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Absolute stability preserving moment matching for Lur'e systems ⋆

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Abstract: In this paper, we tackle the problem of model order reduction for Lur'e systems, that preserves absolute stability, within a sector. The proposed model reduction procedure exploits linear moment matching techniques to reduce the linear part of the Lur'e system. In general, directly applying moment matching does not necessarily preserve absolute stability. For a class of linear systems, we use the characterization of the Popov criterion for absolute stability, in terms of a passivity condition. In the family of linear reduced order models achieving moment matching, we seek the free parameters that preserve the passivity condition such that the reduced order model is absolutely stable in a sector. This reduces to finding the solutions of a linear system and of a linear matrix inequality, which is computationally attractive.

Keywords: Moment matching, absolute stability, Lur'e systems, linear system, passivity, positive realness, linear matrix inequality.

1. INTRODUCTION

In this paper, we consider the class of Lur'e nonlinear systems. Such systems are formed as the feedback interconnection of a linear dynamical system and a static nonlinear function Khalil (2002); Răsvan and Ştefan (2007). In the literature, there are engineering applications, where the plants can be modeled as Lur'e systems such as, e.g., Chua circuits, robotic arms with flexible joints, some hyperchaotic systems, where the nonlinearity of the control system is in the form of a relay or actuator/sensor nonlinearity Khalil (2002), or a ball and wheel apparatus, see Ho and Lu (2005). The static nonlinear function must satisfy a sector boundedness condition.

The main problem for such systems is the absolute stability problem, originally formulated by Lur'e and further studied and solved by Popov in Popov (1961), see also Ho and Lu (2005) and the references therein. The problem is one of robust stability, i.e., finding the sector such that the Lur'e system with any nonlinearity in that sector has an asymptotically stable equilibrium point in the origin. In Popov (1961), see also Khalil (2002); Răsvan and Ştefan (2007), a sufficient frequency domain condition on the linear system is given such that absolute stability is achieved, i.e, the Popov criterion. More general results are represented by the Circle criterion, or the Zames-Falb multiplier-based results Zames and Falb (1968). Fairly recent papers address different types of stability for Lur'e systems, such as input-to-state stability, Sarkans and Logemann (2015); Drummond et al. (2024).

The linear system may be highly dimensional, impairing simulation and control design. Hence model reduction is required for analysis, simulation and control design. For example, balanced truncation model reduction of Lur'e systems with error bounds was developed in Besselink et al. (2009). Moreover, when interconnecting a large number of Lur'e systems, analysis and control of the network are tedious and model reduction of the network is performed, see, e.g., Dou et al. (2024), for model reduction Lur'e networks based on clustering.

For linear time-invariant (LTI) systems, moment matching techniques are a popular method, see, e.g., Antoulas (2005) and the references therein. The reduced order model is constructed as a lower degree rational function that approximates the original rational transfer function, matching the latter and its derivatives at a set of (complex-conjugated) points. The moment of an LTI system is based on the series expansion of the transfer function of the linear system (see, e.g., Antoulas (2005)). Written in terms of Krylov projections, the reduced order model matches a prescribed number of moments Jaimoukha and Kasenally (1997); Gugercin and Antoulas (2004).

More recently, in Astolfi (2010); Ionescu et al. (2014) a characterization of moments in a Sylvester framework has been given, see also Gallivan et al. (2004b). The low-order approximation achieving moment matching at ν points is a

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parametric model, with ν free parameters tuned such that certain properties are imposed, such as stability Astolfi (2010), or, more general, dissipativity, Shakib et al. (2025).

Contributions. In this paper, we tackle the problem of moment matching-based model order reduction of a nonlinear Lur'e system, achieving absolute stability for all the nonlinear functions in a given sector. The problem is formulated as follows: given a Lur'e system with the linear part having poles in the open left half-plane and simple poles on the imaginary axis, find a reduced order linear system achieving moment matching, and a sector condition, such that the reduced-order Lur'e system is absolutely stable. Directly applying linear moment matching both to the linear system and to the linearized version of the Lur'e system does not necessarily preserve absolute stability. In this paper we assume that the nonlinear Lur'e system possesses a passive (positive real) multiplier-based transfer function, stemming from the observed behaviour of the system. For instance, the system may consist of interconnected subsystems with passive multiplier-based functions yielding an absolutely stable full model. A reduced order model that preserves the positive realness of the multiplier-based system is needed.
We further consider linear moment matching such that absolute stability is preserved. In this case, we use the characterization of the Popov criterion for absolute stability (Răsvan and Ştefan, 2007, Section 5.3), in terms of a passivity condition applied to the multiplier-based transfer function formed with the given linear system. In the family of reduced order models achieving moment matching of the given linear system, we focus on finding the free parameters such that the passivity of the multiplier-based transfer function is preserved. Hence, we use moment matching techniques with preservation of positive realness, based on the solution of an optimization problem to obtain a linear reduced order model satisfying the Popov criterion, and the sector where the criterion holds. This reduces to finding the solutions of a linear systems and a linear matrix inequality, thus rendering the proposed method computationally attractive, compared to nonlinear model reduction techniques.
Similar to Besselink et al. (2009), in this paper, we apply model reduction to the linear system, such that absolute stability is preserved for any nonlinearity within a given sector. The model reduction in Besselink et al. (2009) is performed on the linear part of the system.

