Preprints of the 23rd IFAC World Congress
Busan, Republic of Korea, August 23-28, 2026

Distributed Adaptive Estimation with ISS Guarantees for Sensor Networks with Partially Unknown Source Dynamics

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Abstract: This paper studies distributed adaptive estimation over sensor networks with partially unknown source dynamics. We present parallel continuous-time and discrete-time designs in which each node runs a local adaptive observer and exchanges information over a directed graph. For both time scales, we establish stability of the network coupling operators, prove boundedness of all internal signals, and show convergence of each node's estimate to the source despite model uncertainty and disturbances. We further derive input-to-state stability (ISS) bounds that quantify robustness to bounded process noise. A key distinction is that the discrete-time design uses constant adaptive gains and per-step regressor normalization to handle sampling effects, whereas the continuous-time design does not. A unified Lyapunov framework links local observer dynamics with graph topology. Simulations on star, cyclic, and path networks corroborate the analysis, demonstrating accurate tracking, robustness, and scalability with the number of sensing nodes.

Keywords: Adaptive Estimation, Distributed Systems, Input-to-State Stability (ISS), Multi-Agent Systems, Networked Observers

1. INTRODUCTION

Distributed state estimation over sensor networks has emerged as an underlying problem in multi-agent systems, with applications spanning environmental monitoring, surveillance systems, distributed robotics, and cooperative control [Hu et al. (2020); Yang et al. (2022)]. In these scenarios, a network of sensing agents must collectively estimate the state of a target (source) whose dynamics may be known or only partially known. The distributed nature of the problem necessitates algorithms that coordinate local observations and computations over networks.

Existing approaches to distributed state estimation broadly fall into consensus-based observers and distributed Kalman filtering. Consensus-based methods [Wang et al. (2022); Vivas et al. (2015); Rong et al. (2020)] achieve agreement among agents through iterative averaging but typically assume the source dynamics are perfectly known. Distributed Kalman filters [Olfati-Saber (2007); Li et al. (2019); Talebi and Werner (2018)] handle stochastic uncertainty but require knowledge of noise statistics and system matrices. Recent extensions [Zhu et al. (2023)] address time-varying topologies and communication constraints, yet a fundamental assumption persists: the source dynamics matrices are known a priori.

In many practical scenarios, however, the source dynamics contain parametric uncertainty—for instance, an environmental process with unknown reaction rates, or a vehicle with uncertain mass and damping coefficients. Recent works have explored distributed adaptive estimation to address such uncertainties, combining local learning with networked coordination [Wang and Meng (2021); Stegagno and Yuan (2019)]. While these studies achieve asymptotic convergence under ideal or noise-free conditions, they typically neglect bounded process disturbances and do not establish input-to-state stability (ISS).

This motivates us to study adaptive distributed estimation in directed sensor networks where each node must simultaneously estimate the source state and identify unknown dynamics while coordinating only with in-neighbors. This paper develops continuous- and discrete-time frameworks for such settings, providing rigorous input-to-state stability (ISS) guarantees via Lyapunov analysis and linear matrix inequality (LMI) conditions, and explicitly revealing how network coupling, adaptation dynamics, and disturbance robustness interact.

Our main contributions are:

(1) A unified Kronecker-based formulation that separates local observer dynamics from network topology;
(2) Continuous-time adaptive observers with convergence and ISS guarantees under spectral/LMI conditions on the coupling operator $\mathcal{H}$;

⋆ This work was supported by the Office of Naval Research under Grant N00014-21-1-2431 and the National Science Foundation under Grant 2208182.

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(3) Discrete-time adaptive observers with regressor normalization and constrained gains, ensuring convergence and ISS guarantees under conditions on $\mathcal{S}$;
(4) Explicit robustness trade-offs among network connectivity, local damping, and source growth, including norm-bounded uncertainty analysis;
(5) Simulations on star, cyclic, path, and larger graphs validating convergence, robustness, and scalability.

Notations. I_p is the $p \times p$ identity, and $\mathrm{diag}\{A_i\} \in \mathbb{R}^{pn\times pm}$ for $i = 1,\dots,p$ forms a block-diagonal matrix with blocks $A_i \in \mathbb{R}^{n\times m}$. Vectors of ones and zeros in $\mathbb{R}^p$ are $\mathbf{1}_p$ and $\mathbf{0}_p$. The Kronecker product is $A \otimes B$, $\mathrm{tr}(A)$ is the trace, $\lambda(A)$ denotes eigenvalues.

For proofs. All proofs are omitted due to space limitations and are provided in the arXiv version [Wafi et al. (2026)].

2. COMMUNICATION NETWORK

We consider a network of $m+1$ agents indexed by $\mathcal{V} = \{0, 1, \dots, m\}$, where agent 0 is the source (system being estimated) and agents $1, \dots, m$ are sensing nodes that run local adaptive observers and exchange information.

Communication is modeled by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and nonnegative weights $\mathcal{W} = \{w_{ij}\}$. An edge $(i, j) \in \mathcal{E}$ means agent i receives information from agent j with weight $w_{ij} > 0$; otherwise $w_{ij} = 0$. The in-neighborhood of agent i is $\mathcal{N}_i := \{\, j \in \mathcal{V} \mid (i, j) \in \mathcal{E} \,\}$. The weights $\{w_{ij}\}$ are algorithmic design coefficients (not physical signal strengths); how much weight agent i places on information from agent j.

For analysis, we partition $\mathcal{G}$ into two induced subgraphs: (i) the sensing-only subgraph $\mathcal{G}_m = (\mathcal{V}_m, \mathcal{E}_m, \mathcal{W}_m)$ with $\mathcal{V}_m = \{1, \dots, m\}$; (ii) the source-to-sensing subgraph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{W}_0)$ with $\mathcal{V}_0 = \{0\} \cup \{i \in \mathcal{V}_m : (i, 0) \in \mathcal{E}\}$. Thus, $\mathcal{E}_m$ contains sensing-to-sensing links and $\mathcal{E}_0$ contains source-to-sensing links (see Fig. 1).

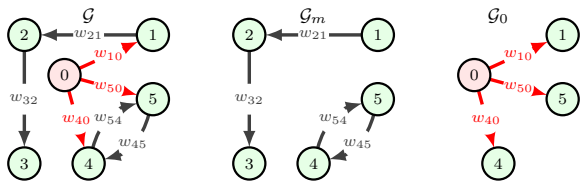


Fig. 1. Example of a graph $\mathcal{G}$ with $m = 5$, sensing-to-sensing subgraph $\mathcal{G}_m$, and source-to-sensing subgraph $\mathcal{G}_0$, showing the decoupling and assignment of w_{ij}.

