Partial Observability and Its Consistency for Linear PDEs ⋆

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Abstract: In this paper, a quantitative measure of partial observability is defined for PDEs. The quantity is proved to be consistent in well-posed approximation schemes. A first order approximation of an unobservability index using empirical gramian is introduced. For linear systems with full state observability, the empirical gramian is equivalent to the observability gramian in control theory. The consistency theorem is exemplified using a Burgers' equation.

1. INTRODUCTION

Observability is a fundamental property of dynamical systems that has an extensive literature (Isidori [1995], Kailath [1980]). It is a measure of well-posedness for the estimation of system states using both the sensor information and other user knowledge about the system. Some interesting work can be found in Gauthier-Kupka [1994], Hermann-Krener [1977], Xia-Gao [1989] for nonlinear systems, Infante-Zuazua [1999] for PDEs, Mohler-Huang [1988] for stochastic systems, and Zheng et al. [2007] for normal forms. For complicated problems, a challenge is to define the concept so that it captures the fundamental nature of observability, and meanwhile, the concept should be practically verifiable. In Kang-Xu [2009a,b], a definition of observability is introduced using dynamic optimization. This concept is developed in a project of optimal sensor placement for data assimilations, a computational method widely used in numerical weather prediction. Different from traditional definitions of observability, the one in Kang-Xu [2009a,b] is able to collectively address several issues in a unified framework, including a quantitative measure of observability, partial observability, and improving observability by using user knowledge. Moreover, computational methods of dynamic optimization provide practical tools of numerically approximating the observability of complicated systems that cannot be treated using analytic approaches.

To extend the definition of observability in Kang-Xu [2009a,b] to systems defined by PDEs, several fundamental issues must be addressed. A partial observability makes perfect sense for infinite dimensional systems such as PDEs. However, its computation must be carried out using finite dimensional approximations. It is known in the literature that an ODE approximation of a PDE may not preserve the property of observability, even if the approximation scheme is convergent and stable (Infante-Zuazua [1999], Hohn-Dee [1988]). Therefore, to apply the concept of observability to PDEs, it is important to understand its consistency in ODE approximations.

⋆ This work was supported in part by NRL, AFOSR, and TDSI

In Section 2, some examples from existing literature are introduced to illustrate the issues being addressed in this paper. Observability is defined for PDEs in Section 3. In Section 4, a theorem on the consistency of observability is proved. The relationship between the unobservability index and an empirical gramian is addressed in Section 5, which serves as a first order approximation of the observability. Then the theory is verified using a Burgers' equation.

2. OBSERVABILITY AND ITS CONSISTENCY

Consider the initial value problem of a heat equation

$$\begin{aligned}&u_t(x,t) = u_{xx}(x,t),\ x \in [0, L], t \in [0, T]\\ &u(0,t) = u(L,t) = 0\\ &u(x, 0) = f(x)\end{aligned}$$

Suppose the measured output is

$$y(t) = u(x_0, t)$$

for some $x_0 \in [0, L]$. In this example, we assume $L = 2\pi$, $T = 10$, and $x_0 = 0.5$. This system can be approximated by ODEs using Fourier spectrum method. The observability of the ODEs can be measured using their gramian matrices (Kailath [1980]) . The smallest eigenvalue, σ^N_{min}, measures the observability of the inital state of the ODEs. A small value of σ^N_{min} implies weak observability.

The system has infinitely many modes in its Fourier expansion. However, it has a single output. The computation shows that the output can make the first mode observable. However, when the number of modes is increased, their observability decreases rapidly as shown in Figure 1. For $N = 1$ we have $\sigma^N_{min} = 1.216$, which implies a reasonably observable $\bar{u}_1(0)$, the first Fourier coefficient. However, when N is increased, the observability decreases rapidly. For $N = 8$, σ^N_{min} is almost zero, i.e

$$\left[\,\bar{u}_1(0)\ \bar{u}_2(0)\ \cdots\ \bar{u}_8(0)\,\right]^T$$

is extremely weakly observable, or practically unobservable. In this case, a small sensor noise results in a big estimation error.