Content. In Section 2, we overview Lur'e systems, absolute stability, and Sylvester equation-based moment matching. In Section 3, we present the application of standard Krylov techniques on the Lur'e system and absolute stability is not preserved. Section 3.3 contains the main result, where we compute a reduced Lur'e system achieving moment matching and preserving absolute stability. Section 4 illustrates the method on a simplified model of the gyroscope. The paper ends with the conclusions in Section 5.

Notation. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of natural, real, and complex numbers, respectively. Then $\mathbb{R}^+$is the set of real, positive numbers. $\mathbb{C}^-$ is the open left-half-plane and $\jmath\mathbb{R}$ is the imaginary axis. $A^\top$ is the transpose of matrix $A \in \mathbb{R}^{n\times m}$. $\sigma(A)$ is the set of eigenvalues of A. $A \succ 0$ means that A is positive definite. $\{s_i\}_{i=1}^{\nu}$ is the set of values s_i, $i = 1 : \nu$.

2. PRELIMINARIES

In this section, we overview positive real linear systems, Lur'e systems and absolute stability, as well as Sylvester equation-based linear moment matching.

2.1 Positive real systems

In this section, we recall the definition of linear positive real systems, of the form $Z(s) = \mathbf{c}^\top(sI - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}$.
Definition 1.(Antoulas, 2005,Section 5.9.1)(Willems, 1972, Theorem 1) A minimal LTI system $Z(s)$ is positive real if $\mathrm{Re}(Z(s)) \geq 0$, for all s with $\mathrm{Re}\, s \geq 0$, s not a pole of $Z(s)$.

The next result presents necessary and sufficient conditions in terms of both as imaginary axis conditions as well as state-space LMI's, for a linear system $Z(s)$ to be positive real, in the sense of (Antoulas, 2005, Theorems 5.30, 5.31), see also, e.g., (Guiver et al., 2017, Theorem 3.7) and (Willems, 1972, Theorem 1).
Theorem 1. [Willems (1972); Antoulas (2005)] Consider an LTI minimal system $Z(s)$. Then, the following statements are equivalent:

(1) $Z(s)$ is passive, i.e., dissipative w.r.t. the supply rate $s = \frac{1}{2}u^\top y$;
(2) $Z(s)$ is positive real in the sense of Definition 1;
(3) the following conditions hold simultaneously
   (a) $Z(j\omega) + Z(-j\omega) \geq 0$, for all $\omega \in \mathbb{R}$ such that $j\omega \notin \sigma(\mathbf{A})$;
   (b) The poles of $Z(s)$ are in $\mathbb{C}^- \cup j\mathbb{R}$ and the poles on the imaginary axis $j\mathbb{R}$ are simple with the associated residues positive;
(4) there exists $X = X^\top \in \mathbb{R}^{n\times n}$, $X \succ 0$ such that the linear matrix inequality (LMI) holds:

$$\Gamma := \begin{bmatrix} \mathbf{A}^\top X + X\mathbf{A} & X\mathbf{b} - \mathbf{c} \\ \mathbf{b}^\top X - \mathbf{c}^\top & -2\mathbf{d} \end{bmatrix} \preceq 0. \tag{1}$$

Throughout the paper, minimal realizations of Z are considered.

