This paper is organized as follows: The next section recalls some well-known nonlinear observer normal forms. Section 3 gives a background on geometrical material to compute the change of coordinates. Section 4 deals with the transformation of the two species model. Section 5 gives the change of coordinates for the three species. The last section is devoted to transform a prey-predator model into an extended nonlinear observer normal form.

2. OBSERVERS FOR A CLASS OF NONLINEAR DYNAMICAL SYSTEMS

There is a class of nonlinear systems for which the design of observer is relatively straight forward. This class is referred to as the so-called nonlinear observer normal forms described as follows

$$\begin{cases} \dot{z}_i = A_i(y,w)z_1 + \beta_i(y,w) \\ \dot{w} = \eta(y,w) \\ y_i = z_{1,r_i} \end{cases} \quad (2)$$

where

$$A_i(y,w) = \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \alpha_{i,2} & 0 & ... & ... & 0 \\ 0 & \alpha_{i,3} & 0 & ... & ... \\ 0 & 0 & \ddots & 0 & ... \\ 0 & ... & 0 & \alpha_{i,r_i} & 0 \end{pmatrix} \quad (3)$$

is a matrix $r_i \times r_i$ where the vector state $z = (z_1^T, ..., z_m^T)^T$ is such that for $i = 1 : m$ $z_i = (z_{i,1}, ..., z_{i,r_i})^T$, $y = (y_1, ..., y_m)^T$ is the measured output or a diffeomorphism of the measured output, α_i are non-vanishing functions of the output and w are auxiliary variables or auxiliary output (the last example of this paper illustrates how this auxiliary variables is introduced).
Indeed, for dynamical systems where A_i are constant namely $\alpha_i = 1$ for all $i = 2 : m$, we use Leunberger's observer.

In those situations where at least one of the α_i is not constant, we could use the high gain observer strategy (Gauthier et al. [1992])

$$\dot{\widehat{z}}_i = A_i\widehat{z} + \beta_i(w,\overline{y}) - \Gamma^{-1}(y)R_{i,\rho}^{-1}C_i^T(C_i\widehat{z} - \overline{y}_i) \quad (4)$$

$$0 = \rho R_{i,\rho} + G_i^T R_{i,\rho} - R_{i,\rho}G_i + C_i^T C_i \quad (5)$$

where G_i, $\Gamma_i$and $R_{i,\rho}$ are the parameters defined, respectively by the $r_i \times r_i$ matrices

$$G_i = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

with

$$\Gamma_i(y) = diag[\prod_{j=2}^{n}\alpha_{i,j}(y), \prod_{j=3}^{n}\alpha_{i,j}(y), \cdots, \alpha_{i,2}(y), 1]$$

and

$$R_{i,\rho}(n+1-i, n+1-j) = \frac{(-1)^{i+j}C_{i+j-2}^{j-1}}{\rho^{i+j-1}}$$

for $1 \leq i, j \leq n$.
Observation error is given by:

$$\dot{e}_i = (A_i(\overline{y}) - \Gamma_i^{-1}(w,\overline{y})R_{i,\rho}^{-1}C_i^T C)e_i$$

It well-known that if the state of this normal form is bounded or its outputs (y, w) are bounded, then the observation error is exponentially stable assuming that ρ is properly chosen Gauthier et al. [1992].

Remark 1. When $\alpha_{i,r_i}(y)$ (see the expression of the matrix (3)) depends only on the output y, then it can be assumed that $\alpha_{i,r_i} = 1$. Indeed, it just takes as new output $z_{i,r_i} = \overline{y}_{r_i} = \int_0^y \frac{ds}{\alpha_{i,r_i}(s)}$. Thus, $\dot{z}_{i,r_i} = z_{i,r_i-1}$. This allows us to reduce the number of $\alpha_{i,j}$ that needs to be computed.

3. GEOMETRICAL BACKGROUND TO COMPUTE THE COORDINATE CHANGE

This section presents an algorithm for computing a change of coordinate which would transform a dynamical system into the nonlinear observer normal form described by (2). Let ω_i for $i = 1 : n$ be differential 1-form which form a co-frame $\omega = (\omega_1, ..., \omega_n)$, and let $\tau = (\tau_1, ..., \tau_n)$ for $i = 1 : n$ be vector fields which form a dual frame of ω. Thus, we have:

$$\omega_i(\tau_j) = \delta_j^i \text{ for } 1 \leq i, j \leq n \quad (6)$$

$\delta_j^i = 0$ for $i = j$ and 0 otherwise

Lemma 1. The following assertions are equivalent:

(1) 1-forms ω_i are closed i.e. $d\omega_i = 0$
(2) vector fields τ_i commute with respect to the Lie brackets i.e. $[\tau_i, \tau_j] = 0$. Note $[,]$ denotes the Lie bracket.

