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Nonlinear analysis of classical phase-locked loops in signal's phase space

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Abstract: Discovery of undesirable hidden oscillations, which cannot be found by the standard simulation, in phase-locked loop (PLL) showed the importance of consideration of nonlinear models and development of rigorous analytical methods for their analysis. In this paper for various signal waveforms, analytical computation of multiplier/mixer phase-detector characteristics is demonstrated, and nonlinear dynamical model of classical analog PLL is derived. Approaches to the rigorous nonlinear analysis of classical analog PLL are discussed.

Keywords: phase-locked loop (PLL), phase detector characteristic, nonlinear analysis, hidden oscillations, hidden attractor, slow-fast systems, coexisting attractors, multistability

1. INTRODUCTION

The Phase locked-loop (PLL) circuits were invented in the first half of the twentieth century and nowadays are widely used in modern telecommunications and computers. A PLL-based circuit behaves as a nonlinear control system and its *physical model in the signal space* can be described by nonlinear nonautonomous difference or differential equations. In practice, numerical simulation is widely used for the analysis of PLL-based circuit nonlinear model (see, e.g., [Best, 2007, Bianchi, 2005, Goldman, 2007, Klapper, 2012, Razavi, 2003, Tranter et al., 2010] and others) since the rigorous analysis of nonlinear nonautonomous equations is often a very difficult task. However for the high-frequency signals, the explicit numerical simulation of the *physical model of PLL-based circuit in the signal space* is very complicated since one has to consider simultaneously both the very fast time scale of the signals and the slow time scale of phase difference between the signals. To analyze the high frequency signals accurately, a very high sampling rate is required, which makes it difficult to perform a simulation in a reasonable time.

To overcome these difficulties, instead of consideration of *physical model in the signal space* one can derive *a mathematical model of PLL-based circuit in the signal's phase space*, which is described by nonlinear autonomous difference or differential equations, and in which only the slow time scale of signal's phases difference is considered (the ideas behind this are traced back to the famous works by F. Gardner and A. Viterbi). Such a consideration requires the computation of phase detector characteristic, which depends on PD physical realization and the waveforms of considered signals [Leonov et al., 2011, 2012]. Note that the derivation of a *mathematical model* and the use of results of its analysis to draw conclusions about the behavior of *physical model* need for a rigorous foundation.

Although PLL is inherently a nonlinear system, in modern literature, devoted to the analysis of PLL-based circuit mathematical models, the main direction [Abramovitch, 2002] is simplified linear models, the methods of linear analysis, empirical rules, and numerical simulation [1]. Note that the linearization without justification and the analysis of linearized models of nonlinear control systems may result in incorrect conclusions [2]. At the same time the attempts to justify analytically the reliability of conclusions, based on such engineering approaches, and to study the nonlinear models of PLL-based circuits are rare [Chicone and Heitzman, 2013, Gelig et al., 1978, Kudrewicz and Wasowicz, 2007, Kuznetsov, 2008, Leonov, 2006, Leonov and Kuznetsov, 2014, Margaris, 2004, Piqueira and Monteiro, 2003, Sarkar et al., 2014, Stensby, 1997, Suarez and Quere, 2003].

Further, in this paper the ideas of F. Gardner and A. Viterbi on nonlinear analysis and design of PLL-based circuits are developed and rigorously justified, a general effective approach to analytical computation of phase detector characteristics is discussed, and nonlinear dynamical models of classical PLL are derived for various nonsinusoidal waveforms.

2. CLASSICAL PLL IN SIGNAL SPACE

Consider classical PLL on the level of electronic realization (Fig. 1). Here $f^1(t) = f^1(\theta^1(t))$ is an input [reference

[1] Remark that the application of standard numerical analysis cannot guarantee to find undesired multiple steady-state solutions in nonlinear control systems: see, e.g., examples of hidden oscillations and coexisting attractors in two-dimensional PLL model, electrical Chua circuits, aircraft control systems, and drilling systems [Andrievsky et al., 2013, Kuznetsov et al., 2013, 2011a, 2010, Leonov and Kuznetsov, 2013]), the presence of which can lead to crashes.