2.2 Nonlinear Lur'e systems and the Popov criterion for absolute stability

In this section, we recall the absolute stability of nonlinear Lur'e systems in the sense of the Popov criterion. Consider the single-input-single-output (SISO) linear system

$$\dot{x} = Ax + bu,\ y = c^\top x, \tag{2}$$

with the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}$, the output $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$. The transfer function of (2) is

$$K(s) = c^\top(sI - A)^{-1}b. \tag{3}$$

Selecting a nonlinear output feedback control law $u(t) = -\varphi(y(t)) + w(t)$, with $\varphi : \mathbb{R} \to \mathbb{R}$ and $w \in \mathbb{R}$, an external input of the closed-loop systems, e.g., reference, such that the sector condition holds for $\beta > 0$:

$$0 \leq \varphi(z)z \leq \beta z^2, \quad \forall z \in \mathbb{R}, \tag{4}$$

we get the model of the closed-loop Lur'e system

$$\dot{x} = Ax - b\varphi(y) + bw,\ y = c^\top x. \tag{5}$$

Note that for all φ satisfying (4), the origin is an equilibrium point of (5), for $w = 0$. For a system (5), we recall the definition of absolute stability as in Khalil (2002).

Definition 2. Consider the system (5), where the static nonlinearity φ satisfies the sector condition (4). If the origin is asymptotically stable for all the static nonlinear functions φ satisfying the sector criterion (4), then the system (5) is called absolutely stable.

For the sake of simplicity, we call any linear system with a transfer function as in (3) absolutely stable when referring to (5) being absolutely stable as in Definition 2. The next result provides a sufficient frequency-domain condition for the absolute stability of (5).

Theorem 2. [(Popov Criterion) Răsvan and Ştefan (2007)] Consider the Lur'e system (5) and assume the following conditions hold.

(i) The transfer function K is strictly proper and irreducible, with the poles in $\mathbb{C}^- \cup \jmath\mathbb{R}$. The poles on $\jmath\mathbb{R}$ can be either a simple pole at zero, a simple pair of complex-conjugate poles, both, or a double pole at zero.
(ii) The class of static nonlinear functions φ satisfy (4).
(iii) There exists $\eta \in \mathbb{R}$ such that

$$\frac{1}{\beta} + \mathrm{Re}[(1 + j\omega\eta)K(j\omega)] \geq 0, \quad \forall \omega \in \mathbb{R} \tag{6}$$

(with a strict inequality if A Hurwitz).

Then the system (2) with the transfer function (3) is absolutely stable, for all φ satisfying (4) .

Following the result in Theorem 1, define the following multiplier-based transfer function, built using the multiplier $1 + \eta s$ (Fu et al., 2005, Example 1):

$$Z(s) = \frac{1}{\beta} + (1 + \eta s)K(s). \tag{7}$$

The multiplier $1 + \eta s$ is not unique, see, e.g., Fu et al. (2005); Zames and Falb (1968), for more types of multipliers and the corresponding absolute stability results. Note that $Z(s)$ is a proper transfer function by construction. It can be shown that the system Z is irreducible if $-1/\eta$ is not a pole of $K(s)$. Throughout the rest of the paper, we make the following working assumption.

Assumption 1. The transfer function $Z(s)$ is irreducible and has the poles in $\mathbb{C}^- \cup \jmath\mathbb{R}$ and the complex conjugated poles on $\jmath\mathbb{R}$ are simple.

Then, under Assumption 1, the frequency-domain condition (6) is equivalent to $Z(j\omega) + Z(-j\omega) \geq 0$, i.e., $Z(s)$ is positive real, see Definition 1 below and also (Antoulas, 2005, Theorem 5.30). Following arguments as in Khalil (2002), we write next a state-space realization of $Z(s)$.

Proposition 3. A state-space realization of $Z(s)$ is $Z(s) = \mathbf{c}^\top(sI - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}$, with

$$\mathbf{A} = A,\ \mathbf{b} = b,\ \mathbf{c}^\top = c^\top(I + \eta A),\ \mathbf{d} = \frac{1}{\beta} + \eta c^\top b. \tag{8}$$

2.3 Moment matching for linear systems

Consider a SISO linear LTI minimal system $K(s)$ as in (3), with the state-space realization (2).

Let us present the notion of moments of K, given by (3), as in Astolfi (2010).

Definition 3. The k-moment of K at $s^* \in \mathbb{C} \setminus \sigma(A)$ is $\eta_k(s^*) = (-1)^k/(k!)\left[\mathrm{d}^k K(s)/\mathrm{d}\, s^k\right]_{s=s^*} \in \mathbb{C}$, $k \in \mathbb{N}$.