Proof 1. In fact, the evaluation of a differential of 1-form ω_i on two vector fields τ_j and τ_k is given by:

$$d\omega_i(\tau_j, \tau_k) = L_{\tau_j}\omega_i(\tau_k) - L_{\tau_k}\omega_i(\tau_j) - \omega_i[\tau_j, \tau_k];$$

as $\omega(\tau) = I_{n\times n}$, then we have:

$$d\omega_i(\tau_j, \tau_k) = -\omega_i[\tau_j, \tau_k].$$

As τ is a basis, then form the above equation we obtain $[\tau_j, \tau_k] = 0$ for all $1 \leq j, k \leq n$ if and only if $d\omega_i = 0$ for any given $1 \leq i \leq n$. Thus, the two assertions in Lemma 1 are equivalent. ∎

Dynamical systems that fulfill the so-called observability rank condition have naturally such frame τ and co-frame ω. In fact, consider the following dynamical system

$$\begin{cases} \dot{x} = F(x) \\ y = h(x) \end{cases} \quad (7)$$

where x is the state and y is the output.
Here, we consider a particular situation that corresponds to the dynamical systems studied in this article. The output $y = h(x) \in \mathbb{R}$ is single or double $h(x) = (h_1(x), h_2(x))^T \in \mathbb{R}^2$. We assume the observability rank condition. Thus there is $r_1 \geq r_2 \geq 0$ such that

(1) the observability indices are such that $r_2 \leq r_1$, $r_1 + r_2 = n$ and $r_1 - r_2 \leq 1$
(2) $\theta_{1,i} = dL_F^{i-1}h_1$ for $i = 1 : r_1$ and $\theta_{2,i} = dL_F^{i-1}h_2$ for $i = 1 : r_2$ are linearly independent, where d is the differential and $L_F^k h_i$ is the k^{th} Lie derivative of h_i in the direction of vector field F.

Therefore, one can build the frame τ as follows:
To begin with we determine two vector fields $\tau_{1,1}$ and $\tau_{2,1}$ from the following algebraic equations

$$\theta_{1,i}(\tau_{1,1}) = 0 \text{ for } 1 \leq i \leq r_1 - 1 \text{ and } \theta_{1,r_1}(\tau_{1,1}) = 1$$
$$\theta_{2,j}(\tau_{1,1}) = 0 \text{ for } 1 \leq j \leq r_2$$
$$\theta_{2,j}(\tau_{2,1}) = 0 \text{ for } 1 \leq j \leq r_2 - 1 \text{ and } \theta_{1,r_2}(\tau_{2,1}) = 1$$
$$\theta_{1,i}(\tau_{2,1}) = 0 \text{ for } 1 \leq i \leq r_2$$

and then, we construct the others vector fields by induction as follows

$$\tau_{1,k} = [\tau_{1,k-1}, F] \text{ for } 2 \le k \le r_1$$
$$\tau_{2,k} = [\tau_{2,k-1}, F] \text{ for } 2 \le k \le r_2$$

Where $[,]$ denotes the Lie bracket. By the observability rank condition, $\tau = (\tau_{1,k}, \tau_{1,j})$ is a frame. Now, to construct the co-frame we set $\Lambda = \theta(\tau)$, the evaluation of the 1-forms θ on the vector fields τ. Then, we have $\omega = \Lambda^{-1}\theta$. It is easy to check that ω and τ fulfill the dual equation (6).
Let us now describe the algorithm that will be used in this work to transform the considered predator-prey models.

3.1 Algorithm

(1) *Nonlinear observer normal form: Krener and Isidori [1983], Krener and Respondek [1985], Xia and Gao [1989]*
If τ fulfils the Lemma 1, then there exists a change of coordinates that will transform the dynamical system (7) into the nonlinear observer normal form (2) where A_i is constant and $z_{1,r_1} = y_1$ and $z_{2,r_2} = y_2 + \nu(y_1)$.