[2] See, e.g., counterexamples to the filter hypothesis, hidden oscillations in counterexamples to Aizerman's and Kalman's conjectures on the absolute stability of nonlinear control systems [Bragin et al., 2011], and the Perron effects of the largest Lyapunov exponent sign reversal for a nonlinear system and its linearization [Kuznetsov and Leonov, 2005, Leonov and Kuznetsov, 2007].

oscillator] signal with phase $\theta^1(t)$, and VCO is a tunable voltage-control oscillator [slave oscillator], which generates signal $f^2(t) = f^2(\theta^2(t))$ with $\theta^2(t)$ as phase. The block $\bigotimes$ is a multiplier (used as PD) of oscillations $f^1(t)$ and $f^2(t)$, and the signal $f^1(\theta^1(t))f^2(\theta(t))$ is its output.

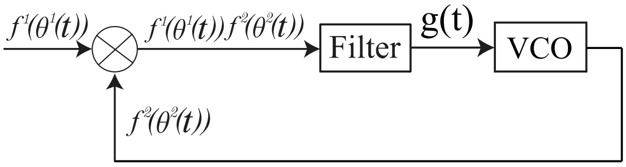

Fig. 1. PLL in the signal space (electronic realization)

The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter is as follows

$$\sigma(t) = \alpha_0(t) + h\xi(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,\mathrm{d}\tau, \qquad (1)$$

where $\gamma(t)$ is an impulse response function of filter and $\alpha_0(t)$ is an exponentially damped function depending on the initial data of filter, h is a constant. By assumption, $\gamma(t)$ is a differentiable function with bounded derivative (this is true for the most considered filters [Best, 2007]).

Suppose that the waveforms $f^{1,2}(\theta)$ are bounded 2π-periodic piecewise differentiable functions [3]. Consider Fourier series representation of such functions

$$f^p(\theta) = \sum_{i=1}^{\infty} \big(a_i^p \sin(i\theta) + b_i^p \cos(i\theta)\big),\ p = 1, 2,$$
$$a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\sin(i\theta)d\theta, b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\cos(i\theta)d\theta.$$

A high-frequency property of signals can be reformulated in the following way. By assumption, the phases $\theta^p(t)$ are smooth functions (this means that frequencies are changing continuously, what corresponds to classical PLL analysis [Best, 2007]). Suppose also that there exists a sufficiently large number ω_{min} such that the following conditions are satisfied on the fixed time interval $[0, T]$:

$$\dot\theta^p(\tau) \geq \omega_{min} > 0, \quad p = 1, 2 \qquad (2)$$

where T is independent of ω_{min} and $\dot\theta^p(t)$ denotes frequencies of signals. The frequencies difference is assumed to be uniformly bounded

$$\big|\dot\theta^1(\tau) - \dot\theta^2(\tau)\big| \leq \Delta\omega,\ \forall\tau \in [0, T], \qquad (3)$$

where $\Delta\omega$ is a constant. Requirements (2) and (3) are obviously satisfied for the tuning of two high-frequency oscillators with close frequencies [Best, 2007]. Let us introduce $\delta = \omega_{min}^{-\frac{1}{2}}$. Consider the relations

$$\begin{aligned} &|\dot\theta^p(\tau) - \dot\theta^p(t)| \leq \Delta\Omega,\ p = 1, 2, \\ &|t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T], \end{aligned} \qquad (4)$$

where $\Delta\Omega$ is independent of δ and t.

3. PHASE-DETECTOR CHARACTERISTIC

Consider two block-diagrams shown in Fig. 2.

[3] Consideration of analog PLL with sinusoidal signals and some special phase detectors [Rosenkranz, 1982] may also be reduced to such a consideration.