The family of solutions of a PDE is an infinite dimensional space. Finite many sensors may not provide adequate

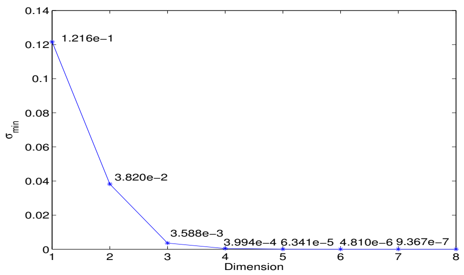

Fig. 1. Observability of the heat equation

information to accurately estimate all modes in an initial state. For the heat equation, a single output makes the first two or three modes observable. All other modes are practically unobservable. On the other hand, in practical applications a finite number of modes is enough to provide accurate approximations. All we need is the observability of these finite modes. This is the reason we would like to *measure* a system's *partial observability.* The concept is useful not only for PDEs. In large scale systems with high (finite) dimensions, it may not be possible or necessary to make the entire state space observable. In many applications a partial observability is all we need.

Another issue to be addressed in this paper is *consistency.* In general the observability for a PDE is numerically computed using a system of ODEs as an approximation. However, it is not automatically guaranteed that the observability of the ODEs is consistent with the observability of the original PDE. In fact, a convergent discretization of a PDE may not preserve its observability. Take the wave equation as an example. It is known that the total energy of the system can be estimated by using the energy concentrated on the boundary. However, in Infante-Zuazua [1999] it is proved that the discretized ODEs do not have the same observability, i.e. the observability of the PDE is not preserved by its discretizations.

In this paper, we introduce a *quantitative measure* of *partial observability* for PDEs. Sufficient conditions are proved for the *consistency* of the observability for well-posed discretization schemes.

3. PROBLEM FORMULATION

Following Canuto et al. [2006], we formulate a linear evolution problem

$$\begin{array}{ll} u_t + \mathcal{L}u = g, & \text{in } \Omega \times (t_0, t_1] \\ u(\cdot, t) \in D(\mathcal{L}) & \text{for } t \in (t_0, t_1] \\ u = u_0 & \text{in } \Omega \times \{t = t_0\} \\ y(t) = \mathcal{H}(u(\cdot, t)) & \end{array} \tag{1}$$

where Ω is an open set in $I\!R^n$, $\mathcal{L}$ is a linear operator, bounded or unbounded, defined in $D(\mathcal{L})$ that is a subspace of a Banach space $(X, ||\cdot||_X)$. In the following, $u(t)$ represents $u(\cdot, t) \in D(\mathcal{L})$. We assume that the boundary conditions of (1) are included in the definition of $D(\mathcal{L})$. We also assume that the initial conditions lie in a subspace of X, denoted by D_0. We consider the family of solutions, in either strict or weak sense, generated by $u(t_0) \in D_0$. For weak solutions, D_0 is not necessarity the same as $D(\mathcal{L})$. The right-hand side g is a continuous function of the variable t with values in X, i.e $g \in C([t_0, t_1], X)$. A solution $u(t)$ for this problem is a X-valued function that is continuous in $[t_0, t_1]$, du/dt exists and is continuous for all $t \in (t_0, t_1]$, satisfying $u(t_0) = u_0$ and $u(t) \in D(\mathcal{L})$ for all $t \in (0, t_1]$. We assume that (1) is a *well-posed problem* in the Hadamard sense (Hille-Phillips [1957], Richtmyer [1978]). More specifically,

- For any $u_0 \in D_0$, (1) has a solution.
- The solution is unique.
- The solution depends continuously on its initial value.

In (1), $y(t) = \mathcal{H}(u(\cdot, t))$ represents the output of the system in which $\mathcal{H}$ is a linear operator from X to $I\!R^p$. The output $y(t) \in C([t_0, t_1])$ has a norm denoted by $||y||_Y$ or $||y(t)||_Y$. Rather than the entire state space, the observability is defined in a finite dimensional subspace. Let

$$W = \text{span}\{e_1, e_2, \cdots, e_s\}$$

be a subspace of D_0 generated by a basis $\{e_1, e_2, \cdots, e_s\}$. In the following, we analyze the observability of $u(t_0)$ using estimates from W. Therefore W is called the *space for estimation.*

Let $u(t)$ be a solution of (1). Suppose $u_w(t_0)$ is the best estimate of $u(t_0)$ in W in the sense that $u_w(t)$ minimizes the following output error,

$$\begin{array}{l} \min ||\mathcal{H}(u_w(t)) - \mathcal{H}(u(t))||_Y \\ \text{subject to} \\ du_w/dt + \mathcal{L}u_w = g \\ u_w(t) \in D(\mathcal{L}) \text{ for all } t \in (t_0, t_1] \\ u_w(t_0) \in W \end{array} \tag{2}$$

Let $u_r(t) = u(t) - u_w(t)$ be the remainder, then

$$u(t) = u_w(t) + u_r(t) \tag{3}$$

If the output $y(t)$ represents the sensor measurement, then it has noise. The data that we use in a estimation process has the following form

$$y(t) + d(t)$$

where $d(t)$ is the measurement error. The observability addressed in this paper is a quantity that defines the sensitivity of the esimation error relative to $d(t)$. From (3), the best estimate $u_w(t)$ has an error that is the remainder $||u_r(t_0)||_X$. This error is not caused by $d(t)$ because the remainder cannot be reduced no matter how accurate the output is measured. This error is due to the choice of W, not the observability of W. Therefore, the following *partial observability* is defined for $u_w(t_0)$ only. Or equivalently, we assume $u(t_0) \in W$ in the definition.

