

On differential passivity

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Abstract: Motivated by developments on differential Lyapunov functions for contraction analysis in Forni, Sepulchre (2012) we propose a definition of differential passivity, based on the geometric framework of the prolongation of a nonlinear system in Crouch, van der Schaft (1987). We explore the ramifications of this definition and its potential uses.

Keywords: Prolonged system, passivity, differential storage function, contraction

1. INTRODUCTION

Over the last years there has been ample interest in extending the notion of Lyapunov stability to various forms of *incremental* stability, in order to cope with, for example, convergence in regulation, synchronization and observer design for nonlinear systems; see in particular Angeli (2000), Aghannan, Rouchon (2003), Lohmiller, Slotine (1998), Pavlov, Pogromsky, van de Wouw, Nijmeijer (2004), Pavlov, van de Wouw, Nijmeijer (2005), Sontag (2010), Forni, Sepulchre (2012) (see also Jouffroy (2005) for references to older work in this area, as well Boyd, Chua (1984)). In such problems the stability with respect to a specific solution (e.g., an equilibrium) needs to be replaced by a stronger notion ensuring the convergence between any pairs of solutions, coined as *contraction analysis* in the influential paper Lohmiller, Slotine (1998).

One of the interesting directions in this area is to 'bypass' the problem of explicit construction of a feasible distance measure by considering some form of *infinitesimal* distance. This idea can be already traced back to work by Demidovich, which was utilized for control purposes in Pavlov, Pogromsky, van de Wouw, Nijmeijer (2004); Pavlov, van de Wouw, Nijmeijer (2005). The infinitesimal distance measure used by Demidovich can be interpreted as a *constant* Riemannian metric, while in Lohmiller, Slotine (1998) the extension was made to general Riemannian metrics on the state space manifold where contraction should take place. In the recent paper Forni, Sepulchre (2012) a further step was taken by extending the concept of Lyapunov functions defined on the state space to so-called Finsler-Lyapunov functions defined on the tangent bundle of the state space; thereby unifying previous forms of infinitesimal distance measures as explored in e.g. Pavlov, Pogromsky, van de Wouw, Nijmeijer (2004); Pavlov, van de Wouw, Nijmeijer (2005); Lohmiller, Slotine (1998); Sontag (2010). In the conclusions of Forni, Sepulchre (2012) the problem is stated of extending the thus developed *differential Lyapunov* framework for contraction analysis to *open and interconnected systems*, aiming at an extension of classical dissipativity theory Willems (1972); van der Schaft (2000) towards *differential* dissipativity theory. Motivated by Forni, Sepulchre (2012) we will explore in

this paper the possibilities for setting up such a differential dissipativity theory¹. For clarity of exposition we mainly restrict ourselves to *differential passivity*.

A first step is to recall, in Section 2, the coordinate-free definition of the *prolonged system* of a nonlinear control system as originally given in Crouch, van der Schaft (1987). Basically, this prolonged system is the original nonlinear system together with all its variational systems, with state, input and output space being the tangent bundle of the original state, input, respectively, output space. This provides an appropriate mathematical framework for exploring notions of differential passivity and dissipativity, simplifying computations which in local coordinates tend to become cumbersome.

Using this geometric notion of the prolonged system we will study in Section 3 the properties of passivity of the prolonged system which may be inherited from properties of passivity of the underlying nonlinear control system. Next, in Section 4 we define in a compact way the notion of *differential passivity*, and study some of its possibilities and limitations. In Section 5 we explore some of the potential uses of this concept, while Section 6 contains the conclusions and outlook.

2. PROLONGATION OF A NONLINEAR SYSTEM

In this section we will recall from Crouch, van der Schaft (1987), see also Cortes, van der Schaft, Crouch (2005), how any nonlinear control system on an n -dimensional state space manifold \mathcal{X} with m inputs and m outputs can be *prolonged* to a system on the $2n$ -dimensional tangent bundle $T\mathcal{X}$ with $2m$ inputs and $2m$ outputs. Let \mathcal{X} be an n -dimensional manifold with tangent bundle $T\mathcal{X}$. Denote by $\mathfrak{X}(\mathcal{X})$ the set of vector fields on \mathcal{X} , and by $C(\mathcal{X})$ the set of functions on \mathcal{X} . Throughout all objects will be assumed to be *smooth* (infinitely differentiable).