(2) *Nonlinear observer normal form with a change on the output: Respondek et al. [2004], Boutat [2007], Boutat et al. [2009]*
If τ doesn't fulfil the Lemma 1, then we build a new frame by setting for $i = 1 : 2$, $\sigma_{i,1} = l_i(y)\tau_{i,1}$, and $\sigma_{i,k} = [F, \sigma_{i,k-1}]$ for $2 \le k \le r_i$. If this new frame σ fulfils the Lemma 1, then there exists a change of coordinates that will transform the dynamical system (7) into the nonlinear observer normal form (2) where A_i constant and $z_{1,r_1} = \nu_1(y)$ and $z_{2,r_2} = \nu_2(y)$ such that $\nu = (\nu_1, \nu_2)^T$ is a diffeomorphism on the output.

(3) *Depending output nonlinear observer normal form: Respondek et al. [2004], Zheng et al. [2007], Wang and Lynch [2010]*
If τ doesn't fulfil the Lemma 1, then we build a new frame by setting for $i = 1 : 2$, $\overline{\tau}_{i,1} = \pi_i(y)\tau_{i,1}$ and $\overline{\tau}_{i,k} = \frac{1}{\alpha_{i,j}}[F, \overline{\tau}_{i,k-1}]$ for $2 \le k \le r_i$, where $\pi_i = \alpha_{i,1}...\alpha_{i,r_i}$ and $\alpha_{i,j}$ are function of the output y as in (2). If this new frame $\overline{\tau}$ fulfils the Lemma 1, then there exists a change of coordinates that will transform the dynamical system (7) into the nonlinear observer normal form (2) where $A_i(y)$s are function of the output.

(4) *Extended nonlinear observer normal form: Noh et al. [2004], Back et al. [2006], Boutat [2007], Boutat and Busawon [2011], Yang and Back [2011]*
If the frame $\overline{\tau}$ doesn't fulfil the Lemma 1, then we add an auxiliary dynamics $\dot{w} = \eta(y, w)$ to the original dynamical system (7). We set $\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = F_1 = \begin{pmatrix} F \\ \eta(y, w) \end{pmatrix}$, and for $i = 1 : 2$, we set $\sigma_{i,1} = l_i(w)\overline{\tau}_{i,1}$ and $\sigma_{i,k} = [F_1, \sigma_{i,k-1}]$ for $2 \le k \le r_i$.
If this new frame σ fulfils the Lemma 1, then there exists a change of coordinates that will transform the dynamical system (7) into the nonlinear observer normal form (2) where $A_i(y, w)$ depends on the output y and auxiliary variable w.

4. OBSERVER FOR TWO SPECIES MODEL

In this section, we consider the following Lotka-Volterra model by assuming per capita growth rates f_i in (1) are linear

$$\begin{cases} \dot{x}_1 = x_1(a - bx_2) \\ \dot{x}_2 = x_2(-c + ex_1) \\ y = x_1 \end{cases} \tag{8}$$

Here, the state x_i are the state variable that are assumed to be > 0, and y is the output and parameters a, b, c, $d > 0$ are defined as:

- Parameter a is the growth rate of the prey in the absence of interaction with predator, and b measures the impact of predation.
- Parameter c is the natural death of predator in the absence of food, and e denotes the efficiency of the predator in interaction with prey.

The observability 1-forms of the model (8) are given by $\theta_1 = dy = dx_1$ and $\theta_2 = (a - bx_2)\, dx_1 - bx_1 dx_2$.
It is clear that these 1-forms are independent. Therefore, the dynamical system (8) fulfils the rank observability condition. A straightforward calculation gives the frame τ as follows
$\tau_1 = \frac{-1}{bx_1}\frac{\partial}{\partial x_2}$ and $\tau_2 = \frac{\partial}{\partial x_1} + (a - bx_2 + -c + ex_1)\,\tau_1$. Now, we have
$[\tau_1, \tau_2] = \frac{1}{x_1}\tau_1 \neq 0$.
Thus the first step in the algorithm is not fulfilled in which case we will use the second step of the algorithm. Let us consider $l(x_1)$ a function of the output, and set
$\sigma_1 = l\tau_1$ and we obtain $\sigma_2 = l\tau_2 - l'\,(x_1(a - bx_2))\,\tau_1$.
A simple calculation gives
$[\sigma_1, \sigma_2] = \left(\frac{l^2}{x_1} - ll'\right)\tau_1 = 0$ if and only if $l = x_1 = y$.
Now, we are ready to compute the change of coordinates. First compute
$\Lambda = \theta(\sigma) = \begin{pmatrix} \theta_1(\sigma_1) & \theta_1(\sigma_2) \\ \theta_2(\sigma_1) & \theta_2(\sigma_2) \end{pmatrix}$, which gives