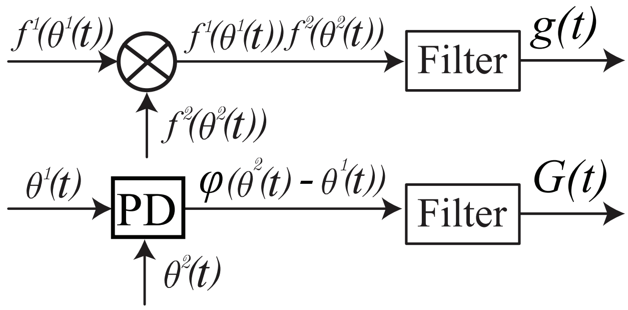

Fig. 2. Phase detector and filter

Here PD is a nonlinear block with characteristic $\varphi(\theta)$. The phases $\theta^p(t)$ are PD block inputs and the output is a function $\varphi(\theta^1(t) - \theta^2(t))$. The PD characteristic $\varphi(\theta)$ depends on the waveforms of input signals. The signal $f^1(\theta^1(t))f^2(\theta^2(t))$ and the function $\varphi(\theta^1(t) - \theta^2(t))$ are the inputs of the same filters with the same impulse response function $\gamma(t)$ and with the same initial state. The outputs of filters are functions $g(t)$ and $G(t)$, respectively. By (1), one can obtain $g(t)$ and $G(t)$

$$\begin{aligned} g(t) =& \alpha_0(t) + hf^1\big(\theta^1(\tau)\big)f^2\big(\theta^2(\tau)\big) + \\ & \int_0^t \gamma(t-\tau)f^1\big(\theta^1(\tau)\big)f^2\big(\theta^2(\tau)\big)d\tau, \\ G(t) =& \alpha_0(t) + h\varphi(\theta^1(t) - \theta^2(t)) + \\ & \int_0^t \gamma(t-\tau)\varphi(\theta^1(\tau) - \theta^2(\tau)\big)d\tau. \end{aligned} \qquad (5)$$

For the loop filter we additionally assume that

$$|\gamma(\tau) - \gamma(t)| = O(\delta),\ |t - \tau| \leq \delta,\ \forall\tau, t \in [0, T]. \qquad (6)$$

The VCO acts as an integrator of its input. Therefore, to prove the equivalence of block-diagrams, it is sufficient to show that the integrals of the filters outputs are close. Then, using the approaches outlined in [Kuznetsov et al., 2011b, Leonov et al., 2011, 2012] the following result can be proved.

Theorem 1. Let conditions (2)–(6) be satisfied and

$$\varphi(\theta) = \frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\Big). \qquad (7)$$

Then the following relation:

$$\left|\int_0^t g(\tau)d\tau - \int_0^t G(\tau)d\tau\right| = O(\delta), \quad \forall t \in [0, T]$$

is valid.

3.1 Proof idea

Consider the difference

$$\int_0^t \big(g(\tau) - G(\tau)\big)d\tau =$$
$$\int_0^t \bigg(h(f^1(\theta^1(\tau))f^2(\theta^2(\tau)) - \varphi(\theta^2(\tau) - \theta^1(\tau))) +$$
$$\int_0^\tau \gamma(\tau - s)\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \varphi\big(\theta^1(s) - \theta^2(s)\big)\Big]ds\bigg)d\tau.$$

Let $\gamma_2(t-s) = h + \int_0^\tau \gamma(\tau - s)ds$. Then

$$\int_0^t \big(g(\tau) - G(\tau)\big)d\tau =$$

$$\int_0^t \gamma_2(t-s)\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \varphi\big(\theta^1(s) - \theta^2(s)\big)\Big]ds.$$

Since property (6) holds for γ_2, then by the theorem from [Leonov et al., 2012] one gets

$$\left|\int_0^t g(\tau)d\tau - \int_0^t G(\tau)d\tau\right| = O(\delta) \qquad (8)$$

■

Roughly speaking, Theorem 1 separates a low-frequency error-correcting signal from parasitic high-frequency oscillations. This result for sinusoidal signals was known to engineers without rigorous justification.

Remark 1. For the most considered waveforms (e.g. sinusoidal, squarewave, sawtooth, polyharmonic), infinite series (7) can be truncated up to the first $\sqrt{\omega_{min}}$ terms.