Definition 1. Given a nominal trajectory u of (1) with $u(t_0) \in W$. For a given $\rho > 0$, define

$$\epsilon = \inf ||\mathcal{H}(\hat{u}(t)) - \mathcal{H}(u(t))||_Y \tag{4}$$

where $\hat{u}$ satisfies

$$\begin{array}{l} \hat{u}_t + \mathcal{L}\hat{u} = g \\ \hat{u}(t) \in D(\mathcal{L}) \text{ for all } t \in (t_0, t_1] \\ \hat{u}(t_0) \in W \\ ||\hat{u}(t_0) - u(t_0)||_X = \rho \end{array} \tag{5}$$

Then ρ/ϵ is called the unobservability index of $u(t_0)$ along the trajectory $u(t)$.

Remark. The ratio ρ/ϵ can be interpreted as follows: if the maximum error of the measured output, or sensor error, is

ϵ, then the worst estimation error of $u(t_0)$ is ρ. Therefore, a small value of ρ/ϵ implies a strong observability of $u(t_0)$.

Remark. If $u(t_0)$ is not in W, then the overall error in the estimation of $u(t_0)$ is bounded by the error of the estimate of u_w plus the remainder, or the truncation error, $||u_r(t_0)||_X$. For $u(t_0)$ to be strongly observable, it requires a strong observability of $u_w(t_0)$ and a small remainder $u_r(t_0)$.

Remark. If $u(t_0) = u_w(t_0) + u_r(t_0)$ is not in W, we can compute the observability of $u_w(t_0)$ without solving (2) first to find $u_w(t_0)$. All the computations can be done based on $u(t_0)$. For the observability of $u_w(t_0)$, consider any solution of (5) in which $u(t_0) = u_w(t_0)$. It can be expressed as

$$\hat{u}(t) = u_w(t) + \delta u(t)$$

where $\delta u(t)$ is a solution of the associated homogeneous PDE and

$$\delta u(t_0) \in W, \ \ ||\delta u(t_0)||_X = \rho$$

Therefore,

$$\begin{aligned}&\hat{u}(t) + u_r(t) = u_w(t) + u_r(t) + \delta u(t)\\&= u(t) + \delta u(t)\end{aligned}$$

Because

$$\begin{aligned}&||\mathcal{H}(\hat{u}(t)) - \mathcal{H}(u_w(t))||_Y\\&= ||\mathcal{H}(\hat{u}(t) + u_r(t)) - \mathcal{H}(u_w(t) + u_r(t))||_Y\\&= ||\mathcal{H}(u(t) + \delta u(t)) - \mathcal{H}(u(t))||_Y\end{aligned}$$

the dynamic optimization (4)-(5) is equivalent to

$$\begin{aligned}&\epsilon = \inf ||\mathcal{H}(\bar{u}(t)) - \mathcal{H}(u(t))||_Y\\&\text{subject to}\\&\bar{u}_t + \mathcal{L}\bar{u} = g\\&\bar{u}(t) \in D(\mathcal{L}) \text{ for all } t \in (t_0, t_1]\\&\bar{u}(t_0) \in u(t_0) + W\\&||\bar{u}(t_0) - u(t_0)||_X = \rho\end{aligned} \tag{6}$$

In the formulation (6), it is not necessary to compute $u_w(t)$.

Remark. It can be shown that, for linear problems, ρ/ϵ is a constant. The expressions in (4)-(5) can be simplified (See Section 5). However, we prefer the form adopted in Definition 1 because it can be easily generalized to nonlinear problems or to problems with user-knowledge (Kang-Xu [2009a]).

To numerically compute a system's observability, (1) is approximated by ODEs. In this paper, we consider a general approximation scheme using a sequence of ODEs,

$$\begin{aligned}&\frac{du^N}{dt} + A^N u^N = g^N, \ u^N \in I\!R^N\\&u^N(t_0) = u_0^N\end{aligned} \tag{7}$$

where $N \geq N_0$ for some integer N_0. The approximation is constructed using two *linear mappings*

$$\begin{aligned}&P^N : D_0 \to I\!R^N\\&\Phi^N : I\!R^N \to X\end{aligned} \tag{8}$$

In addition, a norm, $|| \cdot ||_N$, is defined on $I\!R^N$. The approximation scheme is said to be *well-posed* if it is convergent and the metrics in X and $I\!R^N$ are consistent. More specifically, a well-posed approximation scheme satisfies the following conditions.