Let $\{s_i \in \mathbb{C} \setminus \sigma(A) \mid i = 1 : \varsigma\}$, be a symmetric set of complex numbers (including multiplicities). Take $j_i \geq 0$ such that $\sum_{i=1}^{\varsigma}(j_i + 1) = \nu \ll n$. For each i, let $\eta_0(s_i), ..., \eta_{j_i}(s_i)$ denote the ν moments of orders $0 : j_i$ of K at the given points s_i. Let $S \in \mathbb{R}^{\nu\times\nu}$, with the symmetric spectrum $\sigma(S) = \{s_i \mid i = 1 : \varsigma\}$, be such that $\sigma(S) \cap \sigma(A) = \emptyset$. Let $l^\top \in \mathbb{R}^{1\times\nu}$ be such that the pair $(l^\top, S)$ is observable. Let $\Pi \in \mathbb{R}^{n\times\nu}$ be the solution of the Sylvester equation

$$A\Pi + bl^\top = \Pi S. \tag{9}$$

Since the system is minimal and $\sigma(A) \cap \sigma(S) = \emptyset$, then Π is the unique solution of the equation (9), with rank $\Pi = \nu$, see e.g. Luenberger (1964). Then, the moments of K are uniquely determined by the elements of the vector $c^\top\Pi \in \mathbb{R}^{1\times\nu}$, as the next proposition show.

Proposition 4. [Astolfi (2010)] The ν moments given by $\eta_0(s_i), ..., \eta_{j_i}(s_i)$, $i = 1 : \varsigma$, of K at $\sigma(S)$ are in one-to-one relation with the elements of the vector $c^\top\Pi$.

Consider the LTI system

$$\dot{\xi} = F\xi + gu,\ \psi = h^\top\xi,$$

with $F \in \mathbb{R}^{\nu\times\nu}$, $g \in \mathbb{R}^\nu$ and $h^\top \in \mathbb{R}^{1\times\nu}$, and the corresponding transfer function

$$K_g(s) = h^\top(sI - F)^{-1}g.$$

Let $\hat{\eta}_0(s_i), ..., \hat{\eta}_{j_i}(s_i)$ denote the moments of orders $0 : j_i$ of K_g at s_i, with $i = 1 : \varsigma$. Then, moment matching is defined as follows.

Definition 4. K_g matches ν moments of K at $\{s_1, ..., s_\varsigma\}$, if $\eta_\kappa(s_i) = \hat{\eta}_\kappa(s_i)$, for all $\kappa = 0 : j_i$, $i = 1 : \varsigma$.

The next result gives the necessary and sufficient conditions for a low-order system to achieve moment matching.

Proposition 5. (Ionescu et al. (2022)). Fix $S \in \mathbb{R}^{\nu\times\nu}$ and $l^\top \in \mathbb{R}^{1\times\nu}$, such that the pair $(l^\top, S)$ is observable and $\sigma(S) \cap \sigma(A) = \emptyset$. Furthermore, assume that $\sigma(F) \cap \sigma(S) = \emptyset$. Then, the transfer function K_g matches the moments of K, at $\sigma(S)$, if and only if $h^\top P = c^\top\Pi$, where the matrix $P \in \mathbb{R}^{\nu\times\nu}$ is the unique solution of the Sylvester equation $FP + gl^\top = PS$.

Consequently, the system

$$\Sigma_g : \dot{\xi} = (S - gl^\top)\xi + gu, \quad \psi = c^\top\Pi\xi, \tag{10}$$

with the transfer function

$$K_g(s) = c^\top\Pi(sI - S + gl^\top)^{-1}g, \tag{11}$$

describes a family of ν order models that match ν moments of K, at $\sigma(S)$, in the sense of Definition 4, for all $g \in \mathbb{R}^\nu$ such that $\sigma(S - gl^\top) \cap \sigma(S) = \emptyset$, see, e.g. Astolfi (2010).

Remark 1. The spectrum of S contains the finite frequencies (with multiplicities) of the input signals yielding the desired frequency responses (moments) preserved through model reduction by rational interpolation.

3. MODEL REDUCTION BY MOMENT MATCHING FOR LUR'E SYSTEMS

In this section, we consider the Lur'e system (5). We present moment matching techniques to construct a reduced order model for the given nonlinear Lur'e system.

For (5), we seek a reduced model of order $\nu \ll n$ that preserves the Lur'e structure:

$$\dot{\xi} = F\xi - g\varphi(\psi) + gw,\ \psi = h^\top \xi, \tag{12}$$

such that selected moments are matched. One can consider the following choices to design moment matching based reduced order models (12) for (5): moment matching preserving Lur'e structure, linearization-based moment matching and linear moment matching with preservation of the Lur'e structure and absolute stability.