For $\mathcal{G}_m$, define the in-degree $d_i := \sum_{j:(i,j)\in\mathcal{E}_m} w_{ij}$, $i \in \mathcal{V}_m$ and $\mathbb{D}_m = \mathrm{diag}\{d_1, \dots, d_m\}$. The adjacency and Laplacian of the sensing subgraph are $[\mathbb{A}_m]_{ij} = w_{ij}$ if $(i, j) \in \mathcal{E}_m$ (with 0 otherwise) and $\mathbb{L}_m := \mathbb{D}_m - \mathbb{A}_m$. For $\mathcal{G}_0$, source-to-sensing connections are encoded by a diagonal matrix $\mathbb{A}_0 := \mathrm{diag}\{w_{10}, \dots, w_{m0}\}$ where $[\mathbb{A}_0]_{ii} = w_{i0}$ if $(i, 0) \in \mathcal{E}_0$ and 0 otherwise.

Let $\mathbb{W} := \mathrm{diag}\{w_1, \dots, w_m\}$ with $w_i := d_i + w_{i0}$ (total incoming weight). The augmented Laplacian is

$$\mathbb{L} := \mathbb{L}_m + \mathbb{A}_0 = \mathbb{W} - \mathbb{A}_m.$$

We propose the *balanced* condition $\mathbb{W} = I_m$, or $d_i + w_{i0} = 1$ $(\sum_{j\in\mathcal{N}_i} w_{ij} = 1), \forall i \in \mathcal{V}_m$. Under this condition [1],

$$(\mathbb{L} - \mathbb{A}_0)\mathbf{1}_m = \mathbf{0}_m \quad \Leftrightarrow \quad (\mathbb{A}_m + \mathbb{A}_0)\mathbf{1}_m = \mathbf{1}_m. \tag{1}$$

Remark 1. Assume: (i) at least one node directly accesses the source ($w_{i0} > 0$ for some i), and (ii) every node is reachable from the source through a directed path in $\mathcal{G}$. Using (1), these imply that the augmented Laplacian $\mathbb{L}$ is positive stable, i.e., $\Re\{\lambda_i(\mathbb{L})\} > 0$ for all i.

This property ensures that $\mathbb{L}$ admits a Lyapunov characterization essential for the input-to-state (ISS) analysis.

3. PROBLEM FORMULATION

We study distributed estimation where the source has unknown linear dynamics with bounded perturbations, each node runs a local adaptive observer and exchanges estimates with neighbors, and only a subset of nodes directly senses the source, propagating through $\mathcal{G}$.

We consider both continuous-time and discrete-time cases, which differ fundamentally in adaptive-law construction, particularly regarding constant-gain adaptation and regressor normalization. For clarity, we assume full-state measurements in both cases, as in applications such as GPS-based tracking.

3.1 Problem 1: Unknown Continuous-time Dynamics

The unknown continuous-time source evolves as

$$\dot{x}_0(t) = F_0 x_0(t) + G_0 v_0(t) + d_0(t), \tag{2a}$$
$$y_0(t) = x_0(t), \tag{2b}$$

where $x_0(t) \in \mathbb{R}^n$ is the state, $v_0(t) \in \mathbb{R}^p$ is the reference input and $y_0(t) \in \mathbb{R}^n$ is the output. The process noise $d_0(t) \in \mathbb{R}^n$ is bounded as $\|d_0(t)\| \le d^*$ for some constant $d^* > 0$. The matrices F_0 and G_0 are *unknown*, defined as

$$F_0 := F_* + \Delta F \in \mathbb{R}^{n\times n}, \quad G_0 := G_* + \Delta G \in \mathbb{R}^{n\times p}.$$

The nominal parts F_* and G_* are known; the perturbations ΔF and ΔG are unknown but bounded, $\|\Delta F\|_2 \le f$ and $\|\Delta G\|_2 \le g$, for some constants $f, g > 0$.

Sensing node $i \in \{1, \dots, m\}$ implements

$$\dot{x}_i(t) = H_i x_i(t) + (\hat{F}_i(t) - H_i) z_i(t) + \hat{G}_i(t) v_0(t), \tag{3a}$$
$$z_i(t) = \sum\nolimits_{j\in\mathcal{N}_i} w_{ij} x_j(t) + w_{i0} y_0(t), \quad \mathcal{N}_i \subseteq \mathcal{V}, \tag{3b}$$

where $x_i(t) \in \mathbb{R}^n$ is the i-th sensing node estimate. Both $\hat{F}_i(t) \in \mathbb{R}^{n\times n}$ and $\hat{G}_i(t) \in \mathbb{R}^{n\times p}$ are adaptive estimates of F_0 and G_0, respectively, and $H_i \in \mathbb{R}^{n\times n}$ is user-designed Hurwitz matrix. Here, $z_i(t)$ is weighted aggregate of in-neighbor estimates plus (if applicable) source output.

Main objective: Design a distributed adaptive law such that, under (1) and Remark 1, all sensing node estimates are ISS stable with respect to the bounded disturbance on the source, i.e.,

$$\limsup_{t\to\infty} \|x_i(t) - z_i(t)\| \le c_c d^*, \quad \forall i, \tag{4a}$$

for $c_c > 0$ and, in the disturbance-free case $d^* = 0$

$$\lim_{t\to\infty} e_i(t) := x_i(t) - z_i(t) = 0. \tag{4b}$$

[1] If $\mathbb{W} \neq I_m$, normalization via $\tilde{w}_{ij} := w_{ij}/w_i$ for $(i, j) \in \mathcal{E}_m$ and $\tilde{w}_{i0} := w_{i0}/w_i$ yields $\tilde{\mathbb{W}} = I_m$ and preserves (1); this is not required for implementation but simplifies stability analysis.

3.2 Problem 2: Uncertain Discrete-time Dynamics

To avoid confusion with the continuous-time setting, we adopt separate notation in the discrete-time case. The unknown discrete-time source evolves as

$$x_{0,k+1} = A_0 x_{0,k} + B_0 u_{0,k} + \delta_{0,k}, \tag{5a}$$
$$y_{0,k} = x_{0,k}, \tag{5b}$$

where $x_{0,k} \in \mathbb{R}^n$ defines the source state while $u_{0,k} \in \mathbb{R}^p$ and $y_{0,k} \in \mathbb{R}^n$ denote the reference input and the source output. The term $\delta_{0,k} \in \mathbb{R}^n$ represents process noise, bounded as $\|\delta_{0,k}\| \leq \delta^*$ for some constant $\delta^* > 0$. The matrices A_0 and B_0 are *unknown*, denoted as

$$A_0 \coloneqq A_* + \Delta A \in \mathbb{R}^{n\times n}, \quad B_0 \coloneqq B_* + \Delta B \in \mathbb{R}^{n\times p},$$

yet the pair (A_*, B_*) is known with bounds $\|\Delta A\|_2 \leq a$, $\|\Delta B\|_2 \leq b$, for some constants $a, b > 0$.