- Given any solution of (1), $u(t) : (t_0, t_1] \to D(\mathcal{L})$. Let $u^N(t)$ be a solution of (7) satisfying $u^N(t_0) = P^N(u(t_0))$, then
$$\lim_{N\to\infty} ||\Phi^N(u^N(t)) - u(t)||_X = 0 \tag{9}$$
converges uniformly on $[t_0, t_1]$.
- For any $u \in D_0$, the sequence defined by $u^N = P^N u$ satisfies
$$\lim_{N\to\infty} ||u^N||_N = ||u||_X \tag{10}$$

Given the space for estimation W, we define a sequence of subspaces, $W^N \subseteq I\!R^N$, by

$$W^N = P^N(W)$$

They are used as the space for estimation in $I\!R^N$. If $\{e_1, e_2, \cdots, e_s\}$ is a basis of W, then their projections to W^N are denoted by

$$e_i^N = P^N(e_i), \ \ i = 1, 2, \cdots, s$$

So $W^N = \text{span}\{e_1^N, e_2^N, \cdots, e_s^N\}$.

Example. For a spectral method, approximate solutions can be expressed in terms of an orthonormal basis

$$\{q_N(x) : N = 0, 1, 2, \cdots\}$$

For any function,

$$v(x) = \sum_{k=0}^{\infty} v_k q_N(x) \in D_0$$

one can define

$$P^N(v) = [\, v_0 \ v_1 \ \cdots \ v_N \,]^T \tag{11}$$

Obviously, Φ^N is defined by

$$\Phi^N([\, v_0 \ v_1 \ \cdots \ v_N \,]^T) = \sum_{k=0}^{N} v_k q_N \tag{12}$$

Because the basis is orthonormal, the l^2 norm and the inner product in $I\!R^N$ is consistent with those in $L^2(\Omega)$. □

Example. Some approximation methods, such as finite difference and finite element, are based on a grid defined by a set of points in space, $\{x_k\}_{k=1}^N$ and a basis $\{q_k\}$ satisfying

$$q_k(x_j) = \begin{cases} 1 & k = j \\ 0, & \textit{otherwise} \end{cases} \tag{13}$$

In this case, the mappings in the approximation scheme is defined as follows

$$\begin{aligned}&P^N(v) = [\, v(x_1) \ v(x_2) \ \cdots \ v(x_N) \,]^T\\&\Phi^N([\, v_1 \ v_2 \ \cdots \ v_N \,]^T) = \sum_{k=1}^{N} v_k q_k\end{aligned} \tag{14}$$

The inner product in $I\!R^N$ can be induced from the L^2 space, i.e. for $u, v \in I\!R^N$,

$$< u, v >_N = < \sum_{k=1}^{N} u_k q_k, \sum_{k=1}^{N} v_k q_k >$$

If a function v in D_0 are uniformly continuous, then $\Phi^N(P^N(v(t)))$ converges to $v(t)$ uniformly. It can be shown that $< \cdot, \cdot >_N$ is consistent with the inner product in

$L^2(\Omega)$.□

Following Kang-Xu [2009a], we define the observability for ODE systems.

Definition 2. Given $\rho > 0$ and a trajectory $u^N(t)$ of (7) with $u^N(t_0) \in W^N$. Let

$$\epsilon^N = \inf ||\mathcal{H} \circ \Phi^N(\hat{u}^N(t)) - \mathcal{H} \circ \Phi^N(u^N(t))||_Y$$

where $\hat{u}^N$ satisfies

$$\begin{aligned} &\frac{d\hat{u}^N}{dt} + A^N \hat{u}^N = g^N \\ &\hat{u}^N(t_0) \in W^N \\ &||\hat{u}^N(t_0) - u^N(t_0)||_N = \rho \end{aligned} \tag{15}$$

Then ρ/ϵ^N is called the unobservability index of $u^N(t_0)$ in the space W^N.