3.1 Moment matching preserving Lur'e structure

Consider the matrices $A, b, c^\top$ known in (5). In this case, the reduced order models have to preserve the Lur'e structure and match moments of $K(s) = c^\top (sI - A)^{-1} b$, the linear LTI system driven by $u(t)$. For this purpose, one can use the standard Krylov projection-based moment matching theory Antoulas (2005); Jaimoukha and Kasenally (1997); Gugercin and Antoulas (2004). Let $\{s_i\}_{i=1}^{\nu}$ be interpolation points (distinct, for simplicity). Build right/left projection bases V, W (matrices of size $n \times \nu$) such that, for all $i = 1 : \nu$, we have

$$\text{Right: } (A - s_i I)^{-1} b \in \text{Range}(V),$$
$$\text{Left: } c^\top (A - s_i I)^{-1} \in \text{Range}\left(W^\top\right),$$

that can be implemented using Krylov subspaces. Define the reduced system (12) with the matrices

$$F = W^\top A V,\ g = W^\top b,\ h^\top = c^\top V,$$

with Petrov-Galerkin condition $W^\top V = I_\nu$, where I_ν is the $\nu \times \nu$ identity matrix. Then, this construction guarantees the moment matching for K, see, e.g., Gallivan et al. (2004a); Rafiq and Bazaz (2021). It is also possible to construct reduced order models using two-sided moment matching Antoulas (2005). However, model reduction by moment matching cannot guarantee the preservation of absolute stability, as illustrated by the next example of two-sided moment matching model reduction.

Example 1. Let

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } c = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with $K(s) = 1/(s^2 + 3s + 2)$. Since

$$\min_{\omega \in \mathbb{R}} \text{Re}(K(j\omega)) = \min_{\omega^2 \geq 0} \frac{2 - \omega^2}{4 + 5\omega^2 + \omega^4} = \frac{2}{3}\sqrt{2} - 1,$$

obtained for $\omega_{min}^2 = 2 + 3\sqrt{2}$, absolute stability of this system for φ satisfying (4), with $\beta \leq 9 + 6\sqrt{2}$, follows using the Popov Criterion in Theorem 2 with $\eta = 0$. Then, one obtains a reduced order model of order $1 = \nu < n = 2$, where $F = 1/4$, $g = 1$ and $h = 1/8$ and with the transfer function $K_1(s) = 1/(8s - 2)$, such that $K(s_1) = K_1(s_1)$ at $s_1 = 1$ (note that the moments are also matched at $s_2 = 4$). The reduced order model has an unstable pole at 0.25 and does not preserve the stability property of the original system. Note that stability is not preserved even for the linear case ($\varphi(z) = 0$), a well-known issue of Krylov projection-based moment matching, see, e.g., Antoulas (2005) and the references therein.

3.2 Linearization-based moment matching

Consider the matrices $A, b, c^\top$ and φ known in equation (5). Select an equilibrium point $(x_\star, y_\star)$ that satisfies $Ax_\star - b\varphi(y_\star) + bw_\star = 0$, with $y_\star = c^\top x_\star$, corresponding to a constant input $w_\star$. Define the local slope k (e.g., a representative value within the sector). The linearized system is

$$\dot{\tilde{x}} = \underbrace{\left(A - bkc^\top\right)}_{A_{\text{lin}}} \tilde{x} + b\tilde{w},\ \tilde{y} = c^\top \tilde{x},$$

where $\tilde{x} := x - x_\star$, $\tilde{w} := w - w_\star$. The constant $w_\star$ influences the equilibrium $(x_\star, y_\star)$ since it appears in the equilibrium equation. If φ is differentiable, then $k = \varphi'(y_\star)$. Otherwise, for an uncertain k within a sector $[k_1, k_2]$, one may consider the midpoint $k = (k_1 + k_2)/2$. The transfer function is $K_{\text{lin}}(s) = c^\top (sI - A_{\text{lin}})^{-1} b$. Then the moments of K_{lin} at given $\{s_i\}_{i=1}^{\nu}$ can be matched to enforce the fidelity of the behavior of the small-signal (i.e., for the linearized model around $(x_\star, y_\star)$). Given K_{lin}, one can build V and W for $\left(A_{\text{lin}}, b, c^\top\right)$ at $\{s_i\}_{i=1}^{\nu}$ similarly to the matching problem preserving Lur'e structure. Although in this case V and W are obtained from A_{lin}, we retain (locally) the structure in the reduced model. The resulting reduced transfer function $K_{\text{lin},g}(s) = h^\top (sI - F)^{-1} g$ matches the moments of K_{lin}.

Similarly to Example 1, it is easy to show that model reduction of a linearization-based model by moment matching does not guarantee (absolute) stability preservation. In the next section, we provide sufficient conditions to obtain a reduced order model that achieves both moment matching and preservation of absolute stability.