A remainder $R_{[\frac{1}{\delta}]}$ of series (7) can be estimated as

$$|R_{[\frac{1}{\delta}]}(x)| \leq \sum_{l=[\frac{1}{\delta}]+1}^{\infty} (|a_l^1 a_l^2 + b_l^1 b_l^2| + |a_l^1 b_l^2 - b_l^1 a_l^2|) \leq$$

$$\leq \sum_{l=[\frac{1}{\delta}]+1}^{\infty} (|a_l^1 a_l^2| + |b_l^1 b_l^2| + |a_l^1 b_l^2| + |b_l^1 a_l^2|).$$

Since $a_l^{1,2} = O(\frac{1}{l})$ and $b_l^{1,2} = O(\frac{1}{l})$, then

$$|R_{[\frac{1}{\delta}]}(x)| \leq O\bigg(\sum_{l=[\frac{1}{\delta}]+1}^{\infty} \frac{1}{l^2}\bigg), \qquad (9)$$

the sum is bounded by corresponding integral of function $\frac{1}{x^2}$. Then $|R_{[\frac{1}{\delta}]}(x)| \leq O(\delta)$, what can be easily proved by integration.

4. PLL EQUATIONS IN THE SIGNAL'S PHASE SPACE

From the mathematical point of view, a linear filter can be described [Best, 2007] by a system of linear differential equations

$$\dot{x} = Ax + b\xi(t),\ \sigma = c^*x + h\xi(t). \qquad (10)$$

Here $\xi(t)$ is an input of the filter, $\sigma(t)$ is an output of the filter, A is a constant matrix, $x(t)$ is a state vector of filter, b, h, and c are constant vectors. A solution of this system takes the form (1), where $\gamma(t-\tau) = c^*e^{A(t-\tau)}b$ is an impulse response function of filter and $\alpha_0(t) = c^*e^{At}x_0$ is an exponentially damped function depending on the initial data of filter. By assumption, $\gamma(t)$ is a differentiable function with bounded derivative (this is true for the most considered filters).

The model of VCO generator is usually assumed to be linear [Best, 2007]:

$$\dot{\theta}^2(t) = \omega_{free}^2 + LG(t),\ t \in [0, T]. \qquad (11)$$

Here ω_{free}^2 is a free-running frequency of tunable generator, L is an oscillator gain, and $G(t)$ is an input of VCO. Here it is also possible to use various nonlinear models of VCO.

Suppose that the frequency of master generator is constant $\dot{\theta}^1(t) \equiv \omega^1$. Equation of tunable generator (11) and equations of filter (10) yield

$$\begin{aligned} &\dot{\theta}^1(t) = \omega^1,\\ &\dot{x} = Ax + bf^1(\theta^1(t))f^2(\theta^2(t)),\\ &\dot{\theta}^2 = \omega_{free}^2 + Lc^*x + Lhf^1(\theta^1(t))f^2(\theta^2(t)).\end{aligned} \qquad (12)$$

System (12) is nonautonomous and slow-fast, therefore its analytical and numerical analysis is often a very difficult task. Theorem 1 allows one to compute the phase detector characteristic φ and to derive the equations of PLL in the signal's phase space

$$\begin{aligned} &\dot{z} = Az + b\varphi(\theta),\\ &\dot{\theta} = \omega_{free}^2 - \omega^1 + Lc^*z + Lh\varphi(\theta),\\ &\theta = \theta^2 - \theta^1.\end{aligned} \qquad (13)$$

The averaging method allows one to prove that the solutions of PLL equations in the signal space are close to the solutions of averaged equations in the signal's phase space. For the application of the averaging method consider the following notations

$$\tau = \omega^1 t, \quad \varepsilon = \frac{1}{\omega^1}, \quad y(\tau) = \begin{pmatrix} x(\frac{\tau}{\omega^1}) \\ \theta(\frac{\tau}{\omega^1}) \end{pmatrix},$$

$$F(y,\tau) =$$

$$\begin{pmatrix} Ax(\frac{\tau}{\omega^1}) + bf^1(\tau)f^2(\theta(\frac{\tau}{\omega^1}) + \tau), \\ \omega_{free}^2 - \omega^1 + Lc^*x(\frac{\tau}{\omega^1}) + hf^1(\tau)f^2(\theta(\frac{\tau}{\omega^1}) + \tau). \end{pmatrix}$$

