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Multi-cluster Games in Open Networks ⋆

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Abstract: This paper considers the distributed Nash equilibrium (NE) seeking problem for multi-cluster games in open networks, where players can freely join or leave the game with certain probabilities. In multi-cluster games, within a same cluster, players cooperate to optimize the cost function of the cluster, and different clusters compete, which can be seen as a noncooperative game. To track the time-varying NE of this formulated game timely, based on the inter- and intra- communication networks, a distributed projected gradient tracking algorithm under open networks is designed. Through rigorous theoretical analysis, we prove that the strategy profile sequence of players generated by the proposed algorithm linearly converges to a neighborhood of the time-varying NE in expectation. Finally, a numerical simulation of Cournot competition game is presented to illustrate the theoretical result.

Keywords: Multi-cluster games, freely joining and leaving players, open networks, expected convergence.

1. INTRODUCTION

Distributed optimization and noncooperative game theory have provided guiding schemes for cooperation (Doostmohammadian et al., 2024) and competition (Li et al., 2023) in multi-agent systems, respectively. Recently, the interplay between cooperation and competition have become crucial in various practical field such as energy systems (Chen and Yi, 2024), smart grids (Deng, 2025), and so on, which propel the development of multi-cluster games (Wang et al., 2025). In multi-cluster games, each cluster acts as a virtual player in a noncooperative game, strategically minimizing its cost in response to other clusters. Simultaneously, in the intra-cluster, players collaborate to minimize the aggregate cost, which is defined as the sum of their individual cost functions.

Numerous studies on multi-cluster games have been conducted from both continuous (Huang et al., 2025) and discrete (Wang et al., 2025) perspectives. However, most studies focused on closed multi-agent systems where the composition of players remains unchanged during the entire process. This assumption is inadequate for environments with high-frequency population fluctuations, giving rise to researches on "open networks" and "open multi-agent systems" where players join or leave the systems freely. The continuous arrival and departure of players introduce significant challenges in multi-cluster games. Firstly, population dynamics cause shifts in cost functions, driving uncontrollable variations in Nash equilibrium (NE) seeking, which differs from online multi-cluster games (Liu et al., 2025). Secondly, this frequent player turnover leads to changes in network structures. Although multi-cluster games in time-varying networks were studied (Wang et al., 2025), the network changes introduced by openness are entirely different.

Nowadays, increasing attention is being directed toward open networks. Initially, this problem was considered in computer science (Pinyol and Sabater-Mir, 2013) to address resilience against attacks. Recently, it was extended to control communities, particularly in consensus (Hendrickx and Martin, 2017; Abdelrahim et al., 2017; Dashti et al., 2019) and distributed optimization problems (de Galland et al., 2024; Liu et al., 2024). However, there are generally some limitations in the existing works, such as a finite and known number of players, inheritable information, and reduced variability of optimal solutions, which do not align with certain real-world scenarios. To date, only a few works have addressed games in open networks, which are more complex than optimization problems. One primary reason is that players are self-interested to minimize their own cost functions in response to the strategies of others. Noncooperative games in open net-

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works were studied from continuous-time and discrete-time perspectives respectively in Xu et al. (2025); Liu et al. (2026). However, these works assumed intermittent entry and exit, meaning an unchanged phase.

Motivated by these considerations, this paper investigates multi-cluster games under dynamic player participation. The main contributions are summarized as follows:

(1) Motivated by multi-cluster games with fixed players (Huang et al., 2025; Wang et al., 2025; Liu et al., 2025), a novel multi-cluster game problem is formulated, wherein the number of players in each cluster dynamically changes, leading to variations in NE. Unlike prior studies in open networks (Liu et al., 2024; Xu et al., 2025; Liu et al., 2026), our framework accommodates the probabilistic arrival and departure of players at each stage without fixed time intervals and global population knowledge.

(2) Based on neighbors' information, a distributed projected gradient tracking algorithm in open networks is designed. To address the lack of information inheritability caused by an unknown player population, a novel initialization scheme for new arriving players is proposed in the field of game theory, which is more complex than that in optimization problems (Deplano et al., 2025).

(3) The algorithm's convergence is rigorously analyzed by simultaneously considering the convergence error and the consistency error in expectation. Additionally, the impact of player changes on NE has also been taken into account. It is shown that players' strategy profiles linearly converge to a neighborhood of time-varying NE in expectation.

This paper is organized as follows. Section 2 introduces the multi-cluster game and open characteristics. Section 3 presents the designed algorithm and main result. In Section 4, a numerical simulation is given to illustrate theoretical result. Section 5 concludes this paper.