Next, sensing node $i \in \{1, \dots, m\}$ evolves as

$$x_{i,k+1} = S_i x_{i,k} + (\hat{A}_{i,k} - S_i) z_{i,k} + \hat{B}_{i,k} u_{0,k}, \tag{6a}$$
$$z_{i,k} = \sum\nolimits_{j\in\mathcal{N}_i} w_{ij} x_{j,k} + w_{i0} y_{0,k}, \quad \mathcal{N}_i \subseteq \mathcal{V}, \tag{6b}$$

where $x_{i,k} \in \mathbb{R}^n$ is the i-th sensing node estimate. The terms $\hat{A}_{i,k} \in \mathbb{R}^{n\times n}$ and $\hat{B}_{i,k} \in \mathbb{R}^{n\times p}$ are the estimates of the pair (A_0, B_0) with user-designed Schur-stable matrix $S_i \in \mathbb{R}^{n\times n}$. Finally, $z_{i,k}$ is weighted aggregate of in-neighbor estimates (including the source if $0 \in \mathcal{N}_i$).

Main objective: Design a distributed adaptive law such that, under (1) and Remark 1, all sensing node estimates are ISS stable with respect to the bounded disturbance on the source, i.e.,

$$\limsup_{k\to\infty} \|x_{i,k} - z_{i,k}\| \leq c_{\mathrm{d}} \delta^* \quad \forall i, \tag{7a}$$

for $c_{\mathrm{d}} > 0$ and, in the disturbance-free case $\delta^* = 0$

$$\lim_{k\to\infty} e_{i,k} \coloneqq x_{i,k} - z_{i,k} = 0. \tag{7b}$$

3.3 Unified Notation

To avoid repetition, we define the shorthand notation for the stacked variables (cf. (2a)–(7b)) used in the following sections. For any source signal $\beta_0(t), \beta_{0,k} \in \mathbb{R}^q$,

$$\bar{\beta}_0(t) = \mathbf{1}_m \otimes \beta_0(t) \in \mathbb{R}^{mq}, \tag{8a}$$
$$\bar{\beta}_{0,k} = \mathbf{1}_m \otimes \beta_{0,k} \in \mathbb{R}^{mq}. \tag{8b}$$

For any sensing-node signal $\beta_i(t), \beta_{i,k} \in \mathbb{R}^q$,

$$\bar{\beta}(t) = [\beta_1^\top(t), \dots, \beta_m^\top(t)]^\top \in \mathbb{R}^{mq}, \tag{9a}$$
$$\bar{\beta}_k = [\beta_{1,k}^\top, \dots, \beta_{m,k}^\top]^\top \in \mathbb{R}^{mq}. \tag{9b}$$

For matrices $E_0, E_i, \hat{E}_i(t), \hat{E}_{i,k} \in \mathbb{R}^{q_1\times q_2}$,

$$\mathbf{E}_0 = I_m \otimes E_0 \in \mathbb{R}^{mq_1\times mq_2}, \tag{10a}$$
$$\mathbf{E} = \mathrm{diag}\{E_1, \dots, E_m\} \in \mathbb{R}^{mq_1\times mq_2}, \tag{10b}$$
$$\hat{\mathbf{E}}(t) = \mathrm{diag}\{\hat{E}_1(t), \dots, \hat{E}_m(t)\} \in \mathbb{R}^{mq_1\times mq_2}, \tag{10c}$$
$$\hat{\mathbf{E}}_k = \mathrm{diag}\{\hat{E}_{1,k}, \dots, \hat{E}_{m,k}\} \in \mathbb{R}^{mq_1\times mq_2}. \tag{10d}$$

With these definitions, we are ready to study the adaptive laws and their stability in both time scales.

4. ADAPTIVE ESTIMATION FOR PROBLEM 1

4.1 Error Dynamics in Networked Systems

In network-level form, the source evolves as

$$\dot{\bar{x}}_0(t) = \mathbf{F}_0 \bar{x}_0(t) + \mathbf{G}_0 \bar{v}_0(t) + \bar{d}_0(t), \tag{11a}$$
$$\bar{y}_0(t) = \bar{x}_0(t), \tag{11b}$$

while the sensing nodes obey

$$\dot{\bar{x}}(t) = \mathbf{H}\bar{x}(t) + (\hat{\mathbf{F}}(t) - \mathbf{H})\bar{z}(t) + \hat{\mathbf{G}}(t)\bar{v}_0(t), \tag{12a}$$
$$\bar{z}(t) = (\mathbb{A}_m \otimes I_n)\bar{x}(t) + (\mathbb{A}_0 \otimes I_n)\bar{y}_0(t). \tag{12b}$$

Here, the signals $\bar{x}_0(t)$, $\bar{v}_0(t)$, $\bar{d}_0(t)$, and $\bar{y}_0(t)$ follow (8a) whereas $\bar{x}(t)$ and $\bar{z}(t)$ follow (9a). The matrices $\mathbf{F}_0$ and $\mathbf{G}_0$ are expressed as in (10a) while the matrix $\mathbf{H}$ is denoted as in (10b). Also, both $\hat{\mathbf{F}}(t)$ and $\hat{\mathbf{G}}(t)$ are defined as in (10c).

Before proceeding, we denote the following assumption.

Assumption 1. There exist optimal parameter values of the source dynamics in (11a), denoted as $\hat{\mathbf{F}}^*$ and $\hat{\mathbf{G}}^*$, such that $\hat{\mathbf{F}}^* \coloneqq \mathbf{F}_0$ and $\hat{\mathbf{G}}^* \coloneqq \mathbf{G}_0$.