and transform system (12) to the following form

$$\frac{dy}{d\tau} = \varepsilon F(y,\tau), \quad y(0) = y_0, \qquad (14)$$

where F is T-periodic on τ. For the simplicity assume that F is Lipschitz continuous. Suppose that D is an bounded open set, containing x_0, and choose ε_0 such that $0 < \varepsilon \leq \varepsilon_0$. Introduce averaged equation

$$\frac{dz}{d\tau} = \varepsilon\bar{F}(z), \quad z(0) = y_0, \qquad (15)$$

where

$$\bar{F}(z) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} F(z,\tau)d\tau \qquad (16)$$

where the limit in (16) is assumed uniform in $t_0 \geq 0$. If this limit exists, then one looks for non-negative numbers $\Delta(\varepsilon)$ and $\eta(\varepsilon)$ with the property

$$\frac{1}{\Delta(\varepsilon)}\left|\int_{s_0}^{s_0+\Delta(\varepsilon)} \Big(F(y,\frac{s}{\varepsilon}) - \bar{F}(y)\Big)ds\right| \leq \eta(\varepsilon), \qquad (17)$$

$$\forall y, \quad 0 \leq s_0 \leq 1 - \Delta(\varepsilon),$$

where $s_0 = t_0\varepsilon$. Pair $(\Delta(\varepsilon), \eta(\varepsilon))$ is called a rate of averaging.

Theorem 2. [Mitropolsky and Bogolubov, 1961] Assume that the function $F(y,t)$ is K-Lipschitz in y and bounded by a constant r. Suppose that the equation has a time-independent average $\bar{F}(z)$ and $(\Delta(\varepsilon), \eta(\varepsilon))$ is the rate of averaging. Then

$$|y(t) - z(t)| \leq Te^{2KT}((K+2)r\Delta(\varepsilon) + \eta(\varepsilon))$$

It is of order $\max(\Delta(\varepsilon), \eta(\varepsilon))$.

Theorem 1 gives a time-independent average function $\varphi(\theta)$ such that

$$\left| \int_{s_0}^{s_0+1} \left(f^1(\omega^1 t) f^2(\omega^1 t + \theta) - \varphi(\theta) \right) dt \right| = O\left(\sqrt{\varepsilon}\right),$$
$$\forall s_0, \quad 1 \le s_0 + 1 \le T$$

function that satisfies Theorem 2. Thus the solutions of PLL equations in the signal space (12) are close to the solutions of PLL equations in the signal's phase space (13).

Equations of PLL in phase frequency space correspond to the following block-diagram (Fig. 3).

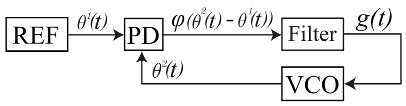

Fig. 3. Phase-locked loop in the signal's phase space

In Fig. 3 PD has the corresponding characteristics. Thus, using asymptotic analysis of high-frequency oscillations, the characteristics of PD can be computed.

By assumption (4), frequency of the signal $\varphi(\theta(t))$ is much smaller than frequency of $f^1(\theta^1(t))f^2(\theta^1(t))$. Typically, the loop bandwidth (maximum frequency difference, $\dot\theta = \dot\theta^2 - \dot\theta^1$) is several orders of magnitude smaller than signal's frequency. Therefore, according to Nyquist-Shannon sampling theorem, it is possible to choose much larger discretization time step for signal's phase model, speeding up the simulation.

The most frequently used in practice filters for the PLL are passive and active lead-lag filters having one pole and one zero [Best, 2007, Pederson and Mayaram, 2008]. Its transfer function $F(s)$ is given by

$$F(s) = \frac{as + \beta}{s + \alpha}, \quad \alpha > 0, \ \beta > 0, \ a \ge 0. \tag{18}$$

For the case $a = 0$ this filter is commonly referred to as a PI filter. Denoting $\tau = \frac{t}{\sqrt{\beta}}$, $z = \sqrt{\beta}z, \hat\alpha = \alpha\sqrt{\beta}$, $\hat a = a\sqrt{\beta}, \gamma = -\frac{\omega_{free}^2 - \omega^1}{La + L\frac{1}{\alpha}(\beta - a\alpha)}$, we get the following equations for PLL with lead-lag filter

$$\begin{aligned} \dot z &= -\hat\alpha z - (1 - \hat a\hat\alpha)(\varphi(\theta) - \gamma), \\ \dot\theta &= Lz + L\hat a(\varphi(\theta) - \gamma). \end{aligned} \tag{19}$$

5. SIMULATION

Since in block-diagram in Fig. 3 and system (13) for the signal's phase space only the slow time change of signal's phases and frequencies is considered, they can be effectively studied numerically.

