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Sampled-Parameter Feedback Control of Discrete-time Linear Stochastic Parameter-Varying Systems ⋆

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Abstract: Feedback stabilization of a discrete-time linear stochastic parameter-varying system is explored. The parameter of the system is modeled as a discrete-time stationary, ergodic, and aperiodic Markov process on a Euclidean space. We develop a stabilizing control framework for the case where the system parameter is observed (sampled) periodically. We obtain sufficient conditions under which almost sure asymptotic stabilization of the closed-loop stochastic parameter-varying system is guaranteed by our proposed control law, which depends only on the sampled version of the system parameter.

Keywords: Parameter-varying systems, Stochastic systems, Sampled-data control

1. INTRODUCTION

The dynamics of various complex real life processes from biology, mechanical engineering, and finance incorporate randomly varying parameters (see De Koning (1984); Cassandras and Lygeros (2006); Bolzern et al. (2008); Zhu and Yin (2009); Xie (2011); and the references therein). Several classes of stochastic parameter-varying system models have been investigated by researchers. Particularly, dynamical systems with Markov jump parameters (also called Markov jump systems) have been studied extensively (de Farias et al., 2000; Costa et al., 2004; Dragan and Morozan, 2008). Markov jump systems incorporate a stochastic parameter, which is modeled as a finite-state Markov chain, to characterize the switching between a number of subsystems (modes) with different dynamics. In addition to Markov jump systems, researchers have also explored linear stochastic systems described by time-varying matrices that form sequences of independent and identically distributed random variables (De Koning, 1982; Farokhi and Johansson, 2012). Furthermore, the case where the dynamics depend on a stationary and ergodic stochastic parameter process is studied by Bolzern et al. (2008) and Xie (2011).

Feedback control of dynamical systems with stochastic parameters have been explored in several studies (Ghaoui and Rami, 1996; El Bouhtouri et al., 1999; de Farias et al., 2000; Fang and Loparo, 2002; Costa et al., 2004; Mao et al., 2007; Sathanantan et al., 2008; Geromel et al., 2009). Most of the documented control frameworks for stochastic parameter-varying systems require the availability of parameter information at all time instants. Note that the parameters of a system usually describe the state of external environment, and may not be directly measurable or may not be observed as frequently as the state of the system itself. Hence, it is important to investigate the control problem for the case where the parameters are not available for control purposes at all time instants.

In our earlier work (Cetinkaya and Hayakawa, 2013a,b), we investigated stabilization problem for Markov jump systems for the case where the controller has access only to *sampled information of the system mode*, which is modeled by a finite-state Markov chain.

In this paper we explore feedback control of discrete-time linear stochastic *parameter-varying* systems under *sampled parameter information*. Specifically, we assume that the parameter of the system, which is modeled as a discrete-time aperiodic, stationary, and ergodic Markov process defined on a Euclidean space, is observed (sampled) periodically. In order to achieve stabilization, we develop a control framework that depends only on the sampled version of the parameter. We obtain sufficient conditions of almost sure asymptotic stabilization of the closed-loop system by utilizing the stationarity and ergodicity properties of a stochastic process that represents the sequences of values that the system parameter takes between consecutive observation instants. We then explore a special class of linear parameter-varying systems where the state matrix is an affine function of the entries of the parameter vector. We show that stabilization for this class of parameter-varying systems can be achieved through a control law with a feedback gain that is an affine function of the entries of the sampled parameter vector.

The paper is organized as follows. In Section 2, we provide the notation and some key results concerning discrete-time stochastic processes. In Section 3, we present the mathematical model for discrete-time linear stochastic parameter-varying systems and explain the feedback control problem under periodically sampled parameter in-

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formation. In Section 4, we obtain sufficient conditions under which our proposed control law guarantees almost sure asymptotic stabilization; furthermore, we discuss almost sure asymptotic stabilization problem for linear parameter-varying systems with affine parameter dependence. Finally, in Section 5 we conclude our paper.

2. MATHEMATICAL PRELIMINARIES

In this section, we provide notation; furthermore, we present several definitions and key results concerning discrete-time stochastic processes. Specifically, we use $\mathbb{N}$ and $\mathbb{N}_0$ in order to denote positive and nonnegative integers, respectively. Moreover, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. We write $\|\cdot\|$ for the Euclidean vector norm, $(\cdot)^{\mathrm{T}}$ for transpose, and $\lfloor\cdot\rfloor$ for the largest integer that is less than or equal to its real argument. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite if $V(x) > 0$, $x \neq 0$, and $V(0) = 0$. The notations $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ respectively denote the probability and expectation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\mathcal{B}(\mathbb{R}^l)$ to denote the Borel σ-algebra associated with $\mathbb{R}^l$.

