Prediction of Partially Synchronous Regimes of Delay-Coupled Nonlinear Oscillators ⋆

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Abstract: We present an approach which allows to accurately predict both the occurrence and type of partially synchronous regimes of delay-coupled non-linear oscillators. Unlike the conventional approach, we build on an analysis of the stability properties of the synchronized equilibrium in the (coupling gain, delay) parameter space. As partially synchronous regimes are closely related to the presence of invariant manifolds, we first present necessary and sufficient conditions for the existence of forward invariant sets. Next, from the existence of these invariant sets and from the characterization of solutions in the unstable manifold of the synchronized equilibrium, we predict which (gain,delay) parameters may result in fully/partially synchronous behavior. We illustrate the approach for a network of delay coupled Hindmarsh-Rose neurons.

Keywords: Partial Synchronization, Delay-coupled non-linear systems, Neuronal networks

1. INTRODUCTION

Synchronous behavior is observed in a wide variety of systems (see Arkady et al. [2001]). There exists different type of synchronous behaviours, for example, if each system in a network behaves in the same fashion, it is called full synchronization. Instead, if the whole system is divided into subgroups consisting of fully synchronized systems which do not synchronize with the systems in different groups is called partial (or cluster) synchronization. Partial synchronization is pervasive in neuronal networks in our brain (Kaneko [1994]).

Partial synchronization in a network of coupled systems via different types of couplings in absence of time-delay has been considered in many works (Pogromsky et al. [2002], Pogromsky [2008], Wu and Chen [2009], Pham and Slotine [2007] and references therein). Partial synchronization in a network of systems coupled via diffusive delay coupling scheme, which arises in diverse interconnected systems (see Steur et al. [2012] and references therein), has been considered in Steur [2011], which was a direct generalization of the approach of Pogromsky et al. [2002] and Pogromsky [2008] for the delay free case. In most of these works, whether or not the coupling scheme is affected by time-delay, sufficient conditions are obtained by using Lyapunov function(al)s, which may make the obtained conditions conservative. Recently, an approach to predict the partially synchronous regimes in a network of non-linear systems coupled via diffusive delay coupling scheme was presented in Ünal and Michiels [2012]. In that work, firstly, forward invariant sets were determined to find the possible partially synchronous regimes. Then, these regimes were predicted by utilizing the forward invariant sets with the stability/instability regions of the synchronized equilibria and structure of the eigenvector of the Laplacian of the network topology. Note that, although the approach of Ünal and Michiels [2012] is based on a local analysis, the partially synchronous regimes were correctly predicted in many cases and different network topologies.

In this paper, we will consider the partial synchronization in a network of diffusively delay coupled systems by using the results of Ünal and Michiels [2012]. We will apply the results considering a network with Hindmarsh-Rose neurons. Due to space limitations, we omit the proofs of some of the theorems. The complete theory, as well as other application examples, can be found in the extended paper Ünal and Michiels [2012].

Throughout, $\mathcal{C}^r(X,Y)$ represents the space of continuous functions from X to Y that are $r \geq 0$ times continuously differentiable. The function in $\mathcal{C}^r(X,Y)$ is called *sufficiently smooth* if sufficiently high order derivatives exist. A non-negative definite function $V : X \to \mathbb{R}_{\geq 0}$ defined on a subset X of $\mathbb{R}^n$ is called radially unbounded if $V(x) \to \infty$ as $\|x\| \to \infty$. A directed graph of order p is represented by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, G\}$, where $\mathcal{V}$ is the set of nodes, p is the number of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the

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set of edges, G is a weighted adjacency matrix with zero diagonal entries and non-negative entries equal to a_{ij} such that $a_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$. For $w_i := \sum_{(i,j)\in\mathcal{E}} a_{ij}$, $L := diag\{w_1, \ldots, w_p\} - G$ is called the weighted Laplacian of $\mathcal{G}$.