Then, we introduce the parameter errors

$$\Phi_{\mathrm{c}}(t) \triangleq \hat{\mathbf{F}}(t) - \mathbf{F}_0, \quad \Psi_{\mathrm{c}}(t) \triangleq \hat{\mathbf{G}}(t) - \mathbf{G}_0. \tag{13}$$

From (4b), the estimation error defined via (9a) is

$$\bar{e}(t) = (\mathbb{L} \otimes I_n)\bar{x}(t) - (\mathbb{A}_0 \otimes I_n)\bar{y}_0(t) \coloneqq \bar{x}(t) - \bar{z}(t). \tag{14}$$

Thus, the estimation error dynamics inspired by [Wafi and Siami (2024)] become

$$\begin{aligned}
\dot{\bar{e}}(t) &= (\mathbb{L} \otimes I_n)\dot{\bar{x}}(t) - (\mathbb{A}_0 \otimes I_n)\dot{\bar{x}}_0(t) \\
&= (\mathbb{L} \otimes I_n)\big[\mathbf{H}\bar{x}(t) + (\hat{\mathbf{F}}(t) - \mathbf{H})\bar{z}(t) + \hat{\mathbf{G}}(t)\bar{v}_0(t)\big] \\
&\quad - (\mathbb{A}_0 \otimes I_n)\big[\mathbf{F}_0\bar{x}_0(t) + \mathbf{G}_0\bar{v}_0(t) + \bar{d}_0(t)\big] \\
&= (\mathbb{L} \otimes I_n)\big[\hat{\mathbf{F}}\bar{z}(t) + \mathbf{H}\bar{e}(t) + \hat{\mathbf{G}}(t)\bar{v}_0(t)\big] \\
&\quad + (\mathbb{A}_m \otimes I_n)\mathbf{F}_0\bar{e}(t) \\
&\quad - (\mathbb{L} \otimes I_n)\big[\mathbf{F}_0\bar{z}(t) + \mathbf{G}_0\bar{v}_0(t) + \bar{d}_0(t)\big] \\
&= \mathcal{H}\bar{e}(t) + (\mathbb{L} \otimes I_n)\Xi_{\mathrm{c}}(t)\bar{\eta}_{\mathrm{c}}(t) - \Omega\bar{d}_0(t),
\end{aligned} \tag{15}$$

where

$$\mathcal{H} \coloneqq \big[(\mathbb{L} \otimes I_n)\mathbf{H} + (\mathbb{A}_m \otimes I_n)\mathbf{F}_0\big], \tag{16}$$

adaptive parameter $\Xi_{\mathrm{c}}(t) \coloneqq [\Phi_{\mathrm{c}}(t), \Psi_{\mathrm{c}}(t)]$, time-varying regressor $\bar{\eta}_{\mathrm{c}}(t) \coloneqq [\bar{z}^\top(t), \bar{v}_0^\top(t)]^\top$, and $\Omega \coloneqq (\mathbb{L} \otimes I_n)$.

4.2 Stability of the Coupling Operator $\mathcal{H}$

The stability of (15) depends on whether or not $\mathcal{H}$ in (16) is Hurwitz. We now establish sufficient conditions.

Proposition 1. Let $\mathcal{H} \coloneqq (\mathbb{L} \otimes I_n)\mathbf{H} + (\mathbb{A}_m \otimes I_n)\mathbf{F}_0$ where $\mathbf{H} = \mathrm{diag}\{H_1, \dots, H_m\}$ and $\mathbf{F}_0 = I_m \otimes F_0$. If there exists $P_{\mathbb{L}} = P_{\mathbb{L}}^\top \succ 0$ such that

$$\begin{aligned}
&\mathbf{H}^\top(\mathbb{L}^\top P_{\mathbb{L}} \otimes I_n) + (P_{\mathbb{L}}\mathbb{L} \otimes I_n)\mathbf{H} \\
&\quad + \mathbf{F}_0^\top(\mathbb{A}_m^\top P_{\mathbb{L}} \otimes I_n) + (P_{\mathbb{L}}\mathbb{A}_m \otimes I_n)\mathbf{F}_0 \prec 0,
\end{aligned}$$

then $\mathcal{H}$ is Hurwitz.

The following corollary provides a more explicit robust sufficient condition in the presence of norm-bounded uncertainty in the source dynamics.

Corollary 1. Let $F_0 = F_* + \Delta F$, where $\|\Delta F\|_2 \leq f$. If there exist $P_{\mathbb{L}} = P_{\mathbb{L}}^\top \succ 0$ and $\tau > 0$ such that

$$\begin{aligned}
&\mathbf{H}^\top(\mathbb{L}^\top P_{\mathbb{L}} \otimes I_n) + (P_{\mathbb{L}}\mathbb{L} \otimes I_n)\mathbf{H} \\
&\quad + \mathbf{F}_*^\top(\mathbb{A}_m^\top P_{\mathbb{L}} \otimes I_n) + (P_{\mathbb{L}}\mathbb{A}_m \otimes I_n)\mathbf{F}_* \\
&\quad + \tau I_m \otimes I_n \prec 0,
\end{aligned} \tag{17}$$

and

$$2\|P_{\mathbb{L}}\mathbb{A}_m\|_2 f < \tau, \tag{18}$$

then $\mathcal{H}$ is Hurwitz for all admissible ΔF.

With the stability of the coupling operator $\mathcal{H}$ ensured by Proposition 1 and Corollary 1, we proceed to design adaptive laws for the parameter estimates.

4.3 Adaptive Laws and Stability

We first retain the disturbance-free result (asymptotic convergence), then extend it to input-to-state stability (ISS) in the presence of bounded source disturbance in.

Theorem 1. Consider the networked system in (11a)–(12a), with error dynamics (15) and $\bar{d}_0(t) = \mathbf{0}_{mn}$. Let the communication graph $(\mathcal{V}, \mathcal{E})$ satisfies the balanced and source-reachability conditions (cf. (1) and Remark 1), and that the matrix $\mathcal{H}$ in (16) is Hurwitz. Let $P_{\mathrm{c}} = P_{\mathrm{c}}^{\top} \succ 0$ and $Q_{\mathrm{c}} = Q_{\mathrm{c}}^{\top} \succ 0$ satisfy the Lyapunov equation

$$\mathcal{H}^{\top} P_{\mathrm{c}} + P_{\mathrm{c}} \mathcal{H} = -Q_{\mathrm{c}}. \tag{19}$$

Using $\Phi_{\mathrm{c}}(t)$ and $\Psi_{\mathrm{c}}(t)$ as in (13), then the adaptive laws

$$\dot{\Phi}_{\mathrm{c}}(t) = -\Gamma_{\phi} (\mathbb{L} \otimes I_n)^{\top} P_{\mathrm{c}} \bar{e}(t) \bar{z}^{\top}(t), \tag{20a}$$

$$\dot{\Psi}_{\mathrm{c}}(t) = -\Gamma_{\psi} (\mathbb{L} \otimes I_n)^{\top} P_{\mathrm{c}} \bar{e}(t) \bar{v}_0^{\top}(t), \tag{20b}$$

with gain matrices $\Gamma_{\phi}, \Gamma_{\psi} \succ 0$, guarantee that all closed-loop signals remain bounded and that $\bar{e}(t) \to 0$ as $t \to \infty$.