Theoretical results are illustrated by simulation of PLL model in Matlab/Simulink for the signal's phase space (see Fig. 4-5)

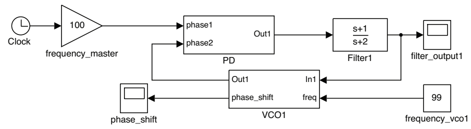

Fig. 4. Simulink models of PLL in the signal's phase space: $\omega_{free} = 99$ Hz, $\omega^1 = 100$ Hz, filter transfer functions $\frac{s+1}{s+2}$

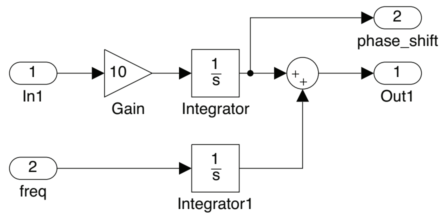

Fig. 5. Simulink models of VCO in the signal's phase space. VCO gain $L = 10$

and the signal space (see Fig. 6-7).

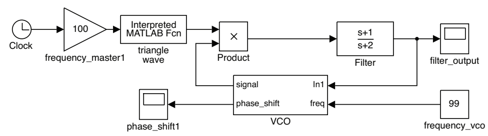

Fig. 6. Simulink models of PLL in the signal space (with triangle waveform)

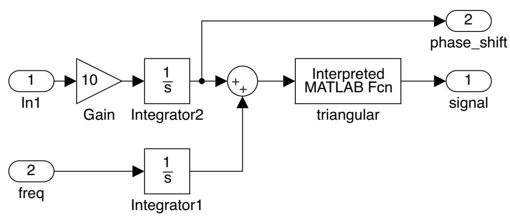

Fig. 7. Simulink models of VCO in the signal space (with triangle waveform)

Here triangle waveform is implemented by standard Matlab function "-sawtooth(u,0.5)", and PD characteristic (7) is implemented by

```
function y = tritri_char(u)
y = 0;
for i=1:100
    y = y + cos((2*i-1)*u)/(2*i-1)^4;
end
y = 32*y/pi^4;
end
```

Unlike the filter output for signal's phase model, the output of filter in the signal space contains additional high-frequency oscillations (see Fig. 8), which interfere with qualitative analysis and efficient simulation of PLL. The analysis of autonomous dynamical model of PLL (in place

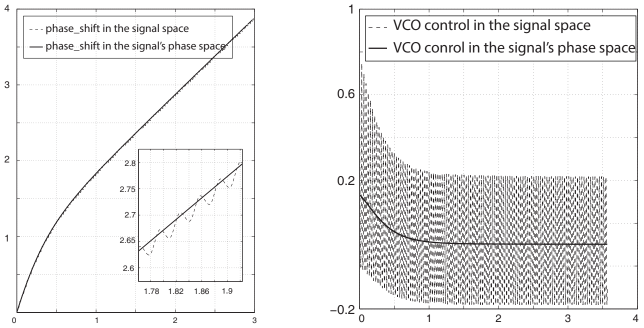

Fig. 8. VCO phase_shift output and VCO control signals. $\omega_{free} = 99$ Hz, $\omega^1 = 100$ Hz

of the nonautonomous one) allows one to overcome the aforementioned difficulties, related with modeling PLL in time domain for hight frequencies, very high sampling rate is required, which makes it difficult to perform a simulation in a reasonable time (see Fig. 9).