2. PRELIMINARIES

2.1 Problem Statement

We consider the partial synchronization of a network of p arbitrarily connected identical non-linear systems described by

$$\dot{x}_i(t) = f(x_i(t)) + Bu_i(t),\ y_i(t) = Cx_i(t),\ i = 1, \ldots, p, \quad (1)$$

where $x_i \in \mathbb{R}^n$, $B, C^T \in \mathbb{R}^{n\times 1}$, $f : \mathbb{R}^n \to \mathbb{R}^n$. We assume that (1) has an unstable equilibrium x^*. The network of systems in (1) is assumed to be described by a strongly connected graph $\mathcal{G}$ of order p, and the coupling between the systems is a diffusive delay coupling function:

$$u_i(t) = k \sum_{(i,j)\in\mathcal{E}} a_{ij}(y_j(t-\tau) - y_i(t-\tau)), \quad (2)$$

where $k > 0$ is the coupling gain, $\tau > 0$ is the transmission delay, a_{ij}, $i, j = 1, \ldots, p$, is the entries of the weighted adjacency matrix.

2.2 Boundedness of Solutions

A prerequisite in the study of synchronization is the boundedness of the solutions of the interconnected system.

Definition 1. (Byrnes et al. [1991]) Consider a system

$$\dot{x}(t) = g(x(t), u(t)),\ \ y(t) = h(x(t)), \quad (3)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathcal{L}_\infty$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are sufficiently smooth functions. The system (3) is called strictly $\mathcal{C}^1$ -semipassive if there exists a $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ nonnegative storage function V satisfying $V(0) = 0$ and

$$\dot{V}(x(t)) \leq y(t)^T u(t) - H(x(t)), \quad (4)$$

where $H \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ satisfies $H(\cdot) > 0$ outside a ball in $\mathbb{R}^n$ with a radius R centered around 0, i.e.,

$$\exists R > 0,\ \ \|x\| \ \geq R \Rightarrow H(x) \geq \varrho(\|x\|),$$

with some positive continuous function $\varrho(\|x\|)$ defined $\forall\ \|x\| \geq R$.

In what follows we make the following assumption.

Assumption 2. The systems (1) are strictly semipassive.

Theorem 3. (Theorem 3.6 in Steur [2011]) Consider a network of p systems in (1) connected via (2) and the network is described by a strongly connected graph. Suppose that each system in (1) is strictly $\mathcal{C}^1$ -semipassive with a radially unbounded storage function $V(x_i)$. Let the functions $H_i(x_i)$ in (4) be such that there exists $R_i > 0$ such that $\|x_i\| > R_i$ implies $H_i(x_i) - 2kd_i\|y_i\| > 0$ with $d_i = \sum_{(i,j)\in\mathcal{E}} a_{ij}$. Then, the solutions of the closed-loop system (1), (2) are ultimately bounded.

2.3 Partial synchronization and forward invariant sets

The existence of partially synchronous regimes is related to symmetry of the network, which can be translated into the presence of forward invariant sets. To make this clear, partially synchronous solutions of (1) and (2) satisfy the property

$$x(t) = (\Phi \otimes I_n)x(t),\ \forall t \geq t_0, \quad (5)$$

where $x(t) := [x_1(t)^T\ \cdots\ x_p(t)^T]^T$, t_0 is the starting time and Φ is a p-by-p permutation matrix. For example, if $p = 4$, solutions satisfying $x_1 \equiv x_2$ and $x_3 \equiv x_4$ are characterized by a relation of the form (5) where

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (6)$$

Note that (5) can be written as

$$x(t) \in \ker\left(I_{pn} - (\Phi \otimes I_n)\right),\ \forall t \geq 0.$$

Therefore, a first step in investigating the occurrence of partial synchronization lies in the detection of symmetries, which can be translated into the presence of permutation matrices for which $\ker\left(I_{pn} - (\Phi \otimes I_n)\right)$ is a forward invariant set for (1)-(2), i.e., for which the implication

$$(I_{pn} - (\Phi \otimes I_n))\phi(s) = 0 \Rightarrow (I_{pn} - (\Phi \otimes I_n))x(\phi)(t) = 0,$$

holds $\forall t \geq 0$, where $x(\phi)(t)$ is the solution at time t with initial condition $x(s) = \phi(s),\ s \in [-\tau,\ 0]$.

Theorem 4. (Ünal and Michiels [2012]) The set $\ker(I_{pn} - (\Phi \otimes I_n))$ is a forward invariant set for (1) and (2) if and only if $\ker(I_p - \Phi)$ is an invariant subspace of the graph Laplacian L.