Remark 2. From LaSalle's invariance principle: the solutions of the closed-loop system approach the largest invariant set in $\mathbb{S} = \{\bar{e}(t) \in \mathbb{R}^{mn} : \dot{V}(\bar{e}(t), \Phi_{\mathrm{c}}(t), \Psi_{\mathrm{c}}(t)) = 0\}$, implying $\bar{e}(t) \to 0$ and bounded $\Phi_{\mathrm{c}}(t), \Psi_{\mathrm{c}}(t)$.

We now establish an ISS property when the source is subject to a bounded disturbance $\|d_0(t)\| \le d^*$ with the same adaptive laws (20).

Theorem 2. Consider the networked system (11a)–(12a) with error dynamics (15), $\Omega \coloneqq (\mathbb{L} \otimes I_n)$ and $\|\bar{d}_0(t)\| \le \bar{d}^*$. Assume the graph satisfies (1) and Remark 1, and $\mathcal{H}$ in (16) is Hurwitz. Let $P_{\mathrm{c}} = P_{\mathrm{c}}^{\top} \succ 0$ and $Q_{\mathrm{c}} = Q_{\mathrm{c}}^{\top} \succ 0$ solve

$$\mathcal{H}^{\top} P_{\mathrm{c}} + P_{\mathrm{c}} \mathcal{H} = -Q_{\mathrm{c}}. \tag{21}$$

Apply the adaptive laws (20) with $\Gamma_{\phi}, \Gamma_{\psi} \succ 0$. Then there exists a constant $c_{\mathrm{c}}(P_{\mathrm{c}}, Q_{\mathrm{c}}, \Omega) > 0$, such that

$$\limsup_{t \to \infty} \|\bar{e}(t)\| \le c_{\mathrm{c}} \bar{d}^*. \tag{22}$$

In particular, if $\bar{d}^* = 0$, then $\bar{e}(t) \to 0$ as $t \to \infty$.

To sum up, Theorem 1 establishes asymptotic convergence in the disturbance-free case, while Theorem 2 provides ISS under bounded disturbances. In discrete time, stability additionally requires fixed gains and regressor normalization.

5. ADAPTIVE ESTIMATION FOR PROBLEM 2

5.1 Error Dynamics in Networked Systems

In network-level form, the discrete-time source evolves as

$$\bar{x}_{0,k+1} = \mathbf{A}_0 \bar{x}_{0,k} + \mathbf{B}_0 \bar{u}_{0,k} + \bar{\delta}_{0,k}, \tag{23a}$$

$$\bar{y}_{0,k} = \bar{x}_{0,k}, \tag{23b}$$

and the sensing node dynamics implement

$$\bar{x}_{k+1} = \mathbf{S} \bar{x}_k + (\hat{\mathbf{A}}_k - \mathbf{S}) \bar{z}_k + \hat{\mathbf{B}}_k \, \bar{u}_{0,k}, \tag{24a}$$

$$\bar{z}_k = (\mathbb{A}_m \otimes I_n) \bar{x}_k + (\mathbb{A}_0 \otimes I_n) \bar{y}_{0,k}. \tag{24b}$$

The quantities $\bar{x}_{0,k}$, $\bar{u}_{0,k}$, $\bar{\delta}_{0,k}$, and $\bar{y}_{0,k}$ are defined via (8b), while $\bar{x}_k$ and $\bar{z}_k$ use the stacking in (9b). The matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ follow (10a) while the matrix $\mathbf{S}$ follows (10b). The estimate matrices $\hat{\mathbf{A}}_k$ and $\hat{\mathbf{B}}_k$ are denoted as in (10d).

We first assume the existence of the optimal estimates.

Assumption 2. There exist optimal parameter values of the source dynamics in (23a), denoted as $\hat{\mathbf{A}}^*$ and $\hat{\mathbf{B}}^*$, such that $\hat{\mathbf{A}}^* \coloneqq \mathbf{A}_0$ and $\hat{\mathbf{B}}^* \coloneqq \mathbf{B}_0$.

Using

$$\bar{e}_k = (\mathbb{L} \otimes I_n) \bar{x}_k - (\mathbb{A}_0 \otimes I_n) \bar{y}_{0,k} = \bar{x}_k - \bar{z}_k, \tag{25}$$

and introducing the parameter errors

$$\Phi_{\mathrm{d},k} \triangleq \hat{\mathbf{A}}_k - \mathbf{A}_0, \quad \Psi_{\mathrm{d},k} \triangleq \hat{\mathbf{B}}_k - \mathbf{B}_0, \tag{26}$$

the estimation error dynamics, with structure analogous to (15), can be expressed as

$$\bar{e}_{k+1} = \mathcal{S} \bar{e}_k + (\mathbb{L} \otimes I_n) \Xi_{\mathrm{d},k} \bar{\eta}_{\mathrm{d},k} - \Omega \bar{\delta}_{0,k}. \tag{27}$$

Recall that $\Omega \coloneqq (\mathbb{L} \otimes I_n)$, then

$$\mathcal{S} \coloneqq \big[(\mathbb{L} \otimes I_n) \mathbf{S} + (\mathbb{A}_m \otimes I_n) \mathbf{A}_0 \big], \tag{28}$$

adaptive parameter $\Xi_{\mathrm{d},k} \coloneqq [\Phi_{\mathrm{d},k}, \Psi_{\mathrm{d},k}]$ and time-varying regressor $\bar{\eta}_{\mathrm{d},k} = [\eta_{\mathrm{d}_1,k}^{\top}, \ldots, \eta_{\mathrm{d}_m,k}^{\top}]^{\top} \coloneqq [\bar{z}_k^{\top}, \bar{u}_{0,k}^{\top}]^{\top}$.

5.2 Stability of the Coupling Operator $\mathcal{S}$

To guarantee exponential convergence in the discrete-time setting, we establish a sufficient condition ensuring that the coupled operator $\mathcal{S}$ is Schur-stable.