4. THE CONSISTENCY OF OBSERVABILITY

In this section, we address the consistency of observability. The theorem is based on a continuity assumption. In the problem formulation, the output mapping $\mathcal{H}$ is not required to be bounded. However, in the proof of the consistency theorem we require $\mathcal{H}$ be continuous in the following subspace of X extended from W

$$W_E = \text{span}\left\{\{e_1, e_2, \cdots, e_s\} \cup \{\Phi^N(e_1^N), \cdots, \Phi^N(e_s^N)\}_{N=N_0}^{\infty}\right\}$$

Output Continuity Assumption: Given a sequence

$$\{v_k(t)\}_{k=k_0}^{\infty} \subset C^1([t_0, t_1], W_E)$$

If $v_k(t)$ converges to $v(t)$ in W_E uniformly on $[t_0, t_1]$, then

$$\lim_{k\to\infty} \mathcal{H}(v_k(t)) = \mathcal{H}(v(t))$$

Remark. The Ouput Continuity Assumption is easy to prove for some special cases. In fact, any one of the following conditions implies this assumption. (a) $\mathcal{H}$ is a bounded linear operator. (b) In a spectral method P^N and Φ^N are defined in (11)-(12). (c) For finite difference or finite element methods P^N and Φ^N are defined in (13)-(14). (d) W_E has a finite dimension.

Theorem 1. Suppose the initial value problem (1) and its approximation scheme (7)-(8) are well-posed. Suppose $\mathcal{H}$ satisfies Output Continuity Assumption. Then,

$$\lim_{N\to\infty} \epsilon^N = \epsilon \tag{16}$$

To prove this theorem, we need the following lemma.

Lemma 1. Given a sequence $\hat{u}^N(t)$, $N \geq N_0$, satisfying (15). Then there exists a subsequence, $\hat{u}^{N_k}(t)$, so that $\{\Phi^{N_k}(\hat{u}^{N_k}(t))\}_{k=1}^{\infty}$ converges uniformly to a solution of (5).

Proof . Let $\bar{u}(t)$ be the solution of the original PDE with the initial value

$$\bar{u}(t_0) = 0$$

and let $h_i(t)$, $i = 1, 2, \cdots, s$, be the solutions of the associated homogeneous PDE satisfying $h_i(t_0) = e_i$, i.e.

$$\begin{aligned} &\partial h_i/\partial t + \mathcal{L}h = 0, \text{ in } \Omega \times (t_0, t_1] \\ &h_i = e_i \qquad\qquad\quad \text{in } \Omega \times \{t = 0\} \end{aligned} \tag{17}$$

Then any solution of the nonhomogeneous PDE with an initial value in W has the form

$$\bar{u} + \sum_{i=1}^{s} a_i h_i \tag{18}$$

For each N, define $\bar{u}^N(t)$ be the solution of an approximating ODE satisfying

$$\bar{u}^N(t_0) = 0$$

We know that $\Phi^N(\bar{u}^N(t))$ approaches $\bar{u}(t)$ uniformly as $N \to \infty$. Let $h_i^N(t)$, $i = 1, 2, \cdots, s$, be the solution of the associated homogeneous ODE with initial value e_i^N, i.e.

$$\begin{aligned} &dh_i^N/dt + A^N h_i^N = 0 \\ &h_i^N(t_0) = e_i^N \end{aligned} \tag{19}$$

Then $\Phi^N(h_i^N(t))$ approaches $h_i(t)$ uniformly as $N \to \infty$. For each $\hat{u}^N(t)$ in Lemma 1, it can be expressed as

$$\hat{u}^N(t) = \bar{u}^N(t) + \sum_{i=1}^{s} a_i^N h_i^N(t)$$

and

$$\hat{u}^N(t_0) = \sum_{i=1}^{s} a_i^N e_i^N$$

From the initial condition in (15), we know that the set

$$\{||\sum_{i=1}^{s} a_i^N e_i^N||_N, \;\; N \geq N_0\}$$

is bounded. Using the compactness of bounded sets in $\mathbb{R}^N$ and the consistency of the norms, we can prove that the sequence $\{(a_1^N, a_2^N, \cdots, a_s^N)^T\}_{N=N_0}^{\infty}$ has a bounded subsequence which converges under the standard norm $||\cdot||_2$. Let $\{(a_1^{N_k}, a_2^{N_k}, \cdots, a_s^{N_k})^T\}_{k=1}^{\infty}$ be the convergent subsequence with a limit $(a_1, a_2, \cdots, a_s)^T$. Then

$$\begin{aligned} &\lim_{k\to\infty} \Phi^{N_k}(\hat{u}^{N_k}(t)) \\ &= \lim_{k\to\infty} (\Phi^{N_k}(\bar{u}^{N_k}(t)) + \sum_{i=1}^{s} a_i^{N_k} \Phi^{N_k}(h_i^{N_k}(t))) \\ &= \bar{u}(t) + \sum_{i=1}^{s} a_i h_i(t) \\ &\triangleq \hat{u}(t) \end{aligned}$$