$$\Lambda = \begin{pmatrix} 0 & x_1 \\ x_1 & l\,(a - bx_2 - c + ex_1) \end{pmatrix},$$

thus the differential of the change of coordinates is as follows

$$dz = \Lambda^{-1}\theta = \begin{pmatrix} d(c\ln x_1 - ex_1 - bx_2) \\ \dfrac{1}{x_1}dx_1 \end{pmatrix}.$$

This implies the following change of coordinates

$$z_1 = c\ln x_1 - ex_1 - bx_2$$
$$z_2 = \ln x_1$$

Finally, the nonlinear observer normal for the dynamical system (8) is given by

$$\begin{cases} \dot{z} = Az + \beta(y) \\ \overline{y} = \ln y = z_2 \end{cases}$$

where $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\beta(y) = \begin{pmatrix} c - ae\exp(\overline{y}) \\ a + e\exp(\overline{y}) - c\overline{y} \end{pmatrix}$

Remark 2. If the output $y = x_2$ was measured instead of x_1, then the same method would result in the following change of coordinates

$$z_1 = -a \ln x_2 + e x_1 + b x_2$$
$$z_2 = \ln x_2$$

which gives the nonlinear observer normal form as follows

$$\begin{cases} \dot{z}_1 = ac - ac \exp \overline{y} \\ \dot{z}_2 = z_1 - c - e \exp \overline{y} + a\overline{y} \\ \overline{y} = \ln y = z_2 \end{cases}$$

5. OBSERVER FOR TWO PREY AND ONE PREDATOR

This section deals with the nonlinear observer normal form for an environment with two prey and one predator (Mamat et al. [2011]). We assume that prey is measured and like to estimate the predator. The model is as follows

$$\begin{cases} \dot{x}_1 = x_1 (a_1 - b_1 x_1 - c_1 x_2 + d_1 x_3) \\ \dot{x}_2 = x_2 (-a_2 + b_2 x_1 + c_2 x_3) \\ \dot{x}_3 = x_3 (a_3 - b_3 x_2 - c_3 x_3 + d_2 x_3) \\ y_1 = x_1 \\ y_2 = x_3 \end{cases} \tag{9}$$

where $x_1 > 0$ and $x_3 > 0$ are two prey in cooperation and $x_2 > 0$ is the predator and a_i, b_i, $c_i > 0$ for all $i = 1:3$ and $d_1, d_2 > 0$.

The observability 1-forms of dynamical system (9) are given by
$\theta_{1,1} = dx_1$, $\theta_{1,2} = -c_1 x_1 dx_2 + (a_1 - 2b_1 x_1 - c_1 x_2 + d_1 x_3) dx_1 + d_1 x_1 dx_3$ and $\theta_{2,1} = dx_3$.
It is clear that they are independent. Thus the dynamical system (9) fulfills the rank observability condition with observability indices $r_1 = 2$ and $r_2 = 1$. A straightforward calculation gives the associated frame

$$\tau_{1,1} = -\frac{1}{x_1 c_1} \frac{\partial}{\partial x_2}$$
$$\tau_{1,2} = (-a_2 + (b_2 - b_1) x_1 + (c_2 + d_1) x_3 + a_1 - c_1 x_2) \tau_{1,1} + \frac{\partial}{\partial x_1} + \frac{b_3 x_3}{x_1 c_1} \frac{\partial}{\partial x_3}.$$

Further calculation shows $[\tau_{1,1}, \tau_{1,2}] = \frac{2}{y_1} \tau_{1,1}$. Now, we use the second step of algorithm to obtain
$\sigma_{1,1} = -\frac{1}{c_1} \frac{\partial}{\partial x_2}$, and $\sigma_{1,2} = x_1 \frac{\partial}{\partial x_1} + (-a_2 + b_2 x_1 + c_2 x_3) \sigma_{1,1} + \frac{b_3 x_3}{c_1} \frac{\partial}{\partial x_3}$.