2.1 Discrete-Time Markov Processes on Euclidean Spaces

A time-homogeneous, discrete-time Markov process defined on a Euclidean state space $\mathbb{R}^l$ is a stochastic process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ characterized by an initial distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ and transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ such that

$$\mathbb{P}[\xi(0) \in S] = \nu(S), \tag{1}$$

$$\mathbb{P}[\xi(k + 1) \in S|\xi(k) = s] = P(s, S), \tag{2}$$

for all $s \in \mathbb{R}^l$, $S \in \mathcal{B}(\mathbb{R}^l)$, $k \in \mathbb{N}_0$. Note that for each $s \in \mathbb{R}^l$, $P(s, \cdot) : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ is a probability measure on the measurable space $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$; furthermore, for each $S \in \mathcal{B}(\mathbb{R}^l)$, $P(\cdot, S) : \mathbb{R}^l \to [0, 1]$ is a measurable function on Euclidean space $\mathbb{R}^l$ (see Athreya and Lahiri (2006); Durrett (2010)).

We define i-step transition probability functions $P^{(i)} : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ by

$$P^{(0)}(s, S) \triangleq \begin{cases} 1, & \text{if } s \in S, \\ 0, & \text{otherwise,} \end{cases}$$

$$P^{(n+1)}(s, S) \triangleq \int_{\mathbb{R}^l} P^{(n)}(\bar{s}, S)P(s, \mathrm{d}\bar{s}), \quad n \in \mathbb{N}.$$

Note that $P^{(1)}(s, S) = P(s, S)$, $s \in \mathbb{R}^l$, $S \in \mathcal{B}(\mathbb{R}^l)$. For a given time $k \in \mathbb{N}_0$ and step size $i \in \mathbb{N}_0$, $P^{(i)}(s, S)$ denotes the conditional probability that the Markov process will take a value inside the set $S \in \mathcal{B}(\mathbb{R}^l)$ at time $k + i$, given that it had the value $s \in \mathbb{R}^l$ at time k, that is

$$\mathbb{P}[\xi(k + i) \in S|\xi(k) = s] = P^{(i)}(s, S), \quad k, i \in \mathbb{N}_0.$$

A probability measure $\pi : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ is a *stationary distribution* of Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ if

$$\int_{\mathbb{R}^l} P(s, S)\pi(\mathrm{d}s) = \pi(S), \quad S \in \mathcal{B}(\mathbb{R}^l). \tag{3}$$

A Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is called *aperiodic* if there is no integer $d \geq 2$ and non-empty subsets $S_i \subseteq \mathbb{R}^l$, $i \in \{1, 2, \ldots, d\}$, such that $S_i \cap S_j = \emptyset$, $i \neq j$, $P(s, S_{i+1}) = 1$, $s \in S_i$, $i \in \{1, 2, \ldots, d-1\}$ and $P(s, S_1) = 1$, $s \in S_d$ (see Rosenthal (2011)).

In Section 3, we employ an aperiodic Markov process defined on a Euclidean space to characterize the parameter of a discrete-time linear stochastic parameter-varying dynamical system.

2.2 Stationarity and Ergodicity of Stochastic Processes

In this section we first give the definition of *stationarity*, then we explain *measure preserving transformations* and *ergodic* stochastic processes.

A stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is called *stationary* if for every $n \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{P}[\zeta(i) \in S_1, \zeta(i + 1) \in S_2, \ldots \zeta(i + n - 1) \in S_n] \\ &\quad = \mathbb{P}[\zeta(j) \in S_1, \zeta(j + 1) \in S_2, \ldots \zeta(j + n - 1) \in S_n],\end{aligned} \tag{4}$$

for all $S_k \in \mathcal{B}(\mathbb{R}^l)$, $k \in \{1, 2, \ldots, n\}$, and $i, j \in \mathbb{N}_0$. Note that for a stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$, the joint distribution of random variables $\zeta(k), \zeta(k + 1), \ldots, \zeta(k + n)$ is the same for all $k \in \mathbb{N}_0$, in other words the joint distribution does not change over time (Athreya and Lahiri, 2006; Klenke, 2008). It is important to note that a time-homogeneous discrete-time Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ characterized with the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ and the initial distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ is stationary if the initial distribution $\nu(\cdot)$ is also a stationary distribution, that is,

$$\int_{\mathbb{R}^l} P(s, S)\nu(\mathrm{d}s) = \nu(S), \quad S \in \mathcal{B}(\mathbb{R}^l). \tag{5}$$