The following corollary of Theorem 4 extends Lemma 4.6 of Steur [2011] (see also Pogromsky et al. [2002], Pogromsky [2008] for the delay-free case), in the sense that the sufficient conditions for the presence an invariant set in terms of the solvability of a matrix equation are shown to be necessary too.

Corollary 5. (Ünal and Michiels [2012]) The set $\ker\left(I_{pn} - (\Phi \otimes I_n)\right)$ is a forward invariant set for (1) and (2) if and only if matrix equation

$$\left(I_p - \Phi\right) L = X \left(I_p - \Phi\right) \quad (7)$$

has a solution.

The property that $\ker\left(I_{pn} - (\Phi \otimes I_n)\right)$ is a forward invariant set by itself does not imply that the corresponding partially synchronized solutions are observed. The latter occurs if the forward invariant set is stable in the sense that it attracts neighboring solutions. At the end of the next section we explain how valuable information can be deduced from the analysis of the synchronized equilibrium.

3. ANALYSIS OF SYNCHRONIZED EQUILIBRIA

In this section, we will explain how the occurrence of partial synchronization can be predicted. For this, firstly, let us define $e_j := x_j - x_1$, $j = 2, \ldots, p$ as in Michiels and Nijmeijer [2009]. Then, (1)-(2) can be written as

$$\dot{x}_1(t) = f(x_1(t)) + kBC \sum_{(1,j)\in\mathcal{E}} a_{1j} e_j(t-\tau), \quad (8)$$

$$\dot{e}(t) := \begin{bmatrix} \dot{e}_2(t) \\ \vdots \\ \dot{e}_p(t) \end{bmatrix} = \begin{bmatrix} f(x_1(t)+e_2(t)) - f(x_1(t)) \\ \vdots \\ f(x_1(t)+e_p(t)) - f(x_1(t)) \end{bmatrix} - k(\widetilde{L} \otimes BC) \begin{bmatrix} e_2(t-\tau) \\ \vdots \\ e_p(t-\tau) \end{bmatrix}, \quad (9)$$

where $\widetilde{L} = [-\mathbf{1}_{p-1} \quad I_{p-1}]L[\mathbf{0}^T \quad I_{p-1}]^T$, $\mathbf{1}_{p-1}$ and $\mathbf{0}$ are appropriate dimensional vector and matrix with entries 1 and zero, respectively. In (8), x_1 describes the dynamics on the (full) synchronization manifold, while e in (9) describes the synchronization of error dynamics. Since all solutions of (8) and (9) are bounded by Assumption 2, full synchronization between agents is achieved locally, if the zero solution of

$$\dot{e}(t) = \left(\frac{\partial}{\partial x} f(x_1(t))\right) \otimes e(t) - k(\widetilde{L} \otimes BC)e(t-\tau), \quad (10)$$

is uniformly asymptotically stable.

The systems in (1) are assumed to be identical and have an equilibrium x^*, hence, $(x^*, \ldots, x^*)$, called the *synchronized equilibrium*, is an equilibrium of the coupled system. If we linerize the coupled system (1)-(2) around the synchronized equilibrium, we obtain

$$\dot{\xi}(t) = (I \otimes A)\xi(t) - k(L \otimes BC)\xi(t-\tau), \quad (11)$$

where $A = \frac{\partial f(x)}{\partial x}\big|_{x=x^*}$, and $\xi(t) = [\xi_1(t)^T \cdots \xi_p(t)^T]^T$. Then, the characteristic function of (11) can be written as

$$f(\lambda; k, \tau) = \det(F(\lambda; k, \tau)) =: \det(I_p \otimes (\lambda I_n - A) + kL \otimes BCe^{-\lambda\tau}). \quad (12)$$

By using similarity transformation as $L = T\Lambda T^{-1}$, where Λ is upper-triangular matrix with main diagonal entries $\lambda_i(L)$, which corresponds to i^{th} eigenvalue of L, (12) can be written as

$$f(\lambda; k, \tau) = \det\left(I \otimes (\lambda I - A) + \Lambda \otimes kBCe^{-\lambda\tau}\right) = \prod_{i=1}^{p} f_i(\lambda; k, \tau),$$

where

$$f_i(\lambda; k, \tau) = \det(\lambda I_n - A + k\lambda_i(L)BCe^{-\lambda\tau}) \quad (13)$$

$$=: \det(F_i(\lambda; k, \tau)),\ i = 1, \ldots, p. \quad (14)$$

Note that, $f_i(\lambda; k, \tau)$ can have complex valued coefficients for complex eigenvalue(s) of L, if any.