The limit converges uniformly. From (18), $\hat{u}(t)$ must be a solution of the PDE in (5). Because

$$||\hat{u}^{N_k}(t_0) - u^{N_k}(t_0)||_{N_k} = \rho$$

and because of the consistency of the norms, we have

$$\begin{aligned} ||\hat{u}(t_0) - u(t_0)||_X &= \lim_{k\to\infty} ||\sum_{i=1}^{s} a_i e_i^{N_k} - u^{N_k}(t_0)||_{N_k} \\ &= \lim_{k\to\infty} ||\sum_{i=1}^{s} a_i^{N_k} e_i^{N_k} - u^{N_k}(t_0)||_{N_k} = \rho \end{aligned}$$

Therefore, $\{\Phi^{N_k}(\hat{u}^{N_k}(t))\}_{k=1}^{\infty}$ converges uniformly to a solution of (5). □

Proof of Theorem 1. First, we prove

$$\liminf \epsilon^N \geq \epsilon \tag{20}$$

Suppose this is not true, then $\liminf \epsilon^N < \epsilon$. There exists $\alpha > 0$ and a subsequence $N_k \to \infty$ so that

$$\epsilon^{N_k} < \epsilon - \alpha$$

for all N_k. From the definition of ϵ^{N_k}, there exist $\hat{u}^{N_k}(t)$ satisfying (15) such that

$$||\mathcal{H} \circ \Phi^{N_k}(\hat{u}^{N_k}(t)) - \mathcal{H} \circ \Phi^{N_k}(u^{N_k}(t))||_Y < \epsilon - \alpha$$

From Lemma 1, we can assume that $\Phi^{N_k}(u^{N_k}(t))$ converges to $\hat{u}(t)$ uniformly and $\hat{u}$ satisfies (5). From Output Continuity Assumption,

$$\lim_{k\to\infty} ||\mathcal{H}\circ\Phi^{N_k}(\hat{u}^{N_k}(t)) - \mathcal{H}\circ\Phi^{N_k}(u^{N_k}(t))||_Y = ||\mathcal{H}(\hat{u}(t)) - \mathcal{H}(u(t))||_Y \leq \epsilon - \alpha$$

However, from the definition of ϵ, we know

$$\epsilon \leq ||\mathcal{H}(\hat{u}(t)) - \mathcal{H}(u(t))||_Y$$

A contradiction is found. Therefore, (20) must hold.

In the next step, we prove

$$\limsup \epsilon^N \leq \epsilon \tag{21}$$

It is adequate to prove the following statement: for any $\alpha > 0$, there exists $N_1 > 0$ so that

$$\epsilon^N < \epsilon + \alpha \tag{22}$$

for all $N \geq N_1$. From the definition of ϵ, there exists $\hat{u}$ satisfying (5) so that

$$||\mathcal{H}(\hat{u}(t)) - \mathcal{H}(u(t))||_Y < \epsilon + \alpha \tag{23}$$

Let $\hat{u}^N$ be a solution of the ODE

$$d\hat{u}^N/dt + A^N\hat{u}^N = g^N$$

with the initial value

$$\hat{u}^N(t_0) = P^N(\hat{u}(t_0))$$

Then the following limit converges uniformly

$$\lim_{N\to\infty} ||\Phi^N(\hat{u}^N(t)) - \hat{u}(t)||_X = 0 \tag{24}$$

A problem with $\hat{u}^N(t_0)$ is that its distance to $u^N(t_0)$ may not be ρ, which is required by (15). Let us define

$$\bar{u}^N(t) = \gamma_N(\hat{u}^N(t) - u^N(t)) + u^N(t)$$
$$\gamma_N = \frac{\rho}{||\hat{u}^N(t_0) - u^N(t_0)||_N}$$

Then $\bar{u}(t)$ satisfies (15). Due to the consistency of the norms and the fact $||\hat{u}(t_0) - u(t_0)||_X = \rho$, we know $\lim_{N\to\infty}\gamma_N = 1$ Because of (24),

$$\Phi^N(\bar{u}^N(t)) - \Phi^N(u^N(t)) = \gamma_N(\Phi^N(\hat{u}^N(t)) - \Phi^N(u^N(t)))$$

converges uniformly to $\hat{u} - u$. Output Continuity Assumption and (23) imply

$$\lim_{N\to\infty} ||\mathcal{H}\circ\Phi^N(\bar{u}^N(t)) - \mathcal{H}\circ\Phi^N(u^N(t))||_Y = ||\mathcal{H}(\hat{u}(t)) - \mathcal{H}(u(t))||_Y < \epsilon + \alpha$$