Proposition 2. Let $\mathcal{S} \coloneqq (\mathbb{L} \otimes I_n) \mathbf{S} + (\mathbb{A}_m \otimes I_n) \mathbf{A}_0$ where $\mathbf{S} = \mathrm{diag}\{S_1, \ldots, S_m\}$ and $\mathbf{A}_0 = I_m \otimes A_0$. If there exists $P_{\mathbb{L}} = P_{\mathbb{L}}^{\top} \succ 0$ such that

$$\mathcal{S}^{\top} (P_{\mathbb{L}} \otimes I_n) \mathcal{S} - (P_{\mathbb{L}} \otimes I_n) \prec 0,$$

then $\mathcal{S}$ is Schur-stable.

The following corollary extends this result to the case of norm-bounded uncertainty in the source dynamics, ensuring robustness of the Schur stability condition.

Corollary 2. Let $A_0 = A_* + \Delta A$, $\|\Delta A\|_2 \le a$, and define $\mathbf{A}_* = I_m \otimes A_*$ and $\Delta \mathbf{A} = I_m \otimes \Delta A$. If there exist $P_{\mathbb{L}} = P_{\mathbb{L}}^{\top} \succ 0$ and $\tau > 0$ such that

$$\mathcal{S}_*^{\top} (P_{\mathbb{L}} \otimes I_n) \mathcal{S}_* - (P_{\mathbb{L}} \otimes I_n) + \tau I_m \otimes I_n \prec 0,$$

where $\mathcal{S}_* = (\mathbb{L} \otimes I_n) \mathbf{S} + (\mathbb{A}_m \otimes I_n) \mathbf{A}_*$, $P = P_{\mathbb{L}} \otimes I_n$, and the uncertainty contribution satisfies the quadratic bound

$$\Delta_{\mathcal{S}}^{\top} P \Delta_{\mathcal{S}} + \mathcal{S}_*^{\top} P \Delta_{\mathcal{S}} + \Delta_{\mathcal{S}}^{\top} P \mathcal{S}_* \preceq \tau I_m \otimes I_n,$$

with $\Delta_{\mathcal{S}} = (\mathbb{A}_m \otimes I_n) \Delta \mathbf{A}$, then $\mathcal{S}$ is Schur-stable for all admissible ΔA.

Next, we analyze how the choice of the adaptive gains and the normalization mechanism affect performance and robustness, two aspects absent in the continuous-time case.

5.3 Normalized Adaptive Laws and Stability

In (27), the estimation and parameter errors are coupled, motivating a regression-based adaptive update for $\Xi_{\mathrm{d},k}$. We first consider the disturbance-free case $\bar{\delta}_{0,k} = \mathbf{0}_{mn}$. Define the prediction mismatch [Fisher and Annaswamy (2023)]

$$\bar{\epsilon}_{k+1} \coloneqq \Xi_{\mathrm{d},k} \bar{\eta}_{\mathrm{d},k} = \big[(\mathbb{L}^{\top} \mathbb{L})^{-1} \mathbb{L}^{\top} \otimes I_n \big] (\bar{e}_{k+1} - \mathcal{S} \bar{e}_k), \tag{29}$$

where $\bar{\epsilon}_{k+1} = [\epsilon_{1,k+1}^{\top}, \ldots, \epsilon_{m,k+1}^{\top}]^{\top}$, with quadratic loss

$$\mathcal{J}_k = \tfrac{1}{2} \|\bar{\epsilon}_{k+1}\|^2 = \tfrac{1}{2} \|\Xi_{\mathrm{d},k} \bar{\eta}_{\mathrm{d},k}\|^2, \tag{30}$$

whose gradient is

$$\nabla \mathcal{J}_k = \bar{\epsilon}_{k+1} \bar{\eta}_{\mathrm{d},k}^{\top}. \tag{31}$$

Since $\mathcal{J}_k$ is generally not strongly convex due to the time-varying regressor norm $\|\bar{\eta}_{\mathrm{d},k}\|^2$, we normalize by

$$N_k = \max\{\varphi, \|\bar{\eta}_{\mathrm{d},k}\|^2\}, \qquad \varphi > 0, \tag{32}$$

(e.g., $\varphi = 0.01$) yielding the normalized cost

$$\mathbf{J}_k = \frac{\|\bar{\epsilon}_{k+1}\|^2}{2N_k}, \qquad \nabla \mathbf{J}_k = \frac{\bar{\epsilon}_{k+1}\bar{\eta}_{\mathrm{d},k}^\top}{N_k}. \tag{33}$$

However, since N_k requires global information (not feasible for networked systems as agent i does not have access to global measurements to compute $\|\bar{\eta}_{\mathrm{d},k}\|^2$), we adopt local greedy estimates $\hat{N}_{i,k} = \max\{\varphi, \|\eta_{\mathrm{d}_i,k}\|^2\}$, leading to

$$\nabla \hat{\mathbf{J}}_k = \mathrm{diag}\left\{ \frac{\epsilon_{1,k+1}\eta_{\mathrm{d}_1,k}^\top}{\hat{N}_{1,k}}, \ldots, \frac{\epsilon_{m,k+1}\eta_{\mathrm{d}_m,k}^\top}{\hat{N}_{m,k}} \right\}. \tag{34}$$

The adaptive law is then

$$\hat{\Xi}_{\mathrm{d},k+1}^\top = \hat{\Xi}_{\mathrm{d},k}^\top - \gamma \nabla \hat{\mathbf{J}}_k, \qquad \gamma > 0. \tag{35}$$

where γ is a constant adaptive gain, common to all agents for simplicity of analysis. We first establish boundedness of the adaptive iterates.

Lemma 1. Under (35), let $\Xi_{\mathrm{d},k} := \hat{\Xi}_{\mathrm{d},k} - \Xi^*_{\mathrm{d},k}$, where $\Xi^*_{\mathrm{d},k}$ is the ideal parameter matrix. Then $\Xi_{\mathrm{d},k}$ remains bounded $\forall k$ if $0 < \gamma < 2$, with Lyapunov function $V_k = \|\Xi_{\mathrm{d},k}\|_F^2$.

We next analyze the closed-loop error dynamics.

Theorem 3. Consider (27) with $\bar{\delta}_{0,k} = \mathbf{0}_{mn}$, together with (29) and (35). Suppose $\varphi > 0$ and $0 < \gamma < 2$. If $\mathcal{S}$ in (28) is Schur-stable, then $\lim_{k\to\infty} \|\bar{e}_k\| = 0$.

Theorem 3 shows asymptotic stability in the disturbance-free case. We now reintroduce bounded process noise in the source and show that the same normalized adaptive law renders the estimation error ISS with respect to $\bar{\delta}_{0,k}$—the discrete-time counterpart of Theorem 2.