Notations: Let $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$ denote the sets of real numbers, n-dimensional real vectors, and $m\times n$ real matrices, respectively. For a positive integer n, define $[n] := \{1,2,\dots,n\}$. For any set S, let $|S|$ denote the Cardinality of S. The identity matrix of size n is denoted by I_n, and $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) represents the n-dimensional all-ones (resp. all-zeros) vector. For a matrix (or vector) A, its transpose is written as $A^\top$. The i-th column and i-th row of A are denoted by $Col_i(A)$ and $Row_i(A)$, respectively. The Kronecker product of matrices A and B is $A\otimes B$. A diagonal matrix with entries $a_1,\dots,a_n$ is written as $\mathrm{diag}(a_1,\dots,a_n)$. For a vector v, $\mathrm{diag}(v)$ is the diagonal matrix formed from v. The symbol $col(z_1,\dots,z_n)$ stacks $z_1,\dots,z_n$ into a column vector or a block matrix. Let $\langle w,v\rangle := \mathrm{trace}(w^\top v)$ for two matrices or vectors w,v. The Frobenius norm $\|\cdot\|_F$ and the Euclidean norm $\|\cdot\|$ are induced by their respective inner products: for $v\in\mathbb{R}^{n\times q}$, $\|v\|_F := \sqrt{\mathrm{trace}(v^\top v)}$, and for $w\in\mathbb{R}^n$, $\|w\| := \sqrt{w^\top w}$. The projection operator onto a set Ω is denoted by $P_\Omega[\cdot]$.

Consider an undirected graph $\mathscr{G} = (\mathscr{V},\mathscr{E},W)$, where $\mathscr{V} = \{1,2,\dots,n\}$ is the node set, $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ is the edge set, and $W=(w_{ij})\in\mathbb{R}^{n\times n}$ denotes the weighted adjacency matrix. The edge weight w_{ij} satisfies $w_{ij}>0$ if $(j,i)\in\mathscr{E}$, and $w_{ij}=0$ otherwise. Each self-loop is assumed to exist, i.e., $w_{ii}>0$ for all $i\in[n]$. Let $\mathcal{N}_i$ be the set of neighbors of player i, i.e., $\mathcal{N}_i=\{j\in\mathscr{V}\mid(i,j)\in\mathscr{E}\}$ and d_i denote the degree of player i, i.e., $d_i=|\mathcal{N}_i|$. A path from node i_1 to i_l consists of a sequence of edges (i_h,i_{h+1}) for $h=1,2,\dots,l-1$. The graph $\mathscr{G}$ is said to be connected if there exists a path between any two distinct vertices.

2. PROBLEM FORMULATION

Consider a multi-cluster game with m clusters. Cluster $i\in[m]$ comprises a player set $\mathscr{V}_i := \{1,\dots,n_i\}$ and the total number of players is $n := \sum_{i=1}^m n_i$. Each player $j\in\mathscr{V}_i$ selects a strategy x_{ij} from a non-empty, closed, and convex set $\Omega_i\subset\mathbb{R}^{q_i}$. Additionally, for inter-cluster communication, player 1 in each cluster is, without loss of generality, designated as the representative (Meng and Li, 2023). Let $x_i := col(x_{i1},\dots,x_{in_i})$ represent the strategy variable of cluster i, and $x_{-i} := col(x_1,\dots,x_{i-1},x_{i+1},\dots,x_m)$ be the collective action of all other clusters. The strategy variable of all players is defined as $x := col(x_1,\dots,x_m)\in\mathbb{R}^b$, where $b=\sum_{i=1}^m n_iq_i$. In the meanwhile, every player j in cluster i has it own cost function $f_{ij}(x_{ij},\Gamma_i(x_{-i})):\mathbb{R}^q\to\mathbb{R}$ which is only available to player j, where $q := \sum_{i=1}^m q_i$ and Γ_i denotes the stacked strategies of the representative players of all the clusters except that of cluster i, i.e., $\Gamma_i(x_{-i}) := col(x_{11},\dots,x_{i-1,1},x_{i+1,1},\dots,x_{m1})$. Thus, given x^*_{-i}, cluster $i\in[m]$ seeks a strategy $x_i^* := col(x_{i1}^*,\dots,x_{in_i}^*)$ with $x_{ij}^*=x_{il}^*$ for $j,l\in[n_i]$ minimizing its own cost $f_i(x_i,x^*_{-i})$, which can be mathematically expressed as

$$\begin{aligned}\min_{x_i} f_i(x_i,x^*_{-i}) &= \frac{1}{n_i}\sum\nolimits_{j=1}^{n_i} f_{ij}(x_{ij},\Gamma_i(x^*_{-i}))\\ \text{s.t.}\quad & x_{ij}=x_{il}\in\Omega_i,\forall j,l\in[n_i].\end{aligned}\tag{1}$$

This optimal strategy is indeed an NE of the noncooperative game among clusters, and $x^*=(x_i^*,x^*_{-i})$ satisfies $x_{ij}^*=x_{il}^*=\tilde{x}_i^*\in\Omega_i$ for all $j,l\in[n_i]$, and for all $i\in[m]$,

$$f_i(x_i^*,x^*_{-i})\le f_i(x_i,x^*_{-i}), x_i=\mathbf{1}_{n_i}\otimes y_i,\forall y_i\in\Omega_i,\tag{2}$$

which is equivalent to the fact that $\forall i\in[m],\forall\xi_i\in\Omega_i$,

$$\left\langle\frac{1}{n_i}\sum\nolimits_{j=1}^{n_i}\nabla_i f_{ij}(\tilde{x}_i^*,\Gamma_i(x^*_{-i})),\xi_i-\tilde{x}_i^*\right\rangle\ge 0.\tag{3}$$

However, in practice, the number of players in the multi-cluster game is not fixed, since some new players may join and existing players may leave at certain times. Here, the number of clusters and their representatives remain fixed. Define a baseline game at $k=0$ with n^0 permanent players (potentially just the representatives) who never leave. At time k, exactly one mutually exclusive event occurs, modeled by $Z^k\in\{\eta_u,\eta_a,\eta_d\}$ denoting a strategy update, arrival, or departure, respectively. These occur with probabilities p_u,p_a,p_d summing to 1. Moreover, the total population is bounded by an unknown number N.