Since the graph of the network topology is assumed to be strongly connected, the eigenvalues of L are non-negative and L has a simple zero eigenvalue with corresponding eigenvector $[1 \ \ldots \ 1]^T$ (see Olfati-Saber and Murray [2004]). Therefore, if the eigenvalues of L are ordered as

$$0 = \lambda_1(L) < |\lambda_2(L)| \leq \ldots \leq |\lambda_p(L)|,$$

then, $f_1(\lambda; k, \tau)$ corresponds to the characteristic equation of the linearization of (8) describing the dynamics on the synchronization manifold, while the functions $f_2(\lambda; k, \tau), \ldots, f_p(\lambda; k, \tau)$ correspond to the linearization of (10).

Now, let us consider the solutions of (11). Let $f_l(\lambda; k, \tau)$ have a zero at $\lambda = \hat{\lambda}$ for some $k > 0$, $\tau > 0$, $l \in \{1, \ldots, p\}$, and assume that $\lambda_l(L)$ is simple and E_l is the corresponding eigenvector. Then, by (14), since there exists $V \in \mathbb{C}^n$ such that $F_l(\hat{\lambda}; k, \tau)V = 0$ and

$$F(\hat{\lambda}; k, \tau)(E_l \otimes V) = E_l \otimes \left(\hat{\lambda} I - A + k\lambda_l(L)BCe^{-\hat{\lambda} l}\right) V,$$

the exponential solution of (11) due to the zero $\hat{\lambda}$ can be written as

$$\begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_p(t) \end{bmatrix} = c(E_l \otimes V)e^{\hat{\lambda} t}, \quad (15)$$

where $c \in \mathbb{C}$ depends on the initial conditions. Similar results can be obtained if $\hat{\lambda}$ is a multiple eigenvalue of F_l and/or λ_l is a multiple eigenvalue of L.

The prediction of partially synchronous regimes, which will be illustrated in Section 4, is based on analyzing the solution of (11) in the *unstable manifold* of the synchronized equilibrium for the given (gain, delay) parameters. In order to explain the main idea in brief, let us assume once more that we have $p = 4$ agents and that all unstable characteristic roots are due to the zeros of f_1 and f_k, $k \in \{2, 3, 4\}$. In addition, let $E_k = [1\ 1\ -1\ -1]^T$ for $k \in \{2, 3, 4\}$ and the set $M = \left\{x \in \mathbb{C}^{4n} : x_1 = x_2;\ x_3 = x_4\right\}$ be a forward invariant for the systems (1)-(2), i.e., Theorem 4 is satisfied with Φ given by (6). By (15), we conclude that close to the synchronized equilibrium the solutions are repelled but in such a way that the synchronization between agents 1 and 2, and the synchronization between agents 3 and 4 are maintained. From this property and the fact that M is a forward invariant set, we predict the corresponding partially synchronized motion. Although the prediction is based on the local behavior of solutions it turns out that in many cases the same type of partially synchronized solutions is observed in the coupled system, as is illustrated in Section 4 as well as in Ünal and Michiels [2012].

Now, we need to determine the stability/instability regions in the (gain,delay)-parameter space of the synchronized equilibria to find for which (k, τ) values $f_l(\lambda; k, \tau)$, $l = 1, \ldots, p$, has zeros in the right-half plane (rhp). Since points on the stability crossing curves correspond to the presence of characteristic roots on the imaginary axis, these curves give the stability/instability regions where the number of rhp zeros of the characteristic function is constant. The stability/instability regions of (11) in delay parameter space for a given k can be determined by Theorem 4 in Ünal and Michiels [2012], which is a slight adaptation of Propositions 3.4-3.5 of Michiels and Nijmeijer [2009] (see also Michiels and Niculescu [2007]). The curves separating the (gain,delay)-parameter space into regions where the number of characteristic roots in the rhp are constant can be determined by repeating Algorithm 1 in Ünal and Michiels [2012] for

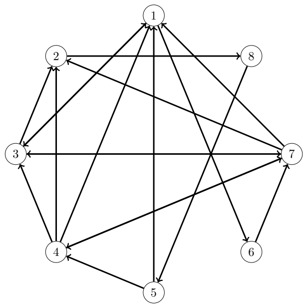


Fig. 1. Network Topology (each connection has weight 1)

a set of k values chosen on a grid. A numerically more efficient way consists of computing these curves as Hopf bifurcation curves of the coupled system using numerical continuation techniques (see, e.g., Seydel [2010]), which need the starting points, however, they can be generated by that algorithm.