¹ *Note added in the final submission.* Related interesting developments can be found in the paper Forni, Sepulchre (2013a), which was submitted after the initial submission of the current paper. Some recent developments can be found in Forni, Sepulchre, van der Schaft (2013).

Consider a nonlinear control system Σ with state space \mathcal{X} , affine in the inputs u , and with an equal number of outputs y , given as

$$\Sigma : \begin{cases} \dot{x} = f(x) + \sum_{j=1}^m u_j g_j(x), \\ y_j = H_j(x), \quad j = 1, \dots, m, \end{cases} \quad (1)$$

where $x \in \mathcal{X}$, and $u = (u_1, \dots, u_m) \in \mathcal{U} \subset \mathbb{R}^m$. The vector fields f, g_1, \dots, g_m on \mathcal{X} are assumed to be complete and H_1, \dots, H_m are real-valued functions on \mathcal{X} . The set \mathcal{U} is the input space, which is assumed to be an open subset of \mathbb{R}^m , containing 0. The function $t \mapsto u(t) = (u_1(t), \dots, u_m(t))$, that we will commonly denote as $u(\cdot)$, belongs to a certain class of functions of time, denoted by \mathfrak{U} , called the set of *admissible controls*. For our purposes, we may restrict the admissible controls to be the piecewise constant right continuous functions. Finally, $\mathcal{Y} = \mathbb{R}^m$ is the output space.

Given an initial state $x(0) = x_0$, take any coordinate neighborhood of \mathcal{X} containing x_0 . Let $t \in [0, T] \mapsto x(t)$ be the solution of (1) corresponding to the input function $t \in [0, T] \mapsto u(t) = (u_1(t), \dots, u_m(t))$ and the initial state $x(0) = x_0$, such that $x(t)$ remains within the selected coordinate neighborhood. Denote the resulting output by $t \in [0, T] \mapsto y(t) = (y_1(t), \dots, y_m(t))$, with $y_j(t) = H_j(x(t))$. Then the *variational system* along the input-state-output trajectory $t \in [0, T] \mapsto (x(t), u(t), y(t))$ is given by the following time-varying system,

$$\begin{aligned} \delta \dot{x}(t) &= \frac{\partial f}{\partial x}(x(t)) \delta x(t) + \\ &\quad \sum_{j=1}^m u_j(t) \frac{\partial g_j}{\partial x}(x(t)) \delta x(t) + \sum_{j=1}^m \delta u_j g_j(x(t)) \quad (2) \\ \delta y_j(t) &= \frac{\partial H_j}{\partial x}(x(t)) \delta x(t), \quad j = 1, \dots, m, \end{aligned}$$

with state $\delta x \in \mathbb{R}^n$, where $\delta u = (\delta u_1, \dots, \delta u_m)$, $\delta y = (\delta y_1, \dots, \delta y_m)$ denote the inputs and the outputs of the variational system. The reason behind the terminology ‘variational’ comes from the following fact: let $(x(t, \epsilon), u(t, \epsilon), y(t, \epsilon))$, $t \in [0, T]$ be a family of input-state-output trajectories of (1) parameterized by $\epsilon \in (-\delta, \delta)$, with $x(t, 0) = x(t)$, $u(t, 0) = u(t)$ and $y(t, 0) = y(t)$, $t \in [0, T]$. Then, the infinitesimal variations

$$\delta x(t) = \frac{\partial x}{\partial \epsilon}(t, 0), \quad \delta u(t) = \frac{\partial u}{\partial \epsilon}(t, 0), \quad \delta y(t) = \frac{\partial y}{\partial \epsilon}(t, 0),$$

satisfy equation (2).

Remark 2.1. For for a *linear* system $\dot{x} = Ax + Bu$, $y = Cx$ the variational systems along any trajectory are simply given as $\delta \dot{x} = A \delta x + B \delta u$, $\delta y = C \delta x$.

The *prolongation or prolonged system* of (1) corresponds to considering the original system (1) *together* with its variational systems, that is the system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^m u_j g_j(x) \\ \delta \dot{x}(t) &= \frac{\partial f}{\partial x}(x(t)) \delta x(t) + \\ &\quad \sum_{j=1}^m u_j(t) \frac{\partial g_j}{\partial x}(x(t)) \delta x(t) + \sum_{j=1}^m \delta u_j(t) g_j(x(t)) \\ y_j &= H_j(x), \quad j = 1, \dots, m \\ \delta y_j(t) &= \frac{\partial H_j}{\partial x}(x(t)) \delta x(t), \quad j = 1, \dots, m \end{aligned} \quad (3)$$

with inputs u_j , δu_j , outputs y_j , δy_j , $j = 1, \dots, m$, and state x , δx .