Remark 1. Unlike fixed-population games (Huang et al., 2025; Wang et al., 2025), dynamic players here induce a time-varying NE. Compared to online game problems, although both involve time-varying NE, they are different in nature. The open networks entail dynamic changes in both player population and network topology, making them inherently more challenging to handle. Additionally, the function average is necessary in each cluster, reflecting the characteristics of openness. This results in very different methods and challenges to tracking the NE of the multi-cluster games with variant number of players.

Here, at time k, the number of players in cluster $i \in [m]$ is n_i^k and the corresponding set of players in cluster $i \in [m]$ is denoted by $\mathcal{V}_i^k$. Similar to Deplano et al. (2025), define the following subsets of players in cluster $i \in [m]$: Remaining player set $\mathcal{R}_i^k := \mathcal{V}_i^k \cap \mathcal{V}_i^{k-1}$: the set of players presenting both at time $k-1$ and time k; Arriving player set $\mathcal{A}_i^k := \mathcal{V}_i^k \backslash \mathcal{V}_i^{k-1}$: the set of new joining players at time k; Departing player set $\mathcal{D}_i^k := \mathcal{V}_i^{k-1} \backslash \mathcal{V}_i^k$: the set of leaving players at time k.

During joining or leaving stage, players within each cluster will undergo changes except baseline players. In light of dynamic population, some standard assumptions are adjusted according to openness properties.

Assumption 1. For every $j \in [n_i], i \in [m]$, local cost function $f_{ij}(x_{ij}, \Gamma_i(x_{-i}))$ is continuously differentiable with respect to x_{ij} and the gradient $\nabla_i f_{ij}(x_{ij}, \Gamma_i(x_{-i})) := \frac{\partial f_{ij}(x_{ij}, \Gamma_i(x_{-i}))}{\partial x_{ij}}$ is Lipschitz continuous on $\mathbb{R}^q$, i.e., for some constant $L_{ij} > 0$,

$$\begin{aligned} &\|\nabla_i f_{ij}(x_{ij}, \Gamma_i(x_{-i})) - \nabla_i f_{ij}(\tilde{x}_{ij}, \Gamma_i(\tilde{x}_{-i}))\| \\ &\leq L_{ij} \left\| \begin{bmatrix} x_{ij} - \tilde{x}_{ij} \\ \Gamma_i(x_{-i}) - \Gamma_i(\tilde{x}_{-i}) \end{bmatrix} \right\|. \end{aligned} \tag{4}$$

Assumption 1 is commonly used in distributed optimization and NE seeking in noncooperative games (Doostmohammadian et al., 2024; Wang et al., 2025).

Define $g_i^k(y_i, y_{-i}) := \frac{1}{n_i^k} \sum_{j=1}^{n_i^k} \nabla_i f_{ij}(y_i, y_{-i})$, where $y_i \in \mathbb{R}^{q_i}, \forall i \in [m]$. Let $F^k(y) : \mathbb{R}^q \to \mathbb{R}^q$ be the pseudo gradient, i.e., $F^k(y) := col(\nabla_1 g_1^k(y), \ldots, \nabla_m g_m^k(y))$, $y \in \mathbb{R}^q$.

Assumption 2. At time $k \geq 0$, the mapping $F^k(y)$ is strongly monotone on $\mathbb{R}^q$ with constant $\mu > 0$, that is, for any $y = col(y_1, \ldots, y_m)$ and $z = col(z_1, \ldots, z_m)$ with $y_i, z_i \in \mathbb{R}^{q_i}, i \in [m]$, the following inequality holds:

$$\begin{aligned} &\sum_{i=1}^{m} \frac{1}{n_i^k} \sum_{j=1}^{n_i^k} [\nabla_i f_{ij}(y_i, y_{-i}) - \nabla_i f_{ij}(z_i, z_{-i})]^T (y_i - z_i) \\ &\geq \mu \|y - z\|^2. \end{aligned} \tag{5}$$

This assumption is standard in the literature on NE seeking algorithms with fast convergence rates and ensures the existence and uniqueness of the NE. Moreover, accounting for player dynamics, strong monotonicity must hold for the cost functions at each stage to ensure the NE exists.

Based on the above description, it is a consensus problem in each cluster, where all agents within the same cluster take the same strategy when reaching the NE x^*, i.e., $x_{ij}^* = x_{il}^* = \tilde{x}_i^* \in \Omega_i$, $\forall j, l \in [n_i], \forall i \in [m]$. Therefore, for simplicity, the NE of the multi-cluster game is denoted by $\tilde{x}^* := col(\tilde{x}_1^*, \ldots, \tilde{x}_m^*) \in \mathbb{R}^q$, which is also used to present the NE of the baseline game comprising n^0 players.

In practice, players will choose to join the game when they share similar motivations, which is practically reasonable, as exemplified by national market regulation. Then, the assumption about new players' cost functions is listed.