3.3 Moment matching with absolute stability preservation

In this section we assume that $A, b, c^\top$ are known in the Lur'e system (5) and the static nonlinear element φ in the feedback, unknown, as in the Popov criterion. Herein we use moment matching techniques with preservation of positive realness, to obtain a linear reduced order model satisfying the Popov criterion, see Theorem 2.

We now formulate the problem of moment matching-based model reduction with preservation of absolute stability. Given $K(s)$ as in (3), satisfying Assumption 1, with $Z(s)$ passive, i.e, there exists $X \succ 0$ satisfying LMI (1), we compute $K_g(s)$ that matches the moments of $K(s)$ at $\sigma(S)$, such that the corresponding $Z_g(s)$ as in (13) is passive.

Consider the family of reduced order models Σ_g as in (10), parametrized in $g \in \mathbb{R}^\nu$, $\nu < n$, matching ν moments of $K(s)$, of orders $k = 0, 1, \ldots$ at $\sigma(S)$, selected as in Remark 1. We seek the vector g and the scalars β and η such that

$$Z_g(s) = \frac{1}{\beta} + (1 + \eta s) K_g(s), \tag{13}$$

with a state-space realization (see Proposition 3 and (10))

$$\mathbf{F} = S - gl^\top, \mathbf{g} = g, \mathbf{h}^\top = c^\top \Pi (I + \eta \mathbf{F}), \Delta = \frac{1}{\beta} + \eta c^\top \Pi g \tag{14}$$

is positive real, equivalent to condition (6) for Z_g. By Theorem 1, Z_g is a positive real system if and only if, there exists $\mathbf{X} \in \mathbb{R}^{\nu \times \nu}, g, \beta$ and η such that the following matrix inequalities hold:

$$\Gamma_g := \begin{bmatrix} (S - gl^\top)^\top \mathbf{X} + \mathbf{X}(S - gl^\top) & \mathbf{X}g - \mathbf{h} \\ g^\top \mathbf{X} - \mathbf{h}^\top & -2\left(\frac{1}{\beta} + \eta c^\top \Pi g\right) \end{bmatrix} \preceq 0$$
$$\mathbf{X} \succ 0. \tag{15}$$

Note that the first matrix inequality in (15) is not linear in the parameters $\mathbf{X}, g, \beta$ and η. To the best of our knowledge, we are not aware of any transformation rendering (15) an LMI. Hence, in the sequel, we reformulate the problem of absolute stability preserving moment matching for $K(s)$, as passivity preserving moment matching for $Z(s)$.

Let $S \in \mathbb{R}^{\nu\times\nu}$ be such that $\sigma(S)\cap\sigma(A) = \emptyset$ and $l^\top \in \mathbb{R}^{1\times\nu}$ such that $(l^\top, S)$ is observable, where the interpolation points $\sigma(S)$ are selected following Remark 1. Let Π be the unique solution of (9). Then a family of reduced order models parameterized in g, matching the moments of $Z(s)$ at $\sigma(S)$ is given by $\widehat{Z}_g(s)$ with a state-space realization

$$\widehat{\mathbf{F}} = S - gl^\top,\ \widehat{\mathbf{g}} = g,\ \widehat{\mathbf{h}}^\top = \mathbf{c}^\top\Pi,\ \widehat{\Delta} = \mathbf{d}. \tag{16}$$

The main difference between (16) and (14) lies in the definition of $(\mathbf{h}, \Delta)$ and $(\widehat{\mathbf{h}}, \widehat{\Delta})$, respectively. Define

$$\mathcal{G} = \{g \in \mathbb{R}^\nu : \sigma(S - gl^\top) \subseteq \mathbb{C}^0 \cup j\mathbb{R}\}. \tag{17}$$

Moreover, we consider the subfamily of models $Z_g(s)$ $\widehat{\Sigma}_{\mathcal{G}} = \{Z_g(s) : g \in \mathcal{G}\}$. We search for $g \in \mathcal{G}$ such that $\widehat{Z}_g$ is passive, i.e, find $\widehat{\mathbf{X}} \in \mathbb{R}^{\nu\times\nu}$, g, β and η such that the following matrix inequalities hold:

$$\widehat{\Gamma}_g := \begin{bmatrix} (S-gl^\top)^\top\widehat{\mathbf{X}} + \widehat{\mathbf{X}}(S-gl^\top) & \widehat{\mathbf{X}}\,g - \widehat{\mathbf{h}} \\ g^\top\widehat{\mathbf{X}} - \widehat{\mathbf{h}}^\top & -2\mathbf{d} \end{bmatrix} \preceq 0$$
$$\widehat{\mathbf{X}} \succ 0. \tag{18}$$

Despite the fact that the first matrix inequality in (18) is not linear in the parameters $\widehat{\mathbf{X}}, g, \beta$ and η, we show below that we can still find these parameters satisfying (18) by convex tools. In detail, the next result provides a selection of g such that $\widehat{Z}_g$ is positive real.

