A remark about linear switched systems in the plane

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Abstract: In this note we prove that if a switched system $\mathcal{F}$ formed by a pair of linear vector fields of $\mathbf{R}^2$ is asymptotically controllable, then the discrete time operator associated to $\mathcal{F}$ admits at least one real eigenvalue λ, with $|\lambda| < 1$. For the particular case at hand, this is an improvement of previous existing results.

Keywords: Switched linear systems, planar systems, asymptotic controllability, stability

1. INTRODUCTION

A key result of linear control theory states that if a linear system is asymptotically controllable (i.e., any initial state can be asymptotically steered to the origin by an open-loop strategy), then it admits a stabilizing feedback (i.e., the same task can be accomplished by a closed-loop strategy). Note that while the definition of asymptotic controllability does not impose any regularity on the dependence of the open-loop controls on the initial state, the stabilizing feedback can be found linear, and hence it preserves the structure of the system (i.e., the closed-loop system has the same regularity as the plant to be controlled).

As well known, such a nice conclusion cannot be extended to nonlinear systems. Indeed, asymptotically controllable smooth (even polynomial) nonlinear systems do not admit, in general, continuous static state stabilizing feedbacks. All the existing results on this subject (see for instance Ancona and Bressan (1999); Clarke et al. (1997); Coron (1995); Goebel et al. (2009); Rifford (2000)) require the introduction of feedback laws which are discontinuous (and so based on some generalized notion of solution), time-varying or hybrid. In any case, the regularity properties of the resulting closed-loop system are lost or weakened.

In Bacciotti and Mazzi (2011), a different point of view is proposed. Instead of looking for closed-loop strategies with weakened properties, one may address the problem by the opposite site, searching open-loop strategies with strengthened properties. More precisely, it is proved in Bacciotti and Mazzi (2011) that, under an additional assumption called radial controllability (see later), any asymptotically controllable bilinear control system represented by a family of linear vector fields admits a stabilizing eventually periodic switching rule. This means that any initial state can be asymptotically steered to the origin by a control input with a special structure: a transient time interval, where the control depends on the initial state, followed by a steady state, where the control is periodic and independent of the initial state.

In the present note, we consider in particular the planar case. We show that the more difficult part of the construction of an eventually periodic stabilizing rule performed in Bacciotti and Mazzi (2011), that is the construction of a discrete time operator with a stable manifold, can be obtained without need of the radial controllability assumption.

The precise statement of the problem is presented in Section 2. The main result is stated in Section 3, while the proof is given in Section 4. The final section contains some conclusive remarks.

2. STATEMENT OF THE PROBLEM

The problem studied in this note is related to the dynamic behavior (especially, stability and asymptotic controllability) of linear switched systems in the plane; a subject which has been recently addressed in a number of papers (Bacciotti and Ceragioli (2006); Boscain (2002); Balde and Boscain (2008); Huang et al. (2010); Xu and Antsaklis (2000)).

Our precise problem can be described in this way. Let $\mathcal{F} = \{A_1, A_2\}$ be any pair of 2×2 real matrices, and consider the linear operator

$$\Phi(t_1, t_2) = e^{t_2 A_2} e^{t_1 A_1}$$

where t_1, t_2 are nonnegative numbers. From the main result [1] of Bacciotti and Mazzi (2011), it follows that:

(A) there exist $t_1, t_2 \geq 0$ such that $\Phi(t_1, t_2)$ has at least one real eigenvalue λ with $|\lambda| < 1$

provided that:

(H1) for each $x \in \mathbf{R}^2$ and each $\varepsilon > 0$, there exist an integer $N \geq 1$ and finite sequences of nonnegative numbers $\{t_{1,n}\}$, $\{t_{2,n}\}$ $(n = 1, \ldots, N)$ such that

[1] The main result of Bacciotti and Mazzi (2011) actually holds without any restriction on the dimension of the state space and on the number of the members of $\mathcal{F}$

$$|\Phi(t_{1,N}, t_{2,N}) \cdots \Phi(t_{1,1}, t_{2,1})x| < \varepsilon$$

(H2) for each $x, y \in \mathbf{R}^2$ ($x \neq 0, y \neq 0$), there exist an integer $M \geq 1$, finite sequences of nonnegative numbers $\{t_{1,m}\}$, $\{t_{2,m}\}$ ($m = 1, \ldots, M$) and $c > 0$ such that

$$\Phi(t_{1,M}, t_{2,M}) \cdots \Phi(t_{1,1}, t_{2,1})x = cy \ .$$

In this note, we show that (A) can be proved under assumption (H1), without need of (H2). To complete this section, some comments are in order.