For the numerical computations presented in Section 4, we used the package DDE-BIFTOOL (Engelborghs et al. [2001]) to determine the stability crossing curves. Thereby the amount of computations was significantly reduced by using the following lemmas.

Lemma 6. For $i \in \{2,\ldots,p\}$ the property $f_i(j\omega_0;k,\tau)=0$ for $\omega_0>0$, $k\in\mathbb{R}$ and $\tau\geq 0$ implies that $f_i\left(j\omega_0;k,\tau+\frac{2\pi l}{\omega_0}\right)=0$, where $l=1,2,\ldots$.

Lemma 7. If Laplacian L has at least two non-zero real eigenvalues $\lambda_i(L)$ and $\lambda_m(L)$ for some $i,m\in\{2,\ldots,p\}$ and $\omega>0$, $f_i(j\omega;k,\tau)=f_m(j\omega;\hat{k},\tau)$, where $\hat{k}=k\frac{\lambda_i(L)}{\lambda_m(L)}$.

In particular, Lemma 6 implies that one computed curve defines a family of curves characterized by a frequency dependent delay shift. By Lemma 7, if L has at least two non-zero real eigenvalues, it is only necessary to compute the stability crossing curves for one real eigenvalue of L, since for the other real eigenvalues, the curves can be simply obtained by re-scaling the k-axis.

4. APPLICATION TO NEURONAL NETWORKS

We consider a network given in Fig. 1 with 8 Hindmarsh-Rose neurons described by

$$\dot{z}_{i,1}(t)=1-5y_i^2(t)-z_{i,1}(t) \tag{16}$$

$$\dot{z}_{i,2}(t)=0.02y_i(t)+0.0324-0.005z_{i,2}(t) \tag{17}$$

$$\dot{y}_i(t)=-y_i(t)^3+3y_i(t)^2+z_{i,1}(t)-z_{i,2}(t)+3.25 +u_i(t), \tag{18}$$

where $z_{i,1}(\cdot)$ and $z_{i,2}(\cdot)$ are internal variables, $y_i(\cdot)$ and $u_i(\cdot)$ are respectively the membrane potential and the external current of the i^{th} neuron (Hindmarsh and Rose [1984]). It is shown in Steur [2011], since the conditions in Theorem 3 hold for Hindmarsh-Rose neurons, all solutions of (16)-(18) coupled via (2) are bounded. In addition, by linearizing the dynamics in (16)-(18) around its unique equilibrium point x^*, A, B, and C matrices in (11) can be obtained as

$$A=\begin{bmatrix}-1 & 0 & -10y_s\\ 0 & -0.005 & 0.02\\ 1 & -1 & -3y_s^2+6y_s\end{bmatrix}, C^T=B=\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{19}$$

where $y_s=-0.722075$. The weighted Laplacian L of the network topology in Fig. 1 is

$$L=\begin{bmatrix}2&0&-1&0&0&-1&0&0\\0&1&0&0&0&0&0&-1\\-1&-1&3&0&0&0&-1&0\\-1&-1&-1&4&0&0&-1&0\\-1&0&0&-1&2&0&0&0\\0&0&0&0&0&1&-1&0\\-1&-1&-1&-1&0&0&4&0\\0&0&0&0&-1&0&0&1\end{bmatrix},$$

where $\lambda_1(L)=0$, $\lambda_2(L)=\overline{\lambda_3(L)}=1.1226+j0.7449$, $\lambda_4(L)=\lambda_5(L)=2$, $\lambda_6(L)=2.7549$, $\lambda_7(L)=4$, $\lambda_8(L)=5$, and the corresponding eigenvectors are given in (20), where E_5 is a *generalized* eigenvector corresponding to the double non-semisimple eigenvalue 2. Now, let us consider the following permutation matrices,