Theorem 6. Consider the system $Z(s)$ as in (7), with a minimal state-space realization defined by (8), assumed passive and let $0 \prec X \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^+$, $\eta \in \mathbb{R}$ be a solution of LMI (1). Consider Π, the unique solution of (9), with rank $\Pi = \nu$, and the family of reduced order models $\widehat{\Sigma}_{\mathcal{G}}$ as in (16), matching ν moments of $Z(s)$, at $\sigma(S)$. Then, for g a solution of the linear system of equations

$$(\Pi^\top X\Pi)g = \Pi^\top Xb, \tag{19}$$

the matrix inequalities (18) are satisfied by

$$0 \prec \widehat{\mathbf{X}} = \Pi^\top X\Pi \in \mathbb{R}^{\nu\times\nu},\ \beta \text{ and } \eta, \tag{20}$$

i.e., $\widehat{Z}_g \in \widehat{\Sigma}_{\mathcal{G}}$, with g satisfying (19) is passive. Moreover, if either $\eta = 0$, or $\eta \neq 0$ and $-1/\eta \notin \sigma(S)$, then the reduced order model

$$\widehat{K}_g(s) = \frac{1}{1+\eta s}\left[\widehat{Z}_g(s) - \frac{1}{\beta}\right], \tag{21}$$

matches ν moments of $K(s)$ at $\sigma(S)$.

Note that, by Theorem 1, if $\widehat{Z}_g$ is passive then $\widehat{Z}_g \in \widehat{\Sigma}_{\mathcal{G}}$.

Proof. The proof is omitted due to space limitations.

Remark 2. The reduced order system $\widehat{K}_g$ in (21) is of order $\nu + 1$ if $-1/\eta$ is not a zero of $\widehat{Z}_g$, i.e., $\widehat{Z}_g(-1/\eta) \neq 0$. Following arguments in (Ionescu et al., 2022, Section 3.2), $-1/\eta$ is a zero of $\widehat{Z}_g$ if $\zeta^\top\Pi g = 0$, where $\zeta \in \mathbb{R}^n$ is such that $-1/\eta\cdot\zeta^\top = \zeta^\top\mathbf{A} + \mathbf{c}^\top$, with $\mathbf{A}, \mathbf{c}^\top$ as in (8). The latter relation yields $\zeta^\top(\mathbf{A} + 1/\eta\cdot I) = -\mathbf{c}^\top$. Since Assumption 1 holds, $\zeta^\top = -\mathbf{c}^\top(\mathbf{A} + 1/\eta\cdot I)^{-1}$. Then, if $-1/\eta$ is a zero of $\widehat{Z}_g$, $b \in \ker(-(\mathbf{A}+1/\eta\cdot I)^{-1}\mathbf{c}^\top\Pi(\Pi^\top X\Pi)^{-1}\Pi^\top X)$. However the latter relation is conservative, therefore, in general, $-1/\eta$ is not a zero of $\widehat{Z}_g(s)$.

The results in this section prove that the reduced order model $\widehat{K}_g$ preserves the absolute stability of the original system K, for any nonlinear function φ satisfying the sector condition

$$0 \leq \varphi(z)z \leq \beta z^2, \quad \forall z \in \mathbb{R}, \tag{22}$$

with the same value of β as in (4) valid for the original Lur'e system. Furthermore, the absolute stability of $\widehat{K}_g$ is preserved with the same value of η as in Theorem 2 for K.

The results involve solving the LMI (1) for the original system. Therefore, a relevant open problem is to directly solve the matrix inequalities (15) or (18), with convex optimization tools. For instance, a possible solution is to fix g as in (19) and then (15) or (18) become an LMI in the unknowns $(\mathbf{X}, \beta, \eta)$ or $(\widehat{\mathbf{X}}, \beta, \eta)$ respectively, of order $\nu + 1 < n + 1$, efficiently solved with convex tools.