In order to formulate a *coordinate-free definition* of the prolonged system (3), we need to introduce the notions of complete and vertical lifts of functions and vector fields; see Yano, Ishihara (1973).

Given a function H on \mathcal{X} , the *complete lift* of H to $T\mathcal{X}$, $H^c : T\mathcal{X} \rightarrow \mathbb{R}$ is defined by $H^c(x, \delta x) = \langle dH, \delta x \rangle(x)$, with $\langle \cdot, \cdot \rangle$ denoting the duality pairing between elements of the co-tangent space and tangent space at $x \in \mathcal{X}$. In local coordinates (x^1, \dots, x^n) for \mathcal{X} and the induced local coordinates $(x^1, \dots, x^n, \delta x^1, \dots, \delta x^n)$ for $T\mathcal{X}$ this reads

$$H^c(x, \delta x) = \sum_{a=1}^n \frac{\partial H}{\partial x^a}(x) \delta x^a.$$

The *vertical lift* of H to $T\mathcal{X}$, $H^v : T\mathcal{X} \rightarrow \mathbb{R}$, is defined by $H^v(x, \delta x) = H \circ \tau_{\mathcal{X}}$, where $\tau_{\mathcal{X}} : T\mathcal{X} \rightarrow \mathcal{X}$ denotes the tangent bundle projection $\tau_{\mathcal{X}}(x, \delta x) = x$. In local coordinates $H^v(x, \delta x) = H(x)$.

Given a vector field f on \mathcal{X} , the *complete lift* of f to $T\mathcal{X}$, $f^c \in \mathfrak{X}(T\mathcal{X})$ is defined as the unique vector field satisfying $L_{f^c} H^c = (L_f H)^c$, for any $H \in C(\mathcal{X})$ (with $L_f H$ denoting the Lie-derivative of the function H along the vector field f , and similarly for $L_{f^c} H^c$). Alternatively, if $\Phi_t : \mathcal{X} \rightarrow \mathcal{X}$, $t \in [0, \epsilon)$, denotes the flow of f , then f^c is the vector field whose flow is given by $(\Phi_t)_* : T\mathcal{X} \rightarrow T\mathcal{X}$. In induced local coordinates $(x^1, \dots, x^n, \delta x^1, \dots, \delta x^n)$ for $T\mathcal{X}$,

$$f^c(x, \delta x) = \sum_{a=1}^n f_a(x) \frac{\partial}{\partial x^a} + \sum_{a,b=1}^n \frac{\partial f_a}{\partial x^b}(x) \delta x^b \frac{\partial}{\partial (\delta x^a)} \quad (4)$$

Finally, the *vertical lift* of f to $T\mathcal{X}$, $f^v \in \mathfrak{X}(T\mathcal{X})$ is the unique vector field such that $L_{f^v} H^c(x, \delta x) = (L_f H)^v(x, \delta x)$, for any $H \in C(\mathcal{X})$. In local induced coordinates for $T\mathcal{X}$

$$f^v(x, \delta x) = \sum_{a=1}^n f_a(x) \frac{\partial}{\partial (\delta x^a)} \quad (5)$$

This enables us to define the prolonged system (3) on the whole tangent space $T\mathcal{X}$ in the following coordinate-free way. Denote the elements of $T\mathcal{X}$ by $z = (x, \delta x)$, where $\tau_{\mathcal{X}}(z) = x \in \mathcal{X}$ with $\tau_{\mathcal{X}} : T\mathcal{X} \rightarrow \mathcal{X}$ again the tangent bundle projection.

Definition 2.2. The prolonged system Σ^p of a nonlinear system Σ of the form (1) is defined as the system

$$\Sigma^p : \begin{cases} \dot{z} = f^c(z) + \sum_{j=1}^m u_j(t)g_j^c(z) + \sum_{j=1}^m \delta u_j(t)g_j^v(z), \\ y_j = H_j^v(z), \quad j = 1, \dots, m \\ \delta y_j = H_j^c(z), \quad j = 1, \dots, m \end{cases} \quad (6)$$

with state $z \in T\mathcal{X}$, inputs $u_j, \delta u_j$ and outputs $y_j, \delta y_j$, $j = 1, \dots, m$.