1. Recall that a *switched trajectory* of $\mathcal{F}$ is any continuous curve which is piecewise coincident with an integral curve either of the linear vector field $f_1(x) = A_1x$ or of the linear vector field $f_2(x) = A_2x$. Our interest in the operator $\Phi(t_1, t_2)$ is motivated by the fact that it describes in discrete terms the dynamic behavior of the switched trajectories of $\mathcal{F}$. More precisely, any finite time switched trajectory of $\mathcal{F}$ can be represented as a composition of these operators, with possibly different choices of t_1 and t_2. In particular, given $x, y \in \mathbf{R}^2$, y is said to be *reachable* from x if x can be joined to y by a switched trajectory or, formally, if there exist an integer K and finite sequences of nonnegative numbers $\{t_{1,k}\}$, $\{t_{2,k}\}$ ($k = 1, \ldots, K$) such that

$$\begin{aligned}&\Phi(t_{1,K}, t_{2,K}) \cdots \Phi(t_{1,1}, t_{2,1})x \\ &\quad = e^{t_{2,K}A_2} e^{t_{1,K}A_1} \ldots e^{t_{2,1}A_2} e^{t_{1,1}A_1} x = y \ .\end{aligned}$$

The set of all the points y reachable from x is denoted by $R(x)$. Note that for each $x \neq 0$, $0 \notin R(x)$.

2. Condition (H1) can be reformulated by saying that for each $x \neq 0$ and each $\varepsilon > 0$, $R(x)$ contains at least one point y such that $|y| < \varepsilon$. In fact, Condition (H1) is clearly equivalent to *asymptotic controllability*, which is more usually defined by saying that every initial state can be steered asymptotically (for $t \to +\infty$) to the origin along a switched trajectory of $\mathcal{F}$.

3. Condition (H2) can be reformulated by saying that for each $x \neq 0$, $R(x)$ has a nonempty intersection with any ray issuing from the origin. This property is called *radial controllability* (Bacciotti (2012); Bacciotti and Mazzi (2011)).

4. As a consequence of (A), $\Phi(t_1, t_2)$ has a stable manifold of dimension at least one. This can be used to transform an open loop switching signal performing asymptotic controllability into a (hybrid) stabilizing feedback rule (see Bacciotti and Mazzi (2011); Bacciotti (2012)).

3. STATEMENT OF THE RESULT AND NOTATION

We begin by the formal statement of the result.

Theorem 1. Assume that a pair of 2×2 real matrices $\mathcal{F} = \{A_1, A_2\}$ satisfies (H1). Then, it satisfies (A).

The proof will be given in the next section. The following notation will be used.

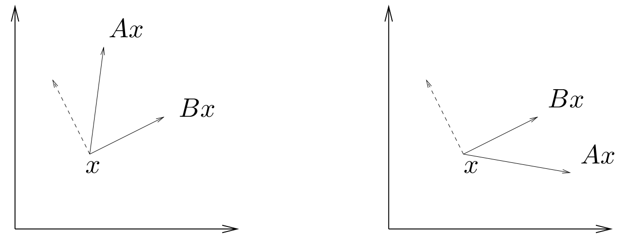

Fig. 1. The case $\omega_{B,A}(x) > 0$ on the left; the case $\omega_{B,A}(x) < 0$ on the right.