Assumption 3. For any new player j in cluster $i \in [m]$, its cost function satisfies $\left\|\nabla_i f_{ij}(\tilde{x}_i^*, \tilde{x}_{-i}^*)\right\| \leq M$.

To prevent dramatic fluctuation in population from hindering the timely transmission of information, the number of players joining and leaving at each stage is constrained.

Assumption 4. For each cluster $i \in [m]$, at most $\bar{A}$ players simultaneously join this game, i.e., $0 \leq |\mathcal{A}_i^k| \leq \bar{A}$.

In the meanwhile, the number of players allowed to leave at one time is also restricted.

Assumption 5. For each cluster $i \in [m]$, at most $\bar{D}$ players simultaneously leave this game, i.e., $0 \leq |\mathcal{D}_i^k| \leq \bar{D}$.

In this game, information is exchanged via a composite network $\mathcal{G}^k = ([n^k], \mathcal{E}^k, W^k)$. This comprises a static inter-cluster network $\mathcal{G}_0 = ([m], \mathcal{E}_0, W_0)$ connecting the fixed representatives, and time-varying intra-cluster networks $\mathcal{G}_i^k = ([n_i^k], \mathcal{E}_i^k, W_i^k)$ for each cluster $i \in [m]$. To prevent communication disruptions caused by population-induced topological changes, some fundamental assumptions about network structure are given as follows.

Assumption 6. At time $k \geq 0$, $\forall i \in [m]$, the communication networks $\mathcal{G}_0$ and $\mathcal{G}_i^k$ are undirected and connected. All adjacency matrices W_0 and W_i^k are doubly stochastic, i.e., $W_0 \mathbf{1}_m = \mathbf{1}_m$, $\mathbf{1}_m^T W_0 = \mathbf{1}_m^T$, $W_i^k \mathbf{1}_{n_i^k} = \mathbf{1}_{n_i^k}$, $\mathbf{1}_{n_i^k}^T W_i^k = \mathbf{1}_{n_i^k}^T$.

To guarantee prompt information access for new arrivals, the following connectivity assumption is made.

Assumption 7. At time $k \geq 1$, every new agent has at least $c \geq 1$ neighbors who are still present at time k.

Assumption 7 characterizes the minimal communication requirements for timely information transmission, rather than imposing a specific network topology.

3. MAIN RESULT

In this section, a distributed algorithm is proposed to track the NE. Based on the partial decision scenario, each player j in cluster i is endowed with a local variable

$$x_{(ij)}^k := col(x_{(1)ij}^k, \ldots, x_{(m)ij}^k) \in \mathbb{R}^q, \tag{6}$$

which is used to estimate representative players' strategies. Here, $q := \sum_{i=1}^m q_i$, $x_{(s)ij}^k \in \mathbb{R}^{q_s}$ for $s \neq i$ is the estimation of x_{s1}^k at the agent j in cluster i and $x_{(i)ij}^k = x_{ij}^k$. Also, the estimates of other representative players except cluster i is denoted by

$$x_{-(ij)}^k := col(x_{(1)ij}^k, \ldots, x_{(i-1)ij}^k, x_{(i+1)ij}^k, \ldots, x_{(m)ij}^k). \tag{7}$$

In the meanwhile, there is a gradient estimation variable $v_{ij}^k \in \mathbb{R}^{q_i}$ to estimate the gradient for cluster i. For convenient analysis, define the following variables: $\mathbf{x}_i^k := [x_{(i1)}^k \cdots x_{(in_i^k)}^k]^T \in \mathbb{R}^{n_i^k \times q}$, $\mathbf{x}^k := col(\mathbf{x}_1^k, \ldots, \mathbf{x}_m^k) \in \mathbb{R}^{n^k \times q}$, $\mathbf{v}_i^k := [v_{i1}^k, \ldots, v_{in_i^k}^k]^T \in \mathbb{R}^{n_i^k \times q_i}$. As defined earlier, we define NE for each stage k as $\tilde{x}^{*,k} := col(\tilde{x}_1^{*,k}, \ldots, \tilde{x}_m^{*,k}) \in \mathbb{R}^q$. Then, denote $\mathbf{x}_{n^k}^{*,k} := \mathbf{1}_{n^k}(\tilde{x}^{*,k})^T \in \mathbb{R}^{n^k \times q}$, $\bar{v}_i^k := (\frac{\mathbf{1}_{n_i^k}^T}{n_i^k} \mathbf{v}_i^k)^T \in \mathbb{R}^{q_i}$ and $\bar{\mathbf{v}}_i^k := \frac{\mathbf{1}_{n_i^k} \mathbf{1}_{n_i^k}^T}{n_i^k} \mathbf{v}_i^k \in \mathbb{R}^{n_i^k \times q_i}$.

To accommodate player dynamics, an initialization method for new players is employed, where the new player j in cluster i initialize its strategies according to the average of its neighbors' information, i.e., $x_{ij}^k = \frac{1}{|\mathcal{N}_{ij}^k|} \sum_{q \in \mathcal{N}_{ij}^k} x_{iq}^k$.

Then, a distributed projected gradient tracking algorithm (DPGT) with dynamic population is designed as in Algorithm 1 to track the NE in multi-cluster games.