Note that Σ^p has state space $T\mathcal{X}$, input space $T\mathcal{U}$ and output space $T\mathcal{Y}$. One can easily check that in any system of local coordinates x for \mathcal{X} and the induced coordinates $x, \delta x$ for $T\mathcal{X}$, the local expression of the system (6) equals (3).

Remark 2.3. For a linear system $\dot{x} = Ax + Bu, y = Cx$ the prolonged system is simply the product of the system with a copy system $\delta x = A\delta x + B\delta u, \delta y = C\delta x$.

3. INHERITED PASSIVITY PROPERTIES OF THE PROLONGED SYSTEM

In this section we investigate to what extent properties of the nonlinear control system Σ such as passivity are being inherited by the prolonged system Σ^p . The relation between local (strong) accessibility and local observability of Σ and Σ^p was already discussed in Crouch, van der Schaft (1987), see also Cortes, van der Schaft, Crouch (2005).

We make extensively use of the following identities for lifts of vector fields and functions, which are proved in Yano, Ishihara (1973) and Crouch, van der Schaft (1987).

Proposition 3.1. For any vector fields $f \in \mathfrak{X}(\mathcal{X})$ and functions $H \in C(\mathcal{X})$ the following identities hold:

$$L_{f^c}H^c = (L_f H)^c \quad (7)$$

$$L_{f^c}H^v = (L_f H)^v = L_{f^v}H^c \quad (8)$$

$$L_{f^v}H^v = 0 \quad (9)$$

Recall Willems (1972); van der Schaft (2000) that a nonlinear control system Σ given by (1) is called *passive* (*lossless*) if there exists a differentiable $S : \mathcal{X} \rightarrow \mathbb{R}^+$ such that

$$\frac{d}{dt}S \leq (=) u^T y \quad (10)$$

where $\frac{d}{dt}S$ denotes the derivative along the system Σ , or equivalently (since S is assumed to be differentiable)

$$\begin{aligned} L_{g_j}S &= H_j, \quad j = 1, \dots, m \\ L_f S &\leq (=) 0 \end{aligned} \quad (11)$$

Furthermore, if this holds for a function $S : \mathcal{X} \rightarrow \mathbb{R}$ (i.e., without the nonnegativity condition), then Σ is called *cyclo-passive*, respectively *cyclo-lossless*. The function S is called the *storage function* and $u^T y$ the (passivity) *supply rate*.

Using (7) we immediately obtain

Proposition 3.2. Let Σ be cyclo-passive or cyclo-lossless with storage function S . Then the prolonged system Σ^p satisfies

$$\frac{d}{dt}S^v = u^T y + (L_f S)^v \quad (12)$$

$$\frac{d}{dt}S^c = u^T \delta y + \delta u^T y + (L_f S)^c \quad (13)$$

Hence if Σ is passive (lossless) then also Σ^p is passive (lossless) for the supply rate $u^T y$ with storage function S^v . The same holds for cyclo-passivity (-losslessness). Furthermore, if Σ is cyclo-lossless then Σ^p is cyclo-lossless with respect to the supply rate $u^T \delta y + \delta u^T y$ with storage function S^c .

Equation (12) expresses a rather obvious result: if Σ is passive then it will *remain* passive with respect to the supply rate $u^T y$ if the variational systems are added. On the other hand, equation (13) can be interpreted as a kind of 'differentiated' version of $\frac{d}{dt}S = u^T y + L_f S$ ('differentiating' $u^T y$ by the product rule to $u^T \delta y + \delta u^T y$).

Note that by definition of the complete lift $S \geq 0$ does *not* imply $S^c \geq 0$. In fact, the function S^c will be always *indefinite* whenever it is non-zero (because of linear dependence on δx). Similarly, $L_f S \leq 0$ does not imply $(L_f S)^c \leq 0$. This prevents us from formulating results regarding inherited passivity or losslessness with respect to the supply rate $u^T \delta y + \delta u^T y$.

4. DIFFERENTIAL PASSIVITY

Using the definition of the prolonged system (Definition 2.2) it is straightforward to define a notion of *differential passivity*, respectively, *differential losslessness*.