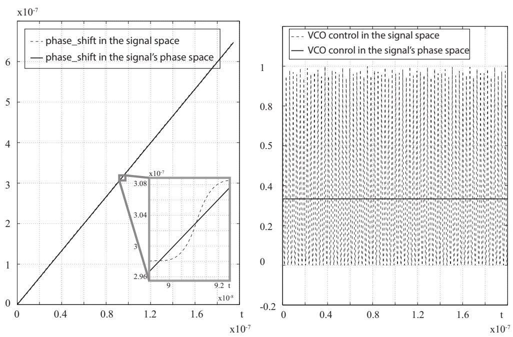

Fig. 9. VCO phase_shift output and VCO control signals. $\omega^1 = 10^9$, $\omega_{free} = 10^9 - 1$ Hz, discretization step $\frac{1}{10\omega_{ref}}$

6. ANALYTICAL STUDY

In the middle of the last century the investigations of dynamical models of phase synchronization system were begun. M. Kapranov [Kapranov, 1956] obtained the conditions of infinite pull-in range for two-dimensional phase-locked loop model (19). In 1961, N. Gubar' [Gubar', 1961] revealed a gap in the proof of Kapranov's results and specified system parameters for which the pull-in range was limited by a periodic solution.

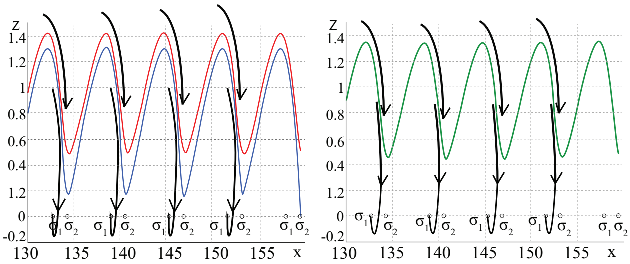

Fig. 10. Coexistence of periodic trajectories and semistable periodic trajectory in classical PLL with sinusoidal signals: $\hat{\alpha} = 0.2513$, $\hat{a} = 4.6203$, $\gamma = 0.716$, $\alpha = 15.7977$, $\beta = 15.7977$, $a = 0.2922$

On the left in Fig. 10 it is shown the coexistence of unstable (blue) periodic solution and stable (red) periodic solution [4] for $(\omega^2_{free} - \omega^1) = 89.5$. Further decreasing of $(\omega^2_{free} - \omega^1)$ results in the birth of semistable periodic trajectory (green) (on the right in Fig. 10). Two close periodic trajectories and semistable trajectory can not be found numerically by the standard computation procedure, therefore from a computational point of view the system considered was globally stable (all the trajectories tend to equilibria), but, in fact, there is a bounded domain of attraction only. The rigorous analysis and explanation of the birth of semistable trajectory can be found in [Leonov and Kuznetsov, 2013].

The phase portrait in Fig. 10 is very sensitive to any disturbances. Thus any simplifications of the mathematical model of PLL and approximate methods of analysis may lead to wrong conclusions about the stability of PLL, e.g., one can read in [Tranter et al., 2010]: "*analysis of the acquisition behavior cannot be accomplished using the simple linear models and nonlinear analysis techniques are necessary*", and in [Lai et al., 2005]: "*the use of linear macromodels can lead to qualitatively incorrect prediction of important PLL phenomena*".

The from of system (19) with one periodic nonlinearity $(\varphi(\theta) - \gamma)$ allows one to use for its nonlinear analysis the special modifications of absolute stability criteria adapted for the cylindrical phase space ([Gelig et al., 1978, Kuznetsov, 2008, Leonov, 2006, Leonov and Kuznetsov, 2014]).

7. CONCLUSION

The considered approach allows one (mathematically rigorously) to compute multiplier PD characteristics and to proceed from analysis of the slow-fast model of PLL in the signal space to effective analysis and simulation of a model of PLL with only slow changing variables in the signal's phase space.

To predict such undesired situation as the existence of hidden oscillations in PLL models, it is necessary to apply the special modifications of absolute stability criteria adapted for the cylindrical phase space since any simplifications of the mathematical model of PLL and approximate methods of analysis may lead to wrong conclusions.

ACKNOWLEDGMENT

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[4] Stable (red) periodic solution is hidden oscillations since its basin of attraction does not intersect with small neighborhoods of equilibria [Leonov and Kuznetsov, 2013].

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