This implies that there exits $N_1 > 0$ so that

$$||\mathcal{H}\circ\Phi^N(\bar{u}^N(t)) - \mathcal{H}\circ\Phi^N(u^N(t))||_Y < \epsilon + \alpha$$

for all $N \geq N_1$. From the definition of ϵ^N, we know

$$\epsilon^N \leq ||\mathcal{H}\circ\Phi^N(\bar{u}^N(t)) - \mathcal{H}\circ\Phi^N(u^N(t))||_Y < \epsilon + \alpha$$

for all $N > N_1$. Therefore, (21) holds. □

5. GRAMIAN MATRIX

In this section, we assume that W^N and the space of $y(t)$ are both Hilbert spaces with inner products $<,>_N$ and $< \cdot,\cdot >_Y$, respectively. Suppose $t_0 = 0$. Then ϵ^N/ρ equals the smallest eigenvalue of a gramian matrix. More specifically, let $\{e_1^N, e_2^N, \cdots, e_s^N\}$ be a set of orthonormal basis of W^N. For any $\hat{u}^N$ satisfying (15), we have $\hat{u}^N(t) - u^N(t) = \sum_{k=1}^{s} a_k e^{-A^N t} e_k^N$ for some coefficients satisfying $\sum_{k=1}^{s} a_k^2 = \rho$ Therefore,

$$< \mathcal{H}\circ\Phi^N(\hat{u}^N(t) - u^N(t)), \mathcal{H}\circ\Phi^N(\hat{u}^N(t) - u^N(t)) >_N = [\, a_1\ a_2\ \cdots\ a_s \,]\, G\, [\, a_1\ a_2\ \cdots\ a_s \,]^T$$

where G is the gramian

$$G = [\, G_{ij} \,]_{s\times s}, \quad G_{ij} = < \mathcal{H}\circ\Phi^N(e^{-A^N t}e_i^N), \mathcal{H}\circ\Phi^N(e^{-A^N t}e_j^N) >_Y \tag{25}$$

This matrix is the same as the observability gramian if W^N is the entire space and if $y(t)$ lies in a L^2-space. It is straightforward to prove

$$(\epsilon^N)^2 = \min < \mathcal{H}\circ\Phi(\hat{u}^N(t) - u^N(t)), \mathcal{H}\circ\Phi(\hat{u}^N(t) - u^N(t)) >_N = \min_{\sum a_k^2 = \rho^2} [\, a_1\ a_2\ \cdots\ a_s \,]\, G\, [\, a_1\ a_2\ \cdots\ a_s \,]^T = \sigma_{min}\rho^2$$

where σ_{min} is the smallest eigenvalue of G. To summarize, if $u^N(0)$ and $y(t)$ lie in Hilbert spaces, then the unobservability index of the discretized system can be computed using the smallest eigenvalue of the gramian (25)

$$\rho/\epsilon^N = \frac{1}{\sqrt{\sigma_{min}}} \tag{26}$$

For the heat equation, the mappings can be defined by

$$P^N(u) = [\, u_1^N, u_2^N, \cdots, u_N^N \,]^T$$
$$u_k^N = \frac{2}{L}\int_0^{2\pi} u(x)\sin\left(\frac{k\pi x}{L}\right)dx$$
$$\Phi^N(u^N) = \sum_{k=1}^{N} u_k^N \sin\left(\frac{k\pi x}{L}\right)$$

If we want to find the observability of the first s modes, Definition 2 is equivalent to the analysis using the traditional observability gramian for $N = s$. In fact, for all $N \geq s$, G is a constant matrix and

$$G = \int_0^T e^{(A^s)'t}(C^s)'C^s e^{A^s t}dt$$

Therefore, $\epsilon^N = \epsilon^s$ for all $N \geq s$ and ϵ^N is consistent.

The idea of gramian matrix can be applied to nonlinear systems as a first order approximation of observability, an approach inspired by the computational method in Krener-Ide [2009].