Definition 4.1. Consider a nonlinear control system Σ given by (1) together with its prolonged system Σ^p given by (6). Then Σ is called *differentially passive* if the prolonged system Σ^p is dissipative with respect to the supply rate $\delta u^T \delta y$, that is, if there exists a function $P : T\mathcal{X} \rightarrow \mathbb{R}^+$ (called the *differential storage function*) satisfying

$$\frac{d}{dt}P \leq \delta u^T \delta y \quad (14)$$

for all $x, u, \delta u$. Furthermore Σ is called *differentially lossless* if (14) holds with equality.

Since the time derivative $\frac{d}{dt}P$ along the prolonged system Σ^p is given as

$$\frac{d}{dt}P(z) = L_{f^c}P(z) + \sum_{j=1}^m u_j(t)L_{g_j^c}P(z) + \sum_{j=1}^m \delta u_j(t)L_{g_j^v}P(z) \quad (15)$$

the following proposition immediately follows.

Proposition 4.2. Consider a nonlinear control system Σ together with its prolonged system Σ^p . Then Σ is differentially passive with storage function $P : T\mathcal{X} \rightarrow \mathbb{R}$ if and only if

$$L_{f^c}P \leq 0, \quad L_{g_j^c}P = 0, \quad L_{g_j^v}P = H_j^c, \quad j = 1, \dots, m \quad (16)$$

The first two conditions $L_{f^c}P \leq 0$ and $L_{g_j^c}P = 0, j = 1, \dots, m$, on the differential storage function P are very close to the conditions for a *differential Lyapunov function* as discussed in Forni, Sepulchre (2012), if we additionally assume that $P : T\mathcal{X} \rightarrow \mathbb{R}$ is a candidate Finsler

Lyapunov function². Indeed, for a fixed input function $\bar{u} : [0, \infty) \rightarrow \mathcal{U}$ the basic requirement for a differential Lyapunov function in the sense of Forni, Sepulchre (2012) amounts in our notation to

$$L_{f^c}P + \sum_{j=1}^m \bar{u}_j(t)L_{g_j^c}P \leq -\alpha(P), \quad (17)$$

for some function $\alpha : [0, \infty) \rightarrow [0, \infty)$. Requiring this uniformly for every \bar{u} amounts to the conditions $L_{f^c}P \leq -\alpha(P)$ and $L_{g_j^c}P = 0, j = 1, \dots, m$. Thus a differential storage function qualifies as a differential Lyapunov function. Hence Definition 4.1 extends the existence of a differential Lyapunov function (in the sense of Forni, Sepulchre (2012)) in a similar way as ordinary passivity extends the existence of an ordinary Lyapunov function.

Remark 4.3. As discussed in Forni, Sepulchre (2012), the existence of a differential Lyapunov function P of the form $P(x, \delta x) = \frac{1}{2}\delta x^T M(x)\delta x$ for some positive definite matrix $M(x)$ amounts to the basic conditions for contraction analysis as derived in Lohmiller, Slotine (1998), while for a *constant* matrix M it amounts to the Demidovich conditions; Pavlov, Pogromsky, van de Wouw, Nijmeijer (2004); Pavlov, van de Wouw, Nijmeijer (2005).

Clearly, the conditions for differential passivity as derived above are demanding. In particular the condition $L_{g_j^c}P = 0, j = 1, \dots, m$, puts heavy restrictions on the existence of P . In the following we will investigate in more detail the consequences of these conditions in the case of differential *losslessness*. In order to do so we first recall the following identities proved in Yano, Ishihara (1973); Crouch, van der Schaft (1987).

Proposition 4.4. For any two vector fields $f_1, f_2 \in \mathfrak{X}(\mathcal{X})$

$$\begin{aligned} (i) \quad [f_1^c, f_2^c] &= [f_1, f_2]^c \\ (ii) \quad [f_1^c, f_2^v] &= [f_1, f_2]^v \\ (iii) \quad [f_1^v, f_2^v] &= 0 \end{aligned} \quad (18)$$

Let us denote by \mathcal{L} the *accessibility algebra* of the nonlinear system Σ , and by \mathcal{O} its *observation space*; see e.g. Nijmeijer, van der Schaft (1990).

Proposition 4.5. Suppose Σ is differentially lossless with differential storage function $P : T\mathcal{X} \rightarrow \mathbb{R}$. Then

$$L_{k^c}P = 0, \quad \text{for all } k \in \mathcal{L} \quad (19)$$

$$L_{k^v}P \in \mathcal{O}^c, \quad \text{for all } k \in \mathcal{L} \quad (20)$$

while for all $H \in \mathcal{O}$ there exists $k \in \mathcal{L}$ such that $L_{k^v}P = H^c$.