4. ILLUSTRATIVE EXAMPLE

Consider the model of a gyroscope, given by the linear unstable system described by the differential equations

$$\dot{x_1} = -x_1 - u_1(t),\ \dot{x_2} = u_1(t),$$
$$\dot{x_3} = (\alpha-1)x_1 + \alpha x_2 + \rho u_1(t),\ y_1 = x_3, \tag{23}$$

with the transfer function $K(s) = (\rho s^2 + (1+\rho)s + \alpha)/(s^2(s+1))$. Note that, since $K(s)$ has two poles on the imaginary axis, then $Z(s)$ as in (7) has two poles on the imaginary axis as well and then Theorem 1 cannot be applied. Hence, we use the perturbed transfer function

$$K_\epsilon(s) = \tfrac{\rho s^2+(1+\rho)s+\alpha}{s(s+\epsilon)(s+1)},$$

such that $\lim_{\epsilon\to 0} K_\epsilon(s) = K(s)$. Taking $\epsilon = 10^{-5}$, the perturbed system has one pole at zero (and another one very close to zero), i.e., the perturbed system is still unstable. In simulations, we choose $\rho = 1$ and $\alpha = 2$, such that $K(s)$ and $K_\epsilon(s)$ are irreducible.

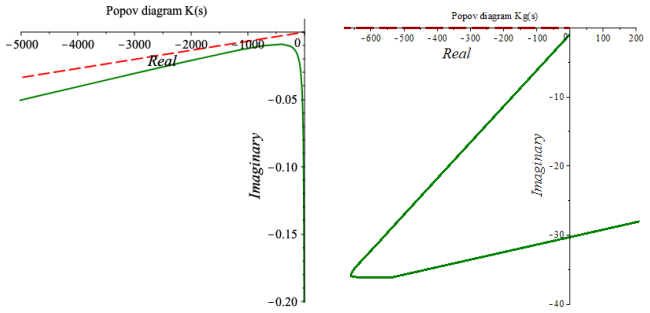
Writing a minimal realization of K_ϵ, using CVX, we solve (1) for the passive system $Z(s)$ as in (8) yielding

$$X = \begin{bmatrix} 4.4371\cdot 10^5 & 2.9608\cdot 10^5 & 2.024 \\ 2.9608\cdot 10^5 & 2.9642\cdot 10^5 & 1.999 \\ 2.024 & 1.999 & 1.5165\cdot 10^4 \end{bmatrix} \succ 0,$$
$$\beta = 0.206,\ \eta = 1.482\cdot 10^5. \tag{24}$$

Hence $K(s)$ is absolutely stable for any static nonlinearity satisfying the sector condition (4), with β and η as in (24), see Figure 1a. We now apply passivity preserving moment matching on $Z(s)$ as in (8). Letting $S = 1/2, l = 1$, we compute the family of first order systems $\{\widehat{Z}_g\}_g$, given by (16), matching the moment of Z in $s = 1/2$, with $\Pi^\top = [0.667\ 1.333\ 2.667]$, the unique solution of (9). Applying (20), we select $g = 0.552$ given by (19). The resulting $\widehat{Z}_g$ is $\widehat{Z}_{0.552}(s) = 10^5\cdot(1.482s + 2.806)/(s + 0.05223)$, passive with $\widehat{X} = \Pi^\top X\Pi = 1.251\cdot 10^6 > 0$, solving LMI (18). Using the multiplier $1 + \eta s$, with η in (24), we construct the second order model

$$\widehat{K}_g(s) = \widehat{K}_{0.552}(s) = 10^5\cdot\tfrac{1.482e05s+2.806e05}{1.482e05s^2+7742s+0.05223}.$$

The system is an approximation of $K(s)$ with the following properties: $\widehat{K}_{0.552}$ is absolutely stable for any nonlinearity satisfying the sector condition (22) with β and η as in (24) and $\widehat{K}_{0.552}(1/2) = K(1/2)$. Figure 1 compares the Popov plots of both the third order system K_ϵ and the second order approximation $\widehat{K}_{0.552}$. Figure 1b shows that absolute stability is preserved by the approximation $\widehat{K}_{0.552}$.



(a) $K(s)$ (b) $K_g(s)$

Fig. 1. Popov plots (Popov diagrams–green solid; Popov lines–dashed red) of: the original 3rd order system K– (a) and the second order approximation K_g– (b)

5. CONCLUSIONS

In this paper, we addressed the problem of moment matching-based model order reduction for Lur'e systems, achieving absolute stability for a given sector. We have exemplified that moment matching directly applied both to the linear system and to the linearized version of the Lur'e system does not necessarily preserve absolute stability. For linear systems with poles in the left complex half-plane and mostly simple poles on the imaginary axis, we use the passivity condition of the Popov criterion for absolute stability. In the family of reduced order models achieving moment matching, we computed the free parameters that preserve the passivity condition such that the reduced order model is absolutely stable in a sector, using a solution of a linear system and a linear matrix inequality.

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