Algorithm 1 DPGT in open networks

Each agent j in cluster i maintains vector variables $x_{(ij)}^k = col\left(x_{(1)ij}^k, \ldots, x_{(m)ij}^k\right) \in \mathbb{R}^q$ and $v_{ij}^k \in \mathbb{R}^{q_i}$ at iteration k.

Initialization: Initialize $x_{ij}^0 \in \Omega_i, x_{-(ij)}^0 \in \mathbb{R}^{q-q_i}$ and let $v_{ij}^0 = \nabla_i f_{ij}\left(x_{ij}^0, x_{-(ij)}^0\right)$, $\forall i \in [m]$, $\forall j \in [n_i^0]$.

Iteration: For $k \geq 1$, each player $j \in [n_i^k]$ in cluster $i \in [m]$:

if $Z^k = \eta_u$ **then**

if $j = 1$ **then**

$$x_{ij}^k = P_{\Omega_i}\left[\frac{1}{2}\Big(\sum_{l=1}^{n_i^k} w_{i,k}^{jl} x_{il}^{k-1} + \sum_{h=1}^{m} w_{0,k}^{ih} x_{(i)hj}^{k-1}\Big) - \alpha_i v_{ij}^{k-1}\right]$$

$$x_{(s)ij}^k = \frac{1}{2}\Big(\sum_{l=1}^{n_i^k} w_{i,k}^{il} x_{(s)il}^{k-1} + \sum_{h=1}^{m} w_{0,k}^{ih} x_{(s)hj}^{k-1}\Big),$$

else

$$x_{ij}^k = P_{\Omega_i}\left[\sum_{l=1}^{n_i^k} w_{i,k}^{il} x_{il}^{k-1} - \alpha_i v_{ij}^{k-1}\right]$$

$$x_{(s)ij}^k = \sum_{l=1}^{n_i^k} w_{i,k}^{jl} x_{(s)il}^{k-1},$$

end if

$v_{ij}^k = \sum_{l=1}^{n_i^k} w_{i,k}^{il} v_{il}^{k-1} + \nabla_i f_{ij}(x_{(ij)}^k) - \nabla_i f_{ij}(x_{(ij)}^{k-1})$,

else if $Z^k = \eta_a$ **then**

if $j \in \mathcal{A}_i^k$ **then**

$x_{ij}^k = \frac{1}{|\mathcal{N}_{ij}^k|}\sum_{q\in\mathcal{N}_{ij}^k} x_{iq}^k$;

$x_{(s)ij}^k = \frac{1}{|\mathcal{N}_{ij}^k|}\sum_{q\in\mathcal{N}_{ij}^k} x_{(s)iq}^k$; $v_{ij}^k = \frac{1}{|\mathcal{N}_{ij}^k|}\sum_{q\in\mathcal{N}_{ij}^k} v_{iq}^k$

else

$x_{ij}^k = x_{ij}^{k-1}$; $x_{(s)ij}^k = x_{(s)ij}^{k-1}$; $v_{ij}^k = v_{ij}^{k-1}$

end if

else

$x_{ij}^k = x_{ij}^{k-1}$; $x_{(s)ij}^k = x_{(s)ij}^{k-1}$; $v_{ij}^k = v_{ij}^{k-1}$

end if

where $w_{0,k}^{ih}$ is the (i,h) element of W_0^k, $w_{i,k}^{jl}$ is the (j,l) element of W_i^k, $i \in [m]$, and $\alpha_i > 0$, $i \in [m]$, are the stepsizes to be determined.

Define a matrix $\mathscr{W}^k \in \mathbb{R}^{n^k \times n^k}$ as follows:

$$\mathscr{W}^k = \begin{bmatrix} \tilde{A}_1^k + B_{11}^k & B_{12}^k & \cdots & B_{1m}^k \\ B_{21}^k & \tilde{A}_2^k + B_{22}^k & \cdots & B_{2m}^k \\ \vdots & \vdots & & \vdots \\ B_{m1}^k & B_{m2}^k & \cdots & \tilde{A}_m^k + B_{mm}^k \end{bmatrix},$$

where $\tilde{A}_i^k \in \mathbb{R}^{n_i^k \times n_i^k}$ matches W_i^k except that the first row $Row_1(\tilde{A}_i^k)$ is $\frac{1}{2}Row_1(W_i^k)$ and $B_{ij}^k \in \mathbb{R}^{n_i^k \times n_j^k}$ has the $(1,1)$ element as $\frac{1}{2}a_0^{ij}$ and other elements as $0, i, j \in [m]$.

Before presenting our main result, some findings regarding this multi-cluster game with a fixed number of players are shown below (Meng and Li, 2023). Lemma 1 characterizes the eigenvectors under this network.

Lemma 1. (Meng and Li (2023)). Under Assumption 6, the matrix $\mathscr{W}^k$ has the left eigenvector $\pi^k \in \mathbb{R}^{n^k}$ as $\pi^k = col(\pi_1^k, \ldots, \pi_m^k)$ with π_i^k being $\pi_i^k = col(\frac{2}{n^k+m}, \frac{1}{n^k+m}, \ldots, \frac{1}{n^k+m}) \in \mathbb{R}^{n_i^k}, i \in [m]$, which corresponds to eigenvalue 1 such that $(\pi^k)^T\mathscr{W}^k = (\pi^k)^T$ and $(\pi^k)^T\mathbf{1}_{n^k} = 1$.