Proof. In order to show (19) we note that by (16)

$$L_{[f^c, g_j^c]}P = L_{f^c}L_{g_j^c}P - L_{g_j^c}L_{f^c}P = 0$$

and thus, making use of (18), $L_{[f, g_j]^c}P = 0, j = 1, \dots, m$. The same holds for all repeated Lie brackets of the vector fields f, g_1, \dots, g_m , and thus (19) results.

In order to show (20) we note that in view of (16)

$$L_{f^c}L_{g_j^v}P = L_{f^c}H_j^c = (L_f H_j)^c, \quad j = 1, \dots, m$$

and hence, since $L_{f^c}P = 0$,

$$L_{[f^c, g_j^v]}P = (L_f H_j)^c, \quad j = 1, \dots, m$$

² This entails, apart from positivity conditions on P , homogeneity conditions in the variational state δx ; see Forni, Sepulchre (2012) for details.

which by (18) is the same as

$$L_{[f, g_j]^v}P = (L_f H_j)^c, \quad j = 1, \dots, m$$

The same reasoning holds for all $k \in \mathcal{L}$. The proof of the final claim uses the same argument. ■

Especially the condition $L_{k^c}P = 0$ for all $k \in \mathcal{L}$ puts severe conditions on the existence of storage functions $P : T\mathcal{X} \rightarrow \mathbb{R}$. Take a basis of vector fields $\{k_1, \dots, k_r\}$ for the accessibility algebra \mathcal{L} (with $r = \dim \mathcal{X}$ if the system is locally accessible, Nijmeijer, van der Schaft (1990)). Then this condition implies that P needs to satisfy the equations
$$\frac{\partial P}{\partial x}(x, \delta x)k_i(x) + \frac{\partial P}{\partial \delta x}(x, \delta x)\frac{\partial k_i}{\partial x}(x)\delta x = 0, \quad i = 1, \dots, r \quad (21)$$

Trying as before a candidate solution $P(x, \delta x) = \frac{1}{2}\delta x^T M(x)\delta x$ for some positive definite matrix $M(x)$ (corresponding to a Riemannian metric) leads to the conditions

$$\frac{\partial}{\partial x} \frac{1}{2}\delta x^T M(x)\delta x k_i(x) + \delta x^T M(x)\frac{\partial k_i}{\partial x}(x)\delta x = 0 \quad (22)$$

for $i = 1, \dots, r$.

If M is a constant matrix this means that all the matrices $M\frac{\partial k_i}{\partial x}(x), i = 1, \dots, r$, should be skew-symmetric, which places a severe condition on M . On the other hand, note that for a linear system $\dot{x} = Ax + Bu$ this reduces to the well-known single condition $A^T M + MA = 0$.

5. USES OF DIFFERENTIAL PASSIVITY

In this section we will discuss some initial ideas about the use of the concept of differential passivity as defined in the previous section.

Of course, a main aim of passivity (and dissipativity) theory is the ability to derive stability results for interconnected systems. Thus passivity theory is a prime example of a compositional analysis technique: properties of the, possibly complex, interconnected system can be inferred from properties of the component systems. In a similar vein, passivity theory can be utilized for the construction of controller systems such that the controlled system remains stable when interacting with unknown, but passive, environments.

The basic passivity theorem, - negative feedback interconnection of two passive systems resulting in a passive closed-loop system -, can be seen to extend to differential passivity as well. Consider two differentially passive nonlinear systems Σ_i with states x_i , inputs $u_i \in \mathbb{R}^m$ and outputs $y_i \in \mathbb{R}^m$, and with differential storage functions $P_i, i = 1, 2$. The standard feedback interconnection is given by

$$u_1 = -y_2 + e_1, \quad u_2 = y_1 + e_2, \quad (23)$$

where $e_1, e_2 \in \mathbb{R}^m$ denote external inputs. The interconnection equations (23) imply that the variational quantities $\delta u_1, \delta u_2, \delta y_1, \delta y_2, \delta e_1, \delta e_2$ satisfy

$$\delta u_1 = -\delta y_2 + \delta e_1, \quad \delta u_2 = \delta y_1 + \delta e_2, \quad (24)$$

implying the basic equality

$$\delta u_1^T \delta y_1 + \delta u_2^T \delta y_2 = \delta e_1^T \delta y_1 + \delta e_2^T \delta y_2 \quad (25)$$

It directly follows that the closed-loop system arising from the feedback interconnection of Σ_1 and Σ_2 is a differentially passive system with regard to the supply rate

$$\delta e_1^T \delta y_1 + \delta e_2^T \delta y_2$$

and storage function $P(z_1, z_2) := P_1(z_1) + P_2(z_2)$. In this way complex differentially passive systems can be built up from simpler ones. Note that this can be extended to more general interconnections as long as the equality (25) remains satisfied.