A straightforward calculation gives $\sigma_{2,1} = x_3 \frac{\partial}{\partial x_3} + \frac{c_1 c_2}{b_3} x_3 \sigma_{1,1}$ which commutes with $\sigma_{1,1}$ and $\sigma_{1,2}$
Now, let us compute the change of coordinates

$$\Lambda = \theta\sigma = \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_1 m & \left(d_1 + \frac{c_1 c_2}{b_3}\right) x_3 x_1 \\ 0 & \frac{b_3 x_3}{c_1} & x_3 \end{pmatrix}$$

where $m = (a_1 - 2b_1 x_1 - c_1 x_2 + d_1 x_3 - a_2 + b_2 x_1 + c_2 x_3 + \frac{d_1 b_3 x_3}{c_1})$.

The change of coordinates is given by $dz = \Lambda^{-1}\theta$ as follows

$$z_{1,1} = -c_1 x_2 - b_2 x_1 + a_2 \ln x_1 - \frac{c_1 c_2}{b_3} x_3$$
$$z_{1,2} = \ln x_1$$
$$z_{2,1} = -\frac{b_3}{c_1} \ln x_1 + \ln x_3$$

Therefore the nonlinear observer normal form of the dynamical system (9) is as follows

$$\begin{cases} \dot{z}_1 = A_1 z_1 + \beta_1(y) \\ \dot{z}_2 = A_2 z_2 + \beta_1(y) \\ \overline{y}_1 = z_{1,2} \\ \overline{y}_2 = -\frac{b_3}{c_1} \ln y_1 + \ln y_2 = -\frac{b_3}{c_1} z_{1,2} + z_{2,1} \end{cases}$$

where $z_1 = (z_{1,1}, z_{1,2})^T$ and $z_2 = (z_{2,1})$, $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $A_2 = 0$ and

$$\beta_{1,1} = a_1 a_2 + b_1 b_2 y_1^2 - (a_1 b_2 + a_2 b_1) y_1 - \left(b_2 d_1 + \frac{1}{b_3} c_1 c_2 d_2\right) y_1 y_2 + \left(a_2 d_1 - \frac{a_3}{b_3} c_1 c_2\right) y_2 + \frac{1}{b_3} c_1 c_2 c_3 y_2^2$$
$$\beta_{1,2} = a_1 + (b_2 - b_1) y_1 - a_2 \ln y_1 + \left(\frac{c_1 c_2}{b_3} + d_1\right) y_2$$
$$\beta_{2,1} = a_3 - a_1 \frac{b_3}{c_1} + \left(b_1 \frac{b_3}{c_1} + d_2\right) y_1 - \left(c_3 + \frac{b_3}{c_1} d_1\right) y_2$$

Remark 3. Thanks to the symmetry of writing (9), it is possible to exchange the roles of the outputs y_1 and y_2 by considering output y_2 that has an observability index of $r_2 = 2$. So we will associate to model (9) another equivalent nonlinear observer normal form.

6. AN EXTENDED NONLINEAR OBSERVER NORMAL FORM

This section deals with the following prey-predator model

$$\begin{cases} \dot{x}_1 = x_1 (a_1 - b_1 x_2) \\ \dot{x}_2 = x_2 (-a_2 + b_2 x_1 - c_2 x_3) \\ \dot{x}_3 = x_3 (-a_3 + b_3 x_2) \\ y = x_2 \end{cases} \tag{10}$$

where species $x_2 > 0$ feeds on species $x_1 > 0$, and species $x_3 > 0$ preys on species x_2. We assume that we measure species $y = x_2$. Now, to simplify the calculation, and inspired by the study of previous models, we can immediately introduce the following diffeomorphism on the output by setting $\xi = \ln x_2$ then the dynamical system is rewritten as follows

$$\begin{cases} \dot{x}_1 = x_1 \left(a_1 - b_1 e^{\xi}\right) \\ \dot{\xi} = -a_2 + b_2 x_1 - c_2 x_3 \\ \dot{x}_3 = x_3 \left(-a_3 + b_3 e^{\xi}\right) \\ \overline{y} = \xi \end{cases} \tag{11}$$