Given the vector $\pi = \mathrm{col}(\pi_1, \ldots, \pi_n)$, the weighted Euclidean and Frobenius norms for $x \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^{n\times q}$ are defined as follows, respectively:

$$\|x\|^\pi := \sqrt{\langle \mathrm{diag}(\pi)x, x\rangle} = \left\|\mathrm{diag}(\sqrt{\pi})x\right\|, \tag{8}$$

$$\|\mathbf{x}\|_F^\pi := \sqrt{\langle \mathrm{diag}(\pi)\mathbf{x}, \mathbf{x}\rangle} = \left\|\mathrm{diag}(\sqrt{\pi})\mathbf{x}\right\|_F, \tag{9}$$

where $\sqrt{\pi} := col(\sqrt{\pi_1}, \ldots, \sqrt{\pi_n})$.

Assuming all players update their strategies at time k, i.e., $Z^k = \eta_u$, we obtain a result similar to Meng and Li (2023).

Lemma 2. If Assumptions 1, 2, 6 hold and $Z^{k+1} = \eta_u$, then there exists a $0 < \gamma < 1$ such that x_{ij}^{k+1} and v_{ij}^{k+1} generated by Algorithm 1 satisfy

$$\begin{aligned} &\left\|\mathbf{x}^{k+1} - \mathbf{x}_{n^{k+1}}^{*,k+1}\right\|_F^{\pi^{k+1}} + \sum\nolimits_{i=1}^m \left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F \\ &\leq \gamma\Big(\left\|\mathbf{x}^k - \mathbf{x}_{n^k}^{*,k}\right\|_F^{\pi^k} + \sum\nolimits_{i=1}^m \left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F\Big). \end{aligned} \tag{10}$$

We first introduce several lemmas bounding the NE shifts induced by player dynamics and the initialization errors of new arrivals. Proofs are omitted due to space constraints. Assuming cluster $i \in [m]$ contains n_i^k players at time $k \geq 0$. Then, the relationship between $\tilde{x}^{*,k}$, the NE at time k, and the NE of the baseline game, $\tilde{x}^*$, can be derived.

Lemma 3. Let Assumptions 1-3 hold. Then, we have

$$\left\|\tilde{x}^{*,k} - \tilde{x}^*\right\| \leq \frac{2mM}{\mu}\Big(1 - \frac{\min_{i\in[m]} n_i^0}{N}\Big). \tag{11}$$

Lemma 3 shows that the change of NE is bounded regardless of how many players join or leave. For simplicity, define $B := \frac{2mM}{\mu}(1 - \frac{\min_{i\in[m]} n_i^0}{N})$ and $e_{ij}^k := \left\|x_{(ij)}^k - \tilde{x}^{*,k}\right\|^2$.

Lemma 4. Let Assumptions 1-7 hold. According to Algorithm 1, at arriving stage, the weighted distance between players' strategies and NE satisfies

$$\left\|\mathbf{x}^{k+1} - \mathbf{x}_{n^{k+1}}^{*,k+1}\right\|_F^{\pi^{k+1}} \leq \sqrt{1 + \frac{\bar{A}}{c}}\left\|\mathbf{x}^k - \mathbf{x}_{n^k}^{*,k}\right\|_F^{\pi^k} + 2B. \tag{12}$$

Lemma 4 shows the distance between players' strategies and NE at arriving stages.

Lemma 5. Let Assumptions 1-7 hold. By Algorithm 1, at arriving stage, the distance between the estimated and average gradients for each cluster $i \in [m]$ satisfies

$$\left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F \leq \sqrt{1 + \frac{\bar{A}}{c}}\left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F. \tag{13}$$

For each cluster i, Lemma 5 claims the change of the distance between $\mathbf{v}_i^k$ and $\bar{\mathbf{v}}_i^k$ at each arriving stage. Before presenting our theorem, define $\mathbb{I}\{\cdot\}$ as the indicator function. For any $\eta \in \{\eta_u, \eta_a, \eta_d\}$, the term $\mathbb{I}\{Z^k = \eta\}$ takes value 1 on the event $\{Z^k = \eta\}$ and 0 elsewhere.

Theorem 1. Let Assumptions 1-7 hold. The sequence of strategy $\mathbf{x}^k$ and estimation of gradient generated by Algorithm 1 satisfies, $\forall k \geq 0$

$$\begin{aligned} &\mathbb{E}[\left\|\mathbf{x}^{k+1} - \mathbf{x}_{n^{k+1}}^{*,k+1}\right\|_F^{\pi^{k+1}} + \sum\nolimits_{i=1}^m \left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F] \\ &\leq \gamma_1^{k+1}\mathbb{E}[\left\|\mathbf{x}^0 - \mathbf{x}_{n^0}^{*,0}\right\|_F^{\pi^0} + \sum\nolimits_{i=1}^m \left\|\mathbf{v}_i^0 - \bar{\mathbf{v}}_i^0\right\|_F] \\ &\quad + \frac{1-\gamma_1^{k+1}}{1-\gamma_1}(p_a + p_d)2B, \end{aligned} \tag{14}$$

where $\gamma_1 = p_u\gamma + p_a\sqrt{1+\frac{\bar{A}}{c}} + p_d\sqrt{1+\frac{m\bar{D}}{n^0+m}}$.