Theorem 4. Consider (23a)–(24a) with error dynamics (27), $\Omega := (\mathbb{L} \otimes I_n)$, and $\|\bar{\delta}_{0,k}\| \le \bar{\delta}^*$. Assume $\mathcal{S}$ is Schur-stable, and let $P_{\mathrm{d}} = P_{\mathrm{d}}^\top \succ 0$ and $Q_{\mathrm{d}} = Q_{\mathrm{d}}^\top \succ 0$ satisfy

$$\mathcal{S}^\top P_{\mathrm{d}} \mathcal{S} - P_{\mathrm{d}} = -Q_{\mathrm{d}}. \tag{36}$$

Applying (35) with $\gamma \in (0, 2)$, there exists $c_{\mathrm{d}} > 0$ such that

$$\limsup_{k\to\infty} \|\bar{e}_k\| \le c_{\mathrm{d}} \bar{\delta}^*. \tag{37}$$

In particular, if $\bar{\delta}^* = 0$, then $\bar{e}_k \to 0$.

In summary, the discrete-time scheme achieves asymptotic convergence without disturbances and ISS under bounded noise, governed by the Schur stability of $\mathcal{S}$, the gain $\gamma \in (0, 2)$, and regressor normalization.

6. NUMERICAL RESULTS

We validate the theoretical results through simulations on three network topologies—*star*, *cyclic*, and *path* (see Fig. 2 for details), each satisfying (1) and Remark 1. For both time scales, we consider $m = 4$ sensing nodes with $n = 2$ dimensional space. For scalability tests, we extend to $m \in \{100, 150, 200, 300\}$ to measure computational complexity [2].

Source dynamics: For continuous time,

$$F_* = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}, \quad G_* = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

[2] All simulations use `MATLAB R2023a`. For complexity, we record total CPU time via `tic`–`toc` per topology and network size.

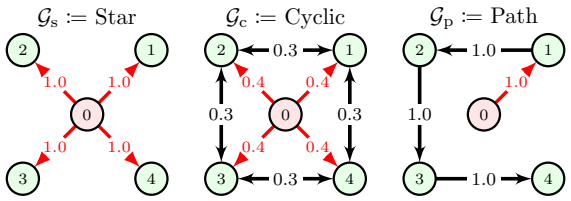


Fig. 2. Three network topologies ($\mathcal{G}_{\mathrm{s}}, \mathcal{G}_{\mathrm{c}}, \mathcal{G}_{\mathrm{p}}$) with weights used in the simulations.

Table 1. Total runtime t_{tot} (s) vs. number of sensing nodes m for each topology.

	Continuous				Discrete		
m	$\mathcal{G}_{\mathrm{s}}$	$\mathcal{G}_{\mathrm{c}}$	$\mathcal{G}_{\mathrm{p}}$	m	$\mathcal{G}_{\mathrm{s}}$	$\mathcal{G}_{\mathrm{c}}$	$\mathcal{G}_{\mathrm{p}}$
100	0.66	0.65	1.65	100	0.45	0.43	0.38
150	0.76	0.86	1.95	150	0.68	0.69	0.71
200	0.98	1.23	2.63	200	0.85	0.84	0.84
300	1.88	2.68	5.38	300	1.17	1.18	1.16

with input $v_0(t) = 0.7\sin(0.5t) + 1.5\cos(1.0t + \pi/6)$, with bounded disturbance $d_0(t) = 5.5[\sin(0.1t),\ 0.5\cos(0.3t)]^\top$. For discrete time,

$$A_* = \begin{bmatrix} 0.9 & 0.1 \\ -0.1 & 0.95 \end{bmatrix}, \quad B_* = \begin{bmatrix} 0.05 \\ 0.10 \end{bmatrix},$$

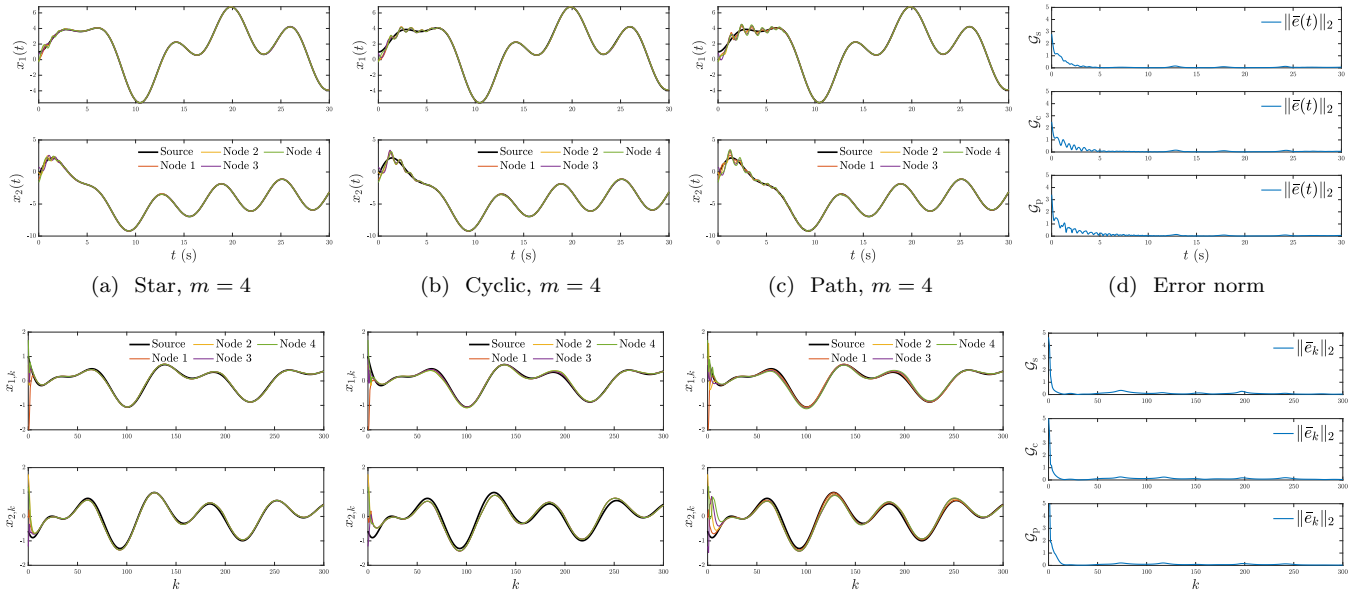
with input $u_{0,k} = 0.9\sin(0.05k) + 0.6\cos(0.1k + \pi/5)$ and disturbance $\delta_{0,k} = 0.05[0.7\sin(0.05k),\ 0.5\cos(0.09k)]^\top$. Unknowns are introduced as $\|\Delta F\|_2 = 0.55$, $\|\Delta G\|_2 = 0.50$, $\|\Delta A\|_2 = 0.1$, and $\|\Delta B\|_2 = 0.07$ to obtain F_0, G_0, A_0, and B_0 respectively.