If E is a nonempty subset of $\mathbf{R}^2$ with $0 \notin E$, we denote by $\ell^+(E)$ the positive cone generated by E, that is the set $\{x \in \mathbf{R}^2 : \exists c > 0 \text{ such that } cx \in E\}$. If $\ell^+(E) = E$, we say that E is a *positive cone*. If E is a singleton, say $E = \{x\}$, the set $\ell^+(\{x\})$ is also called a *ray*. In this case, we use the simplified notation $\ell^+(x)$. Moreover, we denote by $\bar{E}$ the closure of E, and by $\overset{\circ}{E}$ the interior of E.

Given two linear vector fields $f(x) = Ax$ and $g(x) = Bx$, for $x \neq 0$ we denote $\omega_{B,A}(x) = ((Bx)^\perp)^{\mathbf{t}} Ax$, where $x^\perp = (x_1, x_2)^\perp = (-x_2, x_1)$ and $^{\mathbf{t}}$ denotes transposition. The sign of $\omega_{B,A}(x)$ indicates the relative position of the vectors Ax and Bx (see Xu and Antsaklis (2000)). As indicated in Figure 1, if $\omega_{B,A}(x) > 0$ then Ax points in the counterclockwise direction with respect to the vector Bx.

If $B = Id$, we simply write $\omega_A(x)$ instead of $\omega_{Id,A}(x)$. If $A = A_i$ ($i = 1, 2$) we can further simplify the notation by writing $\omega_i(x)$ instead of $\omega_{A_i}(x)$. Clearly, $\omega_A(x) = 0$ if and only if x is an eigenvector of A (provided that $x \neq 0$).

Let $S = \{x \in \mathbf{R}^2 : |x| = 1\}$, where $|x|$ is the Euclidian norm of x. We denote $r(x) = x/|x|$ ($x \neq 0$) the radial projection of $\mathbf{R}^2 \setminus \{0\}$ on S. If $x, y \in \mathbf{R}^2$, we say that x is *parallel* to y if $x = ay$ for some $a \in \mathbf{R}$.

4. THE PROOF

Some preliminary remarks allow us to restrict considerably the discussion.

Remark 1. If at least one between A_1, A_2 has complex conjugate eigenvalues (with nonzero imaginary part), then $\mathcal{F}$ satisfies (H2), as well. Then (A) follows from the main result of Bacciotti and Mazzi (2011). ∎

Remark 2. If at least one between A_1, A_2 has a real eigenvalue μ with $\mu < 0$, then (A) is trivially true (if for instance the matrix with the negative eigenvalue is A_1, we can take any $t_1 > 0$ and $t_2 = 0$). ∎

According with the previous remarks, without loss of generality we can assume from now on:

(H3) the eigenvalues of both A_1 and A_2 are all real and nonnegative.

Remark 3. Under Assumptions (H1), (H3), A_1 and A_2 cannot have a common real eigenvector, $v \neq 0$. Indeed, in the opposite case, we should have for some $\lambda_1, \lambda_2 \geq 0$ and each $t_1, t_2 \geq 0$,

$$e^{t_2 A_2} e^{t_1 A_1} v = e^{t_2 \lambda_1} e^{t_2 \lambda_1} v$$

meaning that $e^{t_2 A_2} e^{t_1 A_1} v$ *is parallel to* v *and that*

$$|e^{t_2 A_2} e^{t_1 A_1} v| \geq |v| \ .$$

This implies in turn that (H1) does not hold with $x = v$.

As a consequence, we also see that the dimension of the space generated by all the eigenvectors corresponding to each eigenvalue of both A_1 *and* A_2 *is exactly 1.*

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Remark 4. Clearly, if $r(R(\bar{x})) = S$ *for each* $\bar{x} \neq 0$*, then* $\mathcal{F}$ *satisfies (H2). Thus, also in this case (A) follows from Bacciotti and Mazzi (2011).*

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We can now start our main argument. By virtue of the previous remark, we can make the following additional restriction:

(H4) $\exists \bar{x} \neq 0$ such that $r(R(\bar{x}))$ is a proper subset of S.

Let us fix a point $\bar{x}$ with the property required in (H4). The set $r(R(\bar{x}))$ is connected; hence, its boundary with respect to S is formed by two points p, q, with possibly $p = q$. We have to distinguish two main cases.