Proof: At leaving stages, for the distance between strategies and NE, we have

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*,k+1}_{n^{k+1}}\right\|_F^{\pi^{k+1}} \\
&\le \sqrt{\frac{1}{n^{k+1}+m}}\sqrt{\sum_{i=1}^m\left[e^k_{i1} + \sum_{j\in[n_i^k]} e^k_{ij} - \sum_{j\in\mathcal{D}_i^{k+1}} e^k_{ij}\right]} \\
&\quad + \left\|\tilde{x}^{*,k+1} - \tilde{x}^{*,k}\right\| \\
&\le \sqrt{\frac{n^k+m}{n^{k+1}+m}}\left\|\mathbf{x}^k - \mathbf{x}^{*,k}_{n^k}\right\|_F^{\pi^k} + \left\|\tilde{x}^{*,k+1} - \tilde{x}^*\right\| \\
&\quad + \left\|\tilde{x}^{*,k} - \tilde{x}^*\right\| \\
&\le \sqrt{1+\frac{m\bar{D}}{n^0+m}}\left\|\mathbf{x}^k - \mathbf{x}^{*,k}_{n^k}\right\|_F^{\pi^k} + 2B, \qquad (15)
\end{aligned}$$

where the first and second inequalities are based on Lemma 1 and the third inequality is derived by Assumption 5 and Lemma 3.

For the distance $\left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|$ in each cluster i, based on the proof similar to Lemma 5, we get

$$\left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F \le \left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F. \qquad (16)$$

From Lemmas 2-5, (15) and (16), one has

$$\begin{aligned}
&\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*,k+1}_{n^{k+1}}\right\|_F^{\pi^{k+1}} + \sum_{i=1}^m\left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F \\
&\le \mathbb{I}\{Z^{k+1}=\eta_u\}\gamma(\left\|\mathbf{x}^k - \mathbf{x}^{*,k}_{n^k}\right\|_F^{\pi^k} + \sum_{i=1}^m\left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F) \\
&\quad + \mathbb{I}\{Z^{k+1}=\eta_a\}\Big[2B + \sqrt{1+\frac{\bar{A}}{c}}\left\|\mathbf{x}^k - \mathbf{x}^{*,k}_{n^k}\right\|_F^{\pi^k} \\
&\quad + \sqrt{1+\frac{\bar{A}}{c}}\left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F\Big] + \mathbb{I}\{Z^{k+1}=\eta_d\}\Big[2B \\
&\quad + \left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F + \sqrt{1+\frac{m\bar{D}}{n^0+m}}\left\|\mathbf{x}^k - \mathbf{x}^{*,k}_{n^k}\right\|_F^{\pi^k}\Big]. \qquad (17)
\end{aligned}$$

Take expectation on both sides, we have

$$\begin{aligned}
&\mathbb{E}[\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*,k+1}_{n^{k+1}}\right\|_F^{\pi^{k+1}} + \sum_{i=1}^m\left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F] \\
&\le \gamma_1\mathbb{E}[\left\|\mathbf{x}^k - \mathbf{x}^{*,k}_{n^k}\right\|_F^{\pi^k}] + \gamma_2\mathbb{E}[\sum_{i=1}^m\left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F] \\
&\quad + (p_a+p_d)2B, \qquad (18)
\end{aligned}$$

where $\gamma_1 = p_u\gamma + p_a\sqrt{1+\frac{\bar{A}}{c}} + p_d\sqrt{1+\frac{m\bar{D}}{n^0+m}}$, $\gamma_2 = p_u\gamma + p_a\sqrt{1+\frac{\bar{A}}{c}} + p_d$. Obviously, $\gamma_1 \ge \gamma_2$. Then, a few algebraic manipulations yield

$$\begin{aligned}
&\mathbb{E}[\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*,k+1}_{n^{k+1}}\right\|_F^{\pi^{k+1}} + \sum_{i=1}^m\left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F] \\
&\le \gamma_1^{k+1}\mathbb{E}[\left\|\mathbf{x}^0 - \mathbf{x}^{*,0}_{n^0}\right\|_F^{\pi^0} + \sum_{i=1}^m\left\|\mathbf{v}_i^0 - \bar{\mathbf{v}}_i^0\right\|_F] \\
&\quad + \frac{1-\gamma_1^{k+1}}{1-\gamma_1}(p_a+p_d)2B. \qquad (19)
\end{aligned}$$

The proof is completed. ■

Remark 2. The convergence depends on the probability of event occurrence, the number of clusters m, openness conditions $\bar{A}$, $\bar{D}$, and network connectivity c. When a large number of players fluctuate, the fluctuation frequency must be reduced, and high network connectivity is required to ensure fast communication and convergence.

Remark 3. Compared with existing works of game theory in open networks (Liu et al., 2026; Xu et al., 2025), we consider a fluctuation with probability rather than a change over time intervals, and in certain circumstances, the frequency of fluctuation is greater in a probabilistic sense and fixed intervals can lead to better convergence results in our paper. Compared with existing works (Hendrickx and Martin, 2017; Deplano et al., 2025), the influence of population fluctuation on the variation of NE is analyzed and the positions of new players' initialization strategies are studied, which is different from optimization problems.