$$\Phi_1=\begin{bmatrix}0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \Phi_2=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1\end{bmatrix},$$

$$\Phi_3=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}, \Phi_4=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1\end{bmatrix},$$

where Φ_1, for instance, is introduced to investigate the synchronization between neurons 1, 5, 6 and the synchronization between neurons 3, 4, 7. Since $\ker(I-\Phi_1)=\operatorname{span}\{E_1,E_2,E_3\}$, $\ker(I-\Phi_2)=\operatorname{span}\{E_1,\ldots,E_6\}$, $\ker(I-\Phi_3)=\operatorname{span}\{E_1,\ldots,E_6,E_8\}$, and $\ker(I-\Phi_4)=\operatorname{span}\{E_1,\ldots,E_7\}$ are invariant subspaces of L, we conclude from Theorem 4 that the sets $M_1:=\{x\in\mathbb{C}^{24}:x_1=x_5=x_6;\ x_3=x_4=x_7\}$, $M_2:=\{x\in\mathbb{C}^{24}:x_3=x_4=x_7\}$, $M_3:=\{x\in\mathbb{C}^{24}:x_3=x_4\}$ and $M_4:=\{x\in\mathbb{C}^{24}:x_4=x_7\}$ are forward invariant sets for the coupled neuronal system. Now, we will predict the (gain,delay) parameters for which the corresponding partially synchronized regimes occur by utilizing the stability crossing curves separating the (k,τ) parameter space of the synchronized equilibrium as shown in Fig. 2.

The Hopf curves labelled as H_{λ_i} in Fig. 2 are determined by the factor $f_i(\lambda;k,\tau)$ in (13) for $\lambda_i(L)$ and the curves labelled as $H^i_{\lambda_i}$ are determined by Lemma 6 (H_{λ_4,λ_5} corresponds both H_{λ_4} and H_{λ_5}, since $\lambda_4(L)=\lambda_5(L)$). The Hopf curves H_{λ_2,λ_3} and $H^s_{\lambda_2,\lambda_3}$ in Fig. 2 are determined by using (12) considering the complex conjugate eigenvalue pair of L. The number of rhp zeros of the corresponding characteristic function is shown by bold numbers in Fig. 2. Since $\lambda_1(L)=0$ and A matrix in (19) has two unstable eigenvalues, by (13), the characteristic function has at least 2 rhp zeros for all (k,τ) pairs. Thus, since $f_i(\lambda;k,\tau)$ has not any rhp zero for (k,τ) parameters lying below the corresponding Hopf curve, the characteristic function has 2 rhp zeros for (k,τ) parameters lying below all the curves, indicated as region **FS** in Fig. 2. Therefore, by (15), the

$$
[E_1\ E_2\ \cdots\ E_8] = \begin{bmatrix} 1 & 0.3066-0.1037i & 0.3066+0.1037i & 1 & -1 & 0.2810 & -207 & 41 \\ 1 & -0.5680 & -0.5680 & -1 & 2 & 0.0912 & 21 & 1 \\ 1 & -0.1148-0.2157i & -0.1148+0.2157i & -1 & 2 & -0.4930 & 357 & -89 \\ 1 & -0.1148-0.2157i & -0.1148+0.2157i & -1 & 2 & -0.4930 & -171 & -89 \\ 1 & 0.3066-0.1037i & 0.3066+0.1037i & -1 & 0 & 0.2810 & 189 & 16 \\ 1 & 0.3066-0.1037i & 0.3066+0.1037i & 1 & -3 & 0.2810 & 57 & -34 \\ 1 & -0.1148-0.2157i & -0.1148+0.2157i & -1 & 2 & -0.4930 & -171 & 136 \\ 1 & 0.0696+0.4231i & 0.0696-0.4231i & 1 & -1 & -0.1601 & -63 & -4 \end{bmatrix} \tag{20}
$$