Case 1. $p = q$. This possibility is ruled out by further distinguishing the two following subcases.

Case 1.1. $r(R(\bar{x})) = \{p\}$. In this case both $A_1 p$ and $A_2 p$ must be parallel to p. In other words, p must be an eigenvector of both A_1 and A_2, which is excluded by Remark 3.

Case 1.2. $r(R(\bar{x})) = S \setminus \{p\}$. If $A_1 p$ is not parallel to $\ell^+(p)$ (and hence in particular not zero) then there exists a neighborhood U of p in S such that $A_1 x$ is not parallel to $\ell^+(x)$ for each $x \in U$. Hence some point $y \in \ell^+(p)$ is reachable from some $z \in R(\bar{x})$, and we should conclude that $p \in r(R(\bar{x}))$. This contradicts (H4). The remaining possibility is that $A_1 p$ is parallel to p. But the same reasoning can be repeated for $A_2 p$, as well. In conclusion, p should be a common eigenvector of A_1 and A_2, and this is excluded by Remark 3, again.

Case 2. $p \neq q$. Let $C = \ell^+(\bar{R}(\bar{x}))$, and let C^c its complement in $\mathbf{R}^2$. We adopt the following agreement, borrowed from Bacciotti and Ceragioli (2006). The point p and q are denoted in such a way that q can be moved toward p by a counterclockwise rotation of an angle less that 2π without leaving $r(\bar{C})$: see Figure 2.

If $\omega_1(p) > 0$, we could reach from $\bar{x}$ some point y such that $y \in C^c$. Hence, $\omega_1(p) \leq 0$. Similarly, we realize that $\omega_2(p) \leq 0$, $\omega_1(q) \geq 0$, $\omega_2(q) \geq 0$. In addition, by Remark 3, we must have $\omega_i(p) < 0$ for at least one index $i = 1, 2$. Without loss of generality, we assume

(H5) $\omega_1(p) < 0$.

Similarly, we must also have $\omega_i(q) > 0$ for at least one index $i = 1, 2$. On the other hand, the case $\omega_i(q) > 0$ for both $i = 1, 2$ is possible only if $\bar{x} \in \ell^+(q)$. But then, we

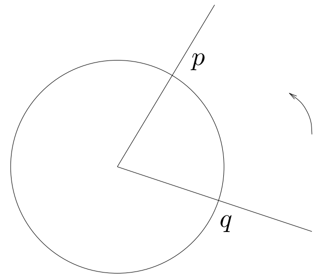

Fig. 2. Location of p and q and relative orientation.

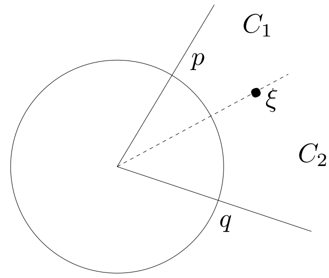

Fig. 3. Splitting C in two cones C_1, C_2.

should have $\omega_2(p) = 0$ ($\omega_2(p) < 0$ would imply that p is not a boundary point of $r(R(\bar{x}))$). In summary, at least one between $\omega_1(q)$, $\omega_2(q)$, $\omega_2(p)$ must be zero. This shows that, possibly exchanging the roles of p and q, we can limit ourselves to the following cases.

Case 2.1. q an eigenvector of A_1 and $\omega_2(q) > 0$.

Case 2.2. q is an eigenvectors of A_2, and $\omega_1(q) > 0$.

In fact, Case 2.2 can be reduced to Case 2.1. Indeed, since $\omega_1(p) < 0$ and $\omega_1(q) > 0$, there exists $\xi \in \mathring{C}$ such that $\omega_1(\xi) = 0$. The ray $\ell^+(\xi)$ divides C in two positive cones C_1 and C_2, and we have $\omega_2(\xi) \neq 0$ because of Remark 3.

Depending on the sign of $\omega_2(\xi)$ (and possibly exchanging the roles of A_1 and A_2), either C_1 or C_2 is in the same situation as C in Case 2.1.