Furthermore, given a differentially passive system Σ with differential storage function P , one can apply negative static output feedback $\delta u_j = -\delta y_j, j = 1, \dots, m$, in order to obtain a closed-loop prolonged system satisfying

$$L_{fc}P - \sum_{j=1}^m H_j^c g_j^c P \leq - \sum_{j=1}^m (H_j^c)^2, \quad L_{g_j^c}P = 0, \quad j = 1, \dots, m \quad (26)$$

Clearly, this opens up several possibilities for strengthening the properties of differential passivity, in particular the properties of the differential Lyapunov function P . This will be left for further research.

6. CONCLUSIONS AND OUTLOOK

Primarily motivated by the recent paper Forni, Sepulchre (2012) we have recalled a geometric, coordinate-free, framework for dealing with systems that are prolonged to the tangent bundle. We investigated how passivity properties are inherited by the prolonged system. Next we proposed a definition of *differential passivity*, which extends the definition of differential Lyapunov functions as recently proposed in Forni, Sepulchre (2012) in pretty much the same way as ordinary passivity theory extends ordinary Lyapunov function theory. Furthermore, the consequences of differential losslessness are explored in greater detail. In the final section we have discussed some initial ideas regarding the use of the concept of differential passivity.

Note that in a similar way differential versions of other notions of dissipativity, such as L_2 -gain $\leq \gamma$, can be defined. Indeed, consider a nonlinear control system Σ given by (1), where the number p of outputs may be different from the number m of inputs, i.e., the output equations are given by $y_j = H_j(x), j = 1, \dots, p$.

Σ will be said to have *differential L_2 -gain $\leq \gamma$* if the prolonged system Σ^p is dissipative with respect to the supply rate $\frac{1}{2}\gamma^2\|\delta u\|^2 - \frac{1}{2}\|\delta y\|^2$, that is, if there exists a function $P : T\mathcal{X} \rightarrow \mathbb{R}^+$ satisfying

$$\frac{d}{dt}P \leq \frac{1}{2}\gamma^2\|\delta u\|^2 - \frac{1}{2}\|\delta y\|^2 \quad (27)$$

Since the time derivative $\frac{d}{dt}P$ along the prolonged system Σ^p is given by (15), this is the same as the satisfaction of

$$L_{fc}P(z) + \sum_{j=1}^m u_j L_{g_j^c}P(z) + \sum_{j=1}^m \delta u_j L_{g_j^c}P(z) \leq \frac{1}{2}\gamma^2\|\delta u\|^2 - \frac{1}{2}\|\delta y\|^2 \quad (28)$$

for all $z = (x, \delta x), u, \delta u$. This in turn holds if and only if $L_{g_j^c}P = 0, j = 1, \dots, m$, together with

$$L_{fc}P(z) + \sum_{j=1}^m \delta u_j L_{g_j^c}P(z) \leq \frac{1}{2}\gamma^2\|\delta u\|^2 - \frac{1}{2}\|\delta y\|^2 \quad (29)$$

for all $z = (x, \delta x), \delta u$. By computing the minimizing δu the last equation can be seen to be satisfied if and only if

$$L_{fc}P(z) + \frac{1}{2}\frac{1}{\gamma^2}\sum_{j=1}^m (L_{g_j^c}P(z))^2 + \frac{1}{2}\sum_{j=1}^p (H_j^c)^2(z) \leq 0$$

Summarizing, Σ has differential L_2 -gain $\leq \gamma$ if and only if there exists a function $P : T\mathcal{X} \rightarrow \mathbb{R}^+$ satisfying

$$L_{g_j^c}P = 0, \quad j = 1, \dots, m, \\ L_{fc}P(z) + \frac{1}{2}\frac{1}{\gamma^2}\sum_{j=1}^m (L_{g_j^c}P(z))^2 + \frac{1}{2}\sum_{j=1}^p (H_j^c(z))^2 \leq 0 \quad (30)$$

In future work we will expand on the uses of this.

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