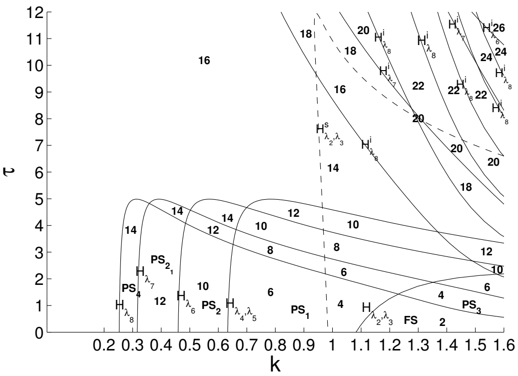
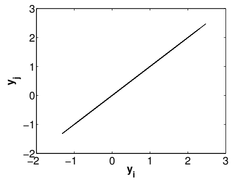


Fig. 2. Stability/instability curves of the coupled systems



$(i, j = 1, \ldots, p,\ i \neq j)$

Fig. 3. Full synchronization for $k = 1.4$ $\tau = 0.3$

corresponding exponential solution of (11) can be written as

$$E_1 \otimes V_1(t),$$

where $V_1(t) \in \mathbb{R}^n$. Because the intersection of all forward invariant sets describes full synchronization and, by (20), all entries of E_1 are non-zero and equal to each other, we may predict fully synchronous behavior in region **FS**. As seen in Fig. 3, all the neurons synchronize for a chosen (k, τ) pair in **FS**. Now, let us consider the region $\mathbf{PS}_1$, which lies under the intersection of H_{λ_8}, H_{λ_7}, H_{λ_6} and H_{λ_4,λ_5} but outside H_{λ_2,λ_3}. In this region, the characteristic function has rhp zeros due to $\lambda_1(L)$, $\lambda_2(L)$ and $\lambda_3(L)$. Then, by (15) and the structure of E_2 and E_3 given in (20), the exponential solution of (11) can be written as follows:

$$E_1 \otimes Z_1(t) + E_2 \otimes Z_2(t), \tag{21}$$

where $Z_i(t) \in \mathbb{R}^n$, $i = 1, 2$. Note that, if the rhp zeros in this region are real and distinct (or with a multiplicity strictly greater than 1), by E_2, the structure of (21) is kept. From (21), solutions which are close to the synchronized equilibrium are repelled, however, the synchronization between neurons 1, 5, 6 and the synchronization between neurons 3, 4, and 7 are preserved. By this observation and the fact that M_1 is a forward invariant set, we may predict such a partial synchronization in region $\mathbf{PS}_1$. As shown in Fig. 4, a chosen (k, τ) pair inside the region $\mathbf{PS}_1$ yields the partially synchronous regime as discussed above. Similarly, let us consider the region $\mathbf{PS}_2$, which is outside H_{λ_4,λ_5} and H_{λ_2,λ_3}, but inside the intersection of H_{λ_8}, H_{λ_7}, and H_{λ_6}. Then, the characteristic function has rhp zeros due to $\lambda_1(L), \ldots, \lambda_5(L)$ such that the introduced rhp zeros due to $\lambda_4(L)$ and $\lambda_5(L)$ have multiplicity two, since $\lambda_4(L) = \lambda_5(L)$. Then, the corresponding unstable solutions of (11) take the form

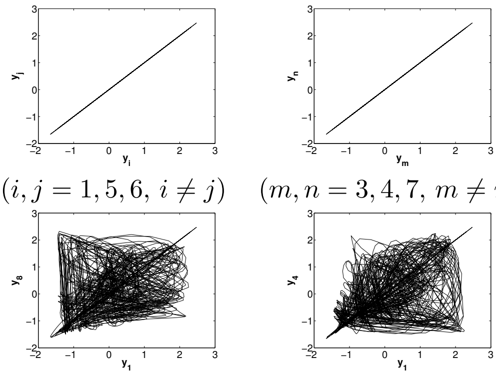
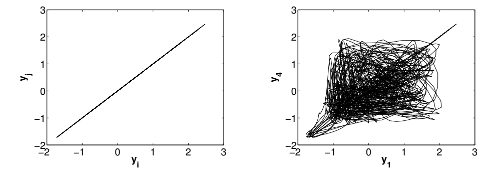