We finally focus on the Case 2.1. Without loss of restrictions, we can also assume:

(H6) for every index $i = 1, 2$ A_i does not have an eigenvector $\xi \in \mathring{C}$.

Indeed, if there exists $\xi \in \mathring{C}$ which is an eigenvector of (say) A_1, then the ray $\ell^+(\xi)$ splits C in two positive cones C_1, C_2: clearly, at least one between C_1 and C_2 falls in at least one between the Cases 2.1, 2.2. Thus, we can restart our argument by replacing C by C_1 or C_2.

In conclusion, it remains to prove that in Case 2.1, assuming (H1),(H3), (H5) and (H6), assertion (A) holds.

Since neither A_1 nor A_2 have eigenvectors lying in $\mathring{C}$, we have $\omega_1(x) < 0$ and $\omega_2(x) > 0$ for each $x \in \mathring{C}$. Because of (H3), the origin is not attractive for the vector field $f_1(x)$; more precisely, since this vector field is linear, there exists $\varepsilon > 0$ such that $|e^{tA_1} p| \geq \varepsilon$ for each $t \geq 0$. Let us consider

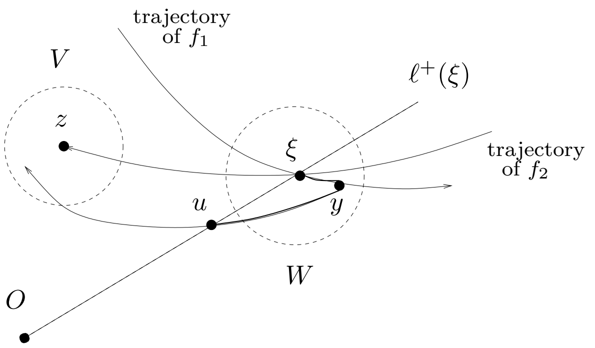

Fig. 4. Switched trajectory starting and ending on $\ell^+(\xi)$.

now the relative orientation of $f_2(x)$ with respect to $f_1(x)$: to do so, we examine the sign of $\omega_{1,2}(x) = \omega_{A_1,A_2}(x)$. Clearly, if $\omega_{1,2}(x) \geq 0$ for each $x \in \overset{\circ}{C}$, the assumption (H1) would be violated: indeed, switching from f_1 to f_2 would lead to increase the distance from the origin (the reasoning is similar to the so-called "worse case" argument, see Margaliot (2006)). Hence, there is a point $\xi \in \overset{\circ}{C}$ such that $\omega_{1,2}(\xi) < 0$. By continuity, the same is true for all x in some neighborhood $\mathcal{O}$ of ξ.

The ray $\ell^+(\xi)$ divides C in two parts, which are both positive cones: let us denote by C_1 the positive cone whose sides are $\ell^+(\xi)$ and $\ell^+(p)$, and by C_2 the positive cone whose sides are $\ell^+(q)$ and $\ell^+(\xi)$ (see Figure 3).

Let $z = e^{\tau A_2}\xi$, with $\tau > 0$ so small that $z \in \mathcal{O}$, and let V be a neighborhood of z, with $V \subset \mathcal{O} \cap C_1$.

Of course, $W = e^{-\tau A_2}V$ is a neighborhood of ξ. Let $t_1 > 0$ be so small that $y = e^{t_1 A_1}\xi \in W$.

By construction, $y \in C_2$ as well, while $e^{\tau A_2}y \in V \subset C_1$. Hence, there exists $t_2 > 0$ such that $u = e^{t_2 A_2}y = e^{t_2 A_2}e^{t_1 A_1}\xi \in \ell^+(\xi)$ (see Figure 4).

By construction, it is also clear that $|u| < |\xi|$. This implies that ξ is an eigenvector of $e^{t_2 A_2}e^{t_1 A_1}$ corresponding to some eigenvalue λ, with $|\lambda| < 1$, as required.

The proof of Theorem 1 is now complete.

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5. FINAL REMARKS

The existence of a stable manifold of the discrete time operator associated to $\mathcal{F}$ can be interpreted as a partial stabilization result. To this respect, Theorem 1 may have some interest, not only from the mathematical point of view.