Sensing nodes: Each node implements a local observer with following details. *Continuous*: $H_i = \mathrm{diag}([-2, -2])$ (Hurwitz), with $\Gamma_\phi = \Gamma_\psi = 10 I_m$; *Discrete*: $S_i = \mathrm{diag}([0.5, 0.5])$ (Schur-stable), with $\gamma = 1.3$ and $\varphi = 0.01$.

Initial conditions: The source state is initialized as $x_0(0) = [1.0, -0.5]^\top$ (continuous) or $x_{0,0} = [1.0, -0.6]^\top$ (discrete). Each node state satisfies $x_i(0) \in \{x \in \mathbb{R}^n : \|x\|_\infty \le 1\}$, $i = 1, \ldots, m$, and in simulations we draw $x_i(0)$ with components independently from Unif$[-1, 1]$. We design $\hat{F}_i(0) = \hat{G}_i(0) = 0$ and $\hat{A}_{i,0} = \hat{B}_{i,0} = 0$ as initial parameter estimates.

For both time scales and all three topologies (star, cyclic, and path), the sensing nodes successfully track the source state and the aggregated error norms remain bounded, as shown in Figs. 3a–3h. The star topology performs best since all nodes directly access source information, while the cyclic topology achieves near-centralized accuracy despite only partial source access through network coordination. The path topology converges more slowly due to weaker connectivity but still maintains bounded ISS-consistent errors. These observations agree with the theoretical stability and robustness guarantees established for $\mathcal{H}$ and $\mathcal{S}$.

Table 1 reports runtime versus network size. Continuous- and discrete-time implementations scale approximately linearly with the number of sensing nodes, while the discrete-time method is faster due to explicit low complexity updates. The path topology requires longer runtimes because weaker connectivity slows information propagation. Overall, the results confirm accurate tracking, ISS robustness, and scalability to large directed networks.



(a) Star, $m = 4$ (b) Cyclic, $m = 4$ (c) Path, $m = 4$ (d) Error norm

(e) Star, $m = 4$ (f) Cyclic, $m = 4$ (g) Path, $m = 4$ (h) Error norm

Fig. 3. Continuous-time cases in (a)–(c), discrete-time cases in (e)–(g), and corresponding error norms in (d) and (h).

7. CONCLUSION

We studied distributed adaptive estimation for both continuous and discrete time scales with a partially unknown source. In both cases we established stability of the network operators—$\mathcal{H}$ (continuous) and $\mathcal{S}$ (discrete)—and proved convergence of the estimation error; moreover, with bounded source disturbances we derived ISS guarantees for both time scales. The main distinction is architectural: continuous-time uses smooth adaptation, whereas discrete-time requires per-step normalization with a constant gain to handle sampling and non–strongly convex regressors. Simulations across multiple topologies corroborate the theory, showing that all estimates accurately track the source despite uncertainties and disturbances. The framework also scales efficiently with the number of sensing nodes, demonstrating both robustness and computational feasibility for larger networks.

REFERENCES

Fisher, P.A. and Annaswamy, A.M. (2023). Discrete-time adaptive control of a class of nonlinear systems using high-order tuners. In *2023 American Control Conference (ACC)*, 2393–2398.

Hu, Z., Hu, J., and Yang, G. (2020). A survey on distributed filtering, estimation and fusion for nonlinear systems with communication constraints: new advances and prospects. *Systems Science & Control Engineering*, 8(1), 189–205. doi:10.1080/21642583.2020.1737846.

Li, C., Dong, H., Li, J., and Wang, F. (2019). Distributed kalman filtering for sensor network with balanced topology. *Systems & Control Letters*, 131, 104500.

Olfati-Saber, R. (2007). Distributed kalman filtering for sensor networks. In *2007 46th IEEE Conference on Decision and Control*, 5492–5498.

Rong, L., Wang, S., Jiang, G.P., and Xu, S. (2020). Distributed observer-based consensus over directed networks with limited communication bandwidth constraints. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50(12), 5361–5368.

Stegagno, P. and Yuan, C. (2019). Distributed cooperative adaptive state estimation and system identification for multi-agent systems. *IET Control Theory & Applications*, 13(6), 815–822.

Talebi, S.P. and Werner, S. (2018). Distributed kalman filtering: Consensus, diffusion, and mixed. In *2018 IEEE Conference on Control Technology and Applications (CCTA)*, 1126–1132. doi:10.1109/CCTA.2018.8511492.

Vivas, C., Siri, S., Ferrara, A., Sacone, S., Cavanna, G., and Rubio, F.R. (2015). Distributed consensus-based switched observers for freeway traffic density estimation. In *2015 54th IEEE Conference on Decision and Control (CDC)*, 3445–3450. doi:10.1109/CDC.2015.7402672.

Wafi, M.K., Hedesh, H.M., and Siami, M. (2026). Distributed adaptive estimation with iss guarantees for sensor networks with partially unknown source dynamics. *arXiv preprint arXiv:2511.07646*.

Wafi, M.K. and Siami, M. (2024). Distributed adaptive control of disturbed interconnected systems with high-order tuners. *IEEE Control Systems Letters*, 8, 1421–1426. doi:10.1109/LCSYS.2024.3407613.

Wang, S. and Meng, X. (2021). Adaptive consensus and parameter estimation of multiagent systems with an uncertain leader. *IEEE Transactions on Automatic Control*, 66(9), 4393–4400. doi:10.1109/TAC.2020.3046215.

Wang, X., Jiang, G.P., Su, H., and Zeng, Z. (2022). Consensus-based distributed reduced-order observer design for lti systems. *IEEE Transactions on Cybernetics*, 52(7), 6331–6341. doi:10.1109/TCYB.2020.3025603.

Yang, G., Barboni, A., Rezaee, H., and Parisini, T. (2022). State estimation using a network of distributed observers with unknown inputs. *Automatica*, 146, 110631.

Zhu, M., Sui, T., and Wang, R. (2023). Distributed kalman filtering over sensor networks with fading measurements and random link failures. *Journal of the Franklin Institute*, 360(4), 2457–2475.