4. SIMULATIONS

As industries diversify, different enterprises establish or close subsidiaries to increase revenue. In this section, a numerical simulation of Cournot competition game with population fluctuation is considered (Meng and Li, 2023).

Here, we model a multi-cluster game comprising 5 large enterprises, each containing n_i^k subsidiary companies at time k. Within the same enterprise, subsidiary companies collaborate to produce goods consistently while competing externally to maximize enterprise profit. Intra-enterprise communication occurs via the graph $\mathscr{G}_i^k$, whereas inter-enterprise communication is conducted through the graph $\mathscr{G}_0^k$ using the first subsidiary of each enterprise as a representative. The production strategy of company j company in i enterprise is denoted by $x_{ij}^k \in [0, 30]$. At each stage k, the system experiences one of three events: an arrival phase (probability $p_a = 0.1$) where at most $\bar{A} = 1$ company joins each enterprise forming at least $c = 2$ internal connections; an exit phase (probability $p_d = 0.1$) where at most $\bar{D} = 1$ non-representative company leaves; or a strategy update phase via Algorithm 1 (probability $p_u = 0.8$). The payoff function of company i in enterprise i is $f_{ij} = c_{ij} - x_{ij}p_{ij}$, comprising the unit price $p_{ij} = 60i - \frac{1}{5}\sum_{h=1}^5 x_{h1}$ and the cost $c_{ij} = (5 - b_{ij}/(i+1))x_{ij}^2 + i5x_{ij} + i$, with $b_{ij} \in [-3, 3]$ capturing random cost fluctuations. Thus, the payoff function of enterprise i is $f_i = \sum_{j=1}^{n_i^k} f_{ij}$.

It is assumed that there are $n_i^0 = 3$ fundamental companies in each enterprise $i \in [5]$ with specific payoff functions and their corresponding NE is calculated as $\tilde{x}_1^{0,*} = 0$, $\tilde{x}_2^{0,*} = 4.41$, $\tilde{x}_3^{0,*} = 9.80$, $\tilde{x}_4^{0,*} = 15.20$, $\tilde{x}_5^{0,*} = 20.59$ by a centralized method. Thus, $\forall i \in [5]$, $\forall j \in [m_i^k]$, the gradient of its payoff function satisfies $\left|\nabla_i f_{ij}(\tilde{x}^{0,*})\right| \le 24.71$. The change in company numbers under expectation is illustrated in Figure 1. Through mathematical calculations, the stepsizes $\alpha_i = \alpha/(n_i^k+1)$ and $\alpha = 0.05$ are selected here, as they satisfy the conditions ensuring algorithm convergence in Theorem 1. Under the above setting, the evolution of the sum of optimal distance $\left\|\mathbf{x}^k - \mathbf{x}^{k,*}_{n^k}\right\|_F^{\pi^k}$ and consensus distance $\sum_{i=1}^5\left\|\mathbf{v}_i^k - \bar{\mathbf{v}}_i^k\right\|_F$ averaged over 100 realizations of process are shown in Figure 2 respectively.

Figure 2 illustrates the convergence of Algorithm 1 and it converges to the neighbor of the sum of consensus and opti-

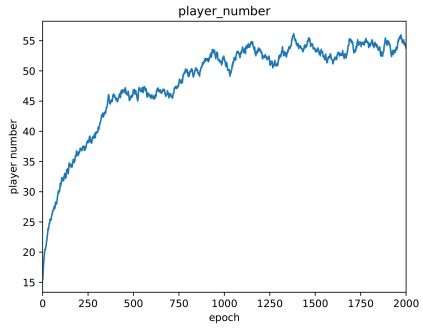


Fig. 1. The average change in the number of companies over 100 experiments.

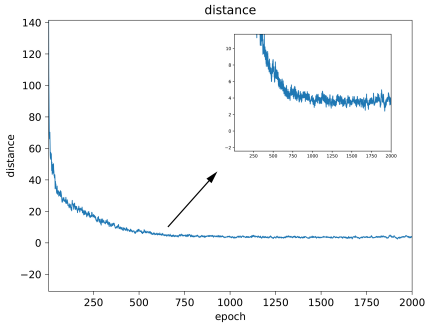


Fig. 2. The average distance over 100 experiments.

mal strategy in expectation, i.e., $\mathbb{E}[\left\|\mathbf{x}^{k+1} - \mathbf{x}^{*,k+1}_{n^{k+1}}\right\|_F^{\pi^{k+1}} + \sum_{i=1}^{m} \left\|\mathbf{v}_i^{k+1} - \bar{\mathbf{v}}_i^{k+1}\right\|_F] \leq 9.84$, which is the worst case.

5. CONCLUSION

This paper investigated the distributed NE seeking for multi-cluster games in open networks. We analyze how player dynamics shift the NE and evaluated our proposed initialization method for bounding the initial errors of new players. Although continuous entry and exit events prevent exact convergence, our algorithm guarantees expected convergence to a neighborhood of the time-varying NE, bounded by the frequency and size of population changes. Finally, considering the variable number of cluster and heterogeneous conditions are meaningful research directions for future work. However, this extension may lead to some challenges, especially in the construction of the network and the selection of representative players.

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