Example. Consider the following Burgers' equation

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \kappa\frac{\partial^2 u(x,t)}{\partial x^2} \\ &u(x,0) = u_0(x), \qquad x \in [0, L] \\ &u(0,t) = u(L,t) = 0, \qquad t \in [0, T] \end{aligned} \tag{27}$$

where $L = 2\pi$, $T = 5$, and $\kappa = 0.14$. The output, $(y_1(t_k), y_2(t_k), y_3(t_k))$ represents three sensors located at $x = L/4$, $2L/4$, and $3L/4$. In the output space

$$||y||_Y = \left(\sum_{k=0}^{N_t}(y_1^2(t_k) + y_2^2(t_k) + y_3^2(t_k))\right)^{1/2}$$

The approximation scheme is based on equally spaced grid-points $x_0 = 0 < x_1 < \cdots < x_N = L$, where $\Delta x = L/N$. System (27) is discretized using a central difference method

$$\begin{aligned} \dot{u}_1^N &= -u_1^N \frac{u_2^N - u_0^N}{2\Delta x} + \kappa \frac{u_2^N + u_0^N - 2u_1^N}{\Delta x^2} \\ &\vdots \\ \dot{u}_{N-1}^N &= -u_{N-1}^N \frac{u_N^N - u_{N-2}^N}{2\Delta x} + \kappa \frac{u_N^N + u_{N-2}^N - 2u_{N-1}^N}{\Delta x^2} \end{aligned} \tag{28}$$

where $u_0^N = u_N^N = 0$. For any $v(x) \in C^1([0, L])$, we define

$$P^N(v) = [\, v(x_1)\ v(x_2)\ \cdots v(x_{N-1})\,] \in I\!R^{N-1}$$

For any $v^N \in I\!R^{N-1}$, define $\Phi^N(v^N) = v(x) \in C^1[0, L]$ be the unique function of cubic spline determined by v^N and $(x_0, x_1, \cdots, x_N)$ satisfying $v(0) = v(L) = 0$. We adopt L^2-norm in $C^1[0, L]$. For any vector $v^N \in I\!R^{N-1}$, its norm is defined as $||v^N||_N^2 = \dfrac{2\pi}{N}\sum_{i=1}^{N-1} v_i^2$.

The space for estimation is defined to be

$$W = \left\{ \alpha_0/2 + \sum_{k=1}^{K_F} \left(\alpha_k \cos(\frac{2k\pi}{L}x) + \beta_k \sin(\frac{2k\pi}{L}x) \right) \right\}$$

where $\alpha_k, \beta_k \in I\!R$, $\alpha_0/2 + \sum_{k=1}^{K_F} \alpha_k = 0$ In this section, $K_F = 2$. This means that we want to find the observability for the first five modes in the Fourier expansion of $u(0)$. Or equivalently, we would like to find the observability of

$$[\, \alpha_0\ \alpha_1\ \beta_1\ \alpha_2\ \beta_2\,] \tag{29}$$

Define $X^N = [\, x_1\ x_2\ \cdots\ x_{N-1}\,]^T$ then

$$W^N = \left\{ \alpha_0/2 + \sum_{k=1}^{K_F} \left(\alpha_k \cos(\frac{2k\pi}{L}X^N) + \beta_k \sin(\frac{2k\pi}{L}X^N) \right) \right\}$$

where α_k and β_k satisfy (29).

In this example, the nominal trajectory has the following initial value

$$u_0(x) = -2 + \cos(x) + \sin(x) + \cos(2x) + \sin(2x)$$

To approximate its observability, we apply the empirical gramian method to (28) in the space W^N. The consistency of observability is verified by the results. The ratio ρ/ϵ^N is approximated for $N = 4k$, $5 \le k \le 21$. The value of unobservability index approaches (Figure 2) $\rho/\epsilon = 6.87$. □

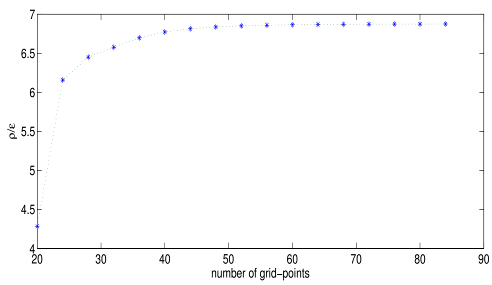

Fig. 2. The observability consistency of Burgers' equation

6. CONCLUSION

A definition of observability using dynamic optimization is introduced for PDEs. Using the concept one can achieve a quantitative measure of partial observability for PDEs. Furthermore, the observability can be numerically approximated. A practical feature of this definition for infinite dimensional systems is that the observability can be numerically computed using well-posed approximation schemes. It is mathematically proved that the approximated observability is consistent with the observability of the original PDE. A first order approximation is derived using empirical gramian matrices. The consistency is verified using an example of a Burgers' equation. Although the results are proved for linear PDEs in this paper, similar results can be generalized to nonlinear PDEs. They will be reported in a separate paper.

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