The observability 1-forms of the above dynamical system (11) are given by

$$\theta_1 = d\xi, \quad \theta_2 = b_2 dx_1 - c_2 dx_3$$
$$\theta_3 = b_2 \left(a_1 - b_1 e^{\xi}\right) dx_1 - c_2 \left(-a_3 + b_3 e^{\xi}\right) dx_3 - (b_1 b_2 x_1 + c_2 b_3 x_3) e^{\xi} d\xi$$

It is clear that this dynamical system has one observability singularity at $y = \frac{a_3 + a_1}{b_3 + b_1}$. Here after, we assume that

$y \neq \frac{a_3+a_1}{b_3+b_1}$. Then a simple calculation gives

$$\tau_1 = \frac{1}{b_2\left(a_1+a_3-(b_3+b_1)\,e^{\xi}\right)}\left(\frac{\partial}{\partial x_1}+\frac{b_2}{c_2}\frac{\partial}{\partial x_3}\right).$$

Expressions of vector fields τ_2 and τ_3, which we omit here, shows that the first step of the algorithm does not work. Now, using Remark 1, the reader can easily check form the step (3) of algorithm that $\alpha_2 = b_2\left(a_1+a_3-(b_3+b_1)\,e^{\xi}\right)$ and $\alpha_3 =$ thus $\pi = \alpha_2 \times \alpha_3 = \alpha_2$.
Therefore we can build $\overline{\tau}_1 = \pi\tau_1 = \frac{\partial}{\partial x_1} + \frac{b_2}{c_2}\frac{\partial}{\partial x_3}$.
However, one can show that the vector fields of the new frame don't commute.
As a result, the fourth step of the algorithm is employed. For this we add an auxiliary dynamics $\dot{w} = \eta(w,y)$. Then, by considering a function $l(w)$ and computing $\sigma_1 = l(w)\overline{\tau}_1$, $\sigma_2 = \frac{1}{l}[\sigma_1, F_1]$ and $\sigma_3 = [\sigma_2, F_1]$ then we obtain (see Boutat [2007], Boutat and Busawon [2011])

$$\begin{array}{c} l(w) = e^{-\int_0^w \frac{ds}{\chi(s)}} \\ \dot{w} = -\chi(w)\left(a_1 - b_1 x_2\right), \end{array} \tag{12}$$

where $\chi(w)$ is to be chosen to render w bounded. Therefore, we have

$$\sigma_1 = e^{-\int_0^w \frac{ds}{\chi(s)}}\left(\frac{\partial}{\partial x_1}+\frac{b_2}{c_2}\frac{\partial}{\partial x_3}\right)$$

$$\sigma_2 = -\frac{1}{l(w)\alpha_2}[\sigma_1, F] = -\frac{1}{c_2}\frac{\partial}{\partial x_3}$$

$$\sigma_3 = \frac{\partial}{\partial \xi} + \frac{1}{c_2}\left(-b_3 e^{\xi} + a_3\right)\frac{\partial}{\partial x_3}$$

As in (Boutat [2007], Boutat and Busawon [2011]) we seek a fourth vector field σ_4 which is independent from $\sigma_i \quad i = 1,2,3$ and which commute with them. It easy to check that the following vector field has the desired properties

$$\sigma_4 = \frac{\partial}{\partial w} - \frac{1}{\chi(w)}x_1\left(\frac{\partial}{\partial x_1}+\frac{b_2}{c_2}\frac{\partial}{\partial x_3}\right).$$

Now, we are ready to compute

$$\Lambda = \theta(\sigma) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 \\ l\alpha_2 & a & b+a^2 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a = \left(-a_3 + b_3 e^{\xi}\right)$, $b = -\left(b_1 b_2 x_1 + c_2 b_3 x_3\right)e^{\xi}$ and $c = -\frac{1}{\chi(w)}x_1\alpha_2$. Now, we compute the differential of change of coordinates $\omega = \phi_*$ as follows

$$\omega = \Lambda^{-1}\theta = \begin{pmatrix} -\frac{b}{l\alpha_2}\theta_1 - \frac{a}{l\alpha_2}\theta_2 + \frac{1}{l\alpha_2}\theta_3 - \frac{c}{l\alpha_2}dw \\ -a\theta_1 + \theta_2 \\ \theta_1 \\ dw \end{pmatrix}$$