At a first sight, the co-existence of Conditions (H1) and (H3) may appear counterintuitive. However, the fact that sometimes a careful switching between unstable systems can result in trajectories converging to the origin, is well known (see Liberzon (2003)). We emphasize that the examples usually reported to illustrate this phenomenon involves matrices with complex eigenvalues. On the contrary, the following example involves matrices with real eigenvalues.

Example 1. Let $\mathcal{F}$ be formed by the unstable matrices

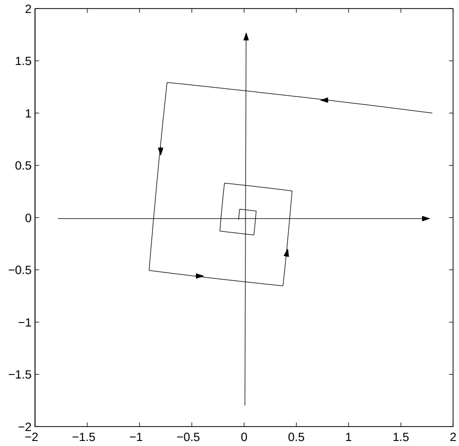

Fig. 5. A trajectory of the system of Example 1.

$$A_1 = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

($a > 0, b > 0$). Setting $t_1 = t_2 = t$, one easily computes

$$\Phi(t,t) = \begin{pmatrix} e^{2at} & \dfrac{(e^{bt} - e^{at})e^{at}}{a-b} \\ \dfrac{(e^{bt} - e^{at})e^{at}}{b-a} & e^{2bt} - \left(\dfrac{e^{bt} - e^{at}}{b-a}\right)^2 \end{pmatrix}. \quad (1)$$

Now consider the numerical values $a = 0.09$, $b = 0.11$, $t = 2.34085$. The eigenvalues of $\Phi(t,t)$ can be numerically computed, which shows that one of them is negative and less than one in absolute value. Figure 5 shows a trajectory, obtained by composition of operators of the form (1), issuing from $\bar{x} = (1.8, 1)$ and approaching the origin.

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We notice that if both $\mathcal{F}$ and $-\mathcal{F}$ satisfy (H1), in general it is not possible to find t_1 and t_2 in such a way that $\Phi(t_1, t_2)$ has one real eigenvalue λ with $|\lambda| < 1$ and one real eigenvalue μ with $|\mu| > 1$. This is proved by the following example suggested by J.C. Vivalda:

$$A_1 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad A_2 = -A_1 .$$

However, if all the eigenvalues of both A_1 and A_2 have nonnegative real part, then for every choice of $t_1 \geq 0$ and $t_2 \geq 0$, $\Phi(t_1, t_2)$ must have[2] at least one eigenvalue μ with $|\mu| \geq 1$. Thus in this case, if in addition $\mathcal{F}$ satisfies (H1), $\Phi(t_1, t_2)$ will have one real eigenvalue λ with $|\lambda| < 1$ (according to Theorem 1) and one real eigenvalue μ with $|\mu| > 1$ for a suitable choice of $t_1 \geq 0$ and $t_2 \geq 0$.

Finally, we point out that the arguments used in the proof of Theorem 1 are very similar to those of Pukhlikov (1998) (see also Margaliot and Branicky (2009)).

[2] The argument can be sketched as follows: if A_1 and A_2 have all the eigenvalues with nonnegative real part, then $e^{t_1 A_1}$ and $e^{t_2 A_2}$ will have all their eigenvalues with modulus greater than or equal to 1, for each $t_1 \geq 0, t_2 \geq 0$. Since the determinant of a matrix is the product of its eigenvalues, this yields $|\det e^{t_1 A_1}| \geq 1$ and $|\det e^{t_2 A_2}| \geq 1$. This in turn implies $\det |\Phi(t_1, t_2)| \geq 1$ and hence, at least one eigenvalue of $\Phi(t_1, t_2)$ must have modulus greater than or equal to 1.

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