Fig. 4. Partial synchronization for $k = 1.0$ $\tau = 0.5$



$(i, j = 3, 4, 7,\ i \neq j)$

Fig. 5. Partial synchronization for $k = 0.5$ $\tau = 0.5$

$$\sum_{i=1}^{5} E_i \otimes U_i(t), \tag{22}$$

where $U_i(t) \in \mathbb{R}^n,\ i = 1, \ldots, 5$. A common property of all terms in (22) is that the third, fourth and seventh components are equal to each other. Hence, from (22) and the fact that M_2 is a forward invariant set, we may expect partial synchronization of neurons 3, 4, and 7 for a chosen (k, τ) pair inside $\mathbf{PS}_2$ (see Fig. 5). In addition, if we pick a (k, τ) inside the region indicated as $\mathbf{PS}_{2_1}$ in Fig. 2, because of the structure of E_6, synchronization of neurons 3, 4, and 7 can also be expected. Now, let us consider the region indicated as $\mathbf{PS}_3$ in Fig. 2. In this region, since rhp zeros of the characteristic function are due to $\lambda_1(L)$ and $\lambda_8(L)$, the corresponding exponential solution takes the form

$$E_1 \otimes \hat{V}_1(t) + E_8 \otimes \hat{V}_2(t), \tag{23}$$

with $\hat{V}_i(t) \in \mathbb{R}^n,\ i = 1, 2$. Note that the third and fourth components in (23) are equal to each other. Thus, close to the synchronized equilibrium solutions are repelled but the synchrony between neurons 3 and 4 is preserved. Since M_3 is a forward invariant set, we may predict such a partial

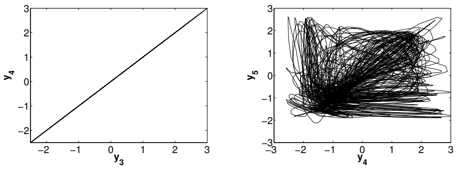


Fig. 6. Partial synchronization for $k = 1.4$ $\tau = 1.3$

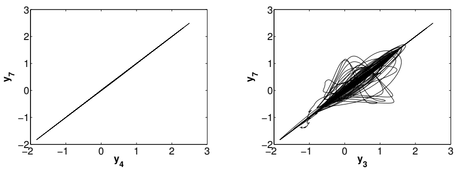


Fig. 7. Partial synchronization for $k = 0.17$ $\tau = 0.15$

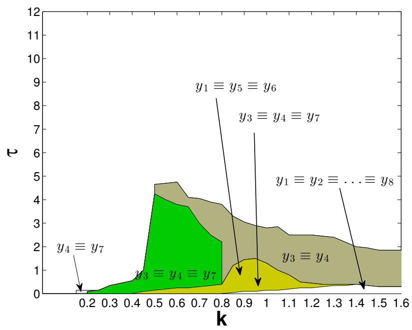


Fig. 8. Regions of (k, τ) pairs resulting different regimes

synchronization of neurons 3 and 4 for a chosen (k, τ) pairs in $\mathbf{PS}_3$ (see Fig. 6). Note that, by the presented approach, if we pick a (k, τ) pair inside the region indicated as $\mathbf{PS}_4$, we should observe the synchronization of neurons 4 and 7, however, we observed it for (k, τ) pairs close to the boundary of $\mathbf{PS}_4$ (see Fig. 7). This can be attributed to using local analysis. In addition, by Lemma 7, since H_{λ_8} can be obtained by scaling H_{λ_7} in k-direction by the ratio of $\lambda_7(L))/\lambda_8(L)$, which is close to 1, the region outside H_{λ_7} but inside H_{λ_8} is narrow, which challenges the prediction of (k, τ) pairs yielding the expected partial synchronization.

Finally, the occurrence of partial synchronization, which is based on numerical simulations for (k, τ)-values on a grid, is depicted in Fig. 8. Note that the type of partially synchronous regimes and the transitions from one regime into another by changing the parameters are correctly predicted by Fig. 2.

5. CONCLUSIONS

We described an approach to analyze the occurrence of partially synchronous regimes for coupled nonlinear oscillators as a function of the coupling delay and the coupling strength. In order to ensure the boundedness of solutions, under mild conditions, the considered subsystems were assumed to be strictly semi-passive. The predictions for the occurrence and type of partially synchronous solutions are based on : existence of the forward invariant sets of (1)-(2), stability analysis of synchronized equilibria, and the structure of the solutions in the unstable manifold which depends on the eigenvector of the weighted Laplacian of network topology. Although the analysis of synchronized equilibria is only local, we have shown that both qualitatively and quantitatively valuable predictions can be made.

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