which gives the following change of coordinates

$$z_1 = e^{\int_0^w \frac{ds}{\chi(s)}}x_1,$$
$$z_2 = -c_2 x_3 - b_3 x_2 + a_3 \ln x_2 + b_2 x_1,$$
$$z_3 = \ln x_2$$

that transforms the nonlinear dynamical system (11) into the following extended nonlinear observer normal form

$$\begin{cases} \dot{z}_1 = 0 \\ \dot{z}_2 = \alpha_2 z_1 + \beta_2 \\ \dot{z}_3 = z_2 + \beta_3(y) \\ \dot{w} = -\chi(w)\left(a_1 - b_1 y\right) \end{cases} \tag{13}$$

where $\alpha_2(y,w) = b_2\left(a_1 + a_3 - (b_1+b_3)y\right)e^{-\int_0^w \frac{ds}{\chi(s)}}$,

$\beta_2(y) = -a_2 a_3 + a_2 b_3 y$ and $\beta_3(y) = b_3 y - a_3 \ln y - a_2$

Remark 4. Instead of the expression of extended dynamics given in (12), one can work with

$$\dot{w} = -\chi(w)\left(-a_3 + b_3 e^{\xi}\right).$$

to obtain a another equivalent extended nonlinear observer normal form.

Simulation 1. Hereafter, we give a simulation of the model (10). We will apply the observer (4) to the nonlinear observer normal form (13) with the following parameter matrices

$$R_\rho = \begin{pmatrix} \frac{6}{\rho^5} & -\frac{3}{\rho^4} & \frac{1}{\rho^3} \\ -\frac{3}{\rho^4} & \frac{2}{\rho^3} & -\frac{1}{\rho^2} \\ \frac{1}{\rho^3} & -\frac{1}{\rho^2} & \frac{1}{\rho} \end{pmatrix} \text{ and } \Gamma_{3\times3} = diag[\alpha_2, \quad 1, \quad 1].$$

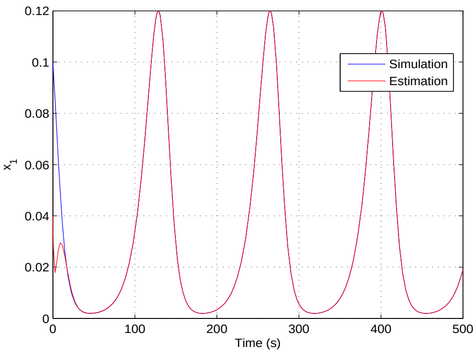

Fig. 1. Evolution of x_1 and its estimation $\hat{x}_1$

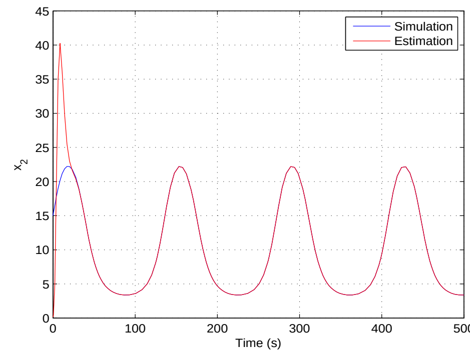

Fig. 2. Evolution of x_2 and its estimation $\hat{x}_2$

The simulation results as illustrated in the above figures, clearly demonstrate that the nonlinear observer designed based on the proposed algorithm in this paper, is fully capable of providing an accurate estimate of the non-measurable states.

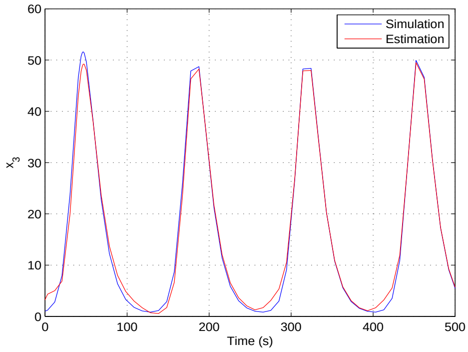

Fig. 3. Evolution of x_3 and its estimation $\hat{x}_3$

7. CONCLUSIONS

This paper considered the problem of observer design for prey-predator ecological systems governed by nonlinear dynamics. The underlying approach to observer design consisted of proposing a change of coordinates for transforming the nonlinear system into the observer nonlinear normal form in systems of interaction between populations. A number of examples illustrated the applicability of the approach. The results presented opens the door to another application area for nonlinear observers in food web (or food chain).

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