Now consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A measurable function $T : \Omega \to \Omega$ is called *measure preserving transformation* if

$$\mathbb{P}[T^{-1}(F)] = \mathbb{P}[F], \quad F \in \mathcal{F},$$

where

$$T^{-1}(F) \triangleq \{\omega \in \Omega : T(w) \in F\}, \quad F \in \mathcal{F}. \tag{6}$$

Note that every stationary stochastic process is associated with a measure preserving transformation (Athreya and Lahiri, 2006; Klenke, 2008). We define the measure preserving transformation associated with the stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ in the following way. First, let $\Omega \triangleq (\mathbb{R}^l)^{\mathbb{N}_0}$ denote the space that includes all infinite-sequences of $\mathbb{R}^l$-valued vectors, and let $\mathcal{F} \triangleq \mathcal{B}\big((\mathbb{R}^l)^{\mathbb{N}_0}\big)$ denote the product σ-algebra (see Athreya and Lahiri (2006); Klenke (2008)). Furthermore, let $\mathbb{P}$ be the probability measure induced by $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. Note that all sequences of the form $\omega \triangleq \{\omega(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ are included in Ω; moreover, $\mathcal{F}$ includes all sets of the form $\{\omega \in \Omega : \omega(i) \in S_1, \omega(i + 1) \in S_2, \ldots \omega(i + n - 1) \in S_n\}$, for all $S_k \in \mathcal{B}(\mathbb{R}^l)$, $k \in \{1, 2, \ldots, n\}$, and $i \in \mathbb{N}_0$. For a fixed $\omega \in \Omega$, the stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is given by $\zeta(k) = \omega(k), k \in \mathbb{N}_0$.

Now, we define $T_\zeta : \Omega \to \Omega$ by

$$T_\zeta(\{\omega(k)\}_{k\in\mathbb{N}_0}) \triangleq \{\omega(k+1)\}_{k\in\mathbb{N}_0}, \quad \omega \in \Omega. \tag{7}$$

Note that $T_\zeta : \Omega \to \Omega$ shifts the sequence $\omega \in \Omega$. The stationarity of the stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ implies that the function $T_\zeta : \Omega \to \Omega$ is a measure preserving transformation (Athreya and Lahiri, 2006). For the measure preserving transformation $T_\zeta : \Omega \to \Omega$, we define $T_\xi^i : \Omega \to \Omega$, by $T_\zeta^0(\omega) = \omega$ and $T_\zeta^{i+1}(\omega) = T_\zeta(T_\zeta^i(\omega))$, $i \in \mathbb{N}_0$.

Consider the stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ and the associated measure preserving transformation $T_\zeta : \Omega \to \Omega$ defined in (7). The stationary stochastic process $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is called *ergodic* if $\mathbb{P}[F] = 0$ or $\mathbb{P}[F] = 1$ for all $F \in \mathcal{F}$ such that $T_\zeta^{-1}(F) = F$.

Now let $\{\zeta(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ be a stationary and ergodic stochastic process. Furthermore, let $f : \mathbb{R}^l \to \mathbb{R}$ be a measurable function such that $\mathbb{E}[|f(\zeta(0))|] < \infty$. *Ergodic Theorem* (Athreya and Lahiri, 2006; Klenke, 2008) states that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\zeta(k)) = \mathbb{E}[f(\zeta(0))]$, almost surely.

In Section 3 below, we consider a discrete-time linear stochastic parameter-varying dynamical system. The parameter of the dynamical system is modeled as an *aperiodic*, *stationary,* and *ergodic* discrete-time Markov process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. We investigate the stabilization problem for the case where the parameter process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is observed (sampled) at every $\tau \in \mathbb{N}$ time steps. The sequences of values that the parameter $\xi(\cdot)$ takes between consecutive observation instants are characterized through the stochastic process $\{\hat{\xi}(n) \in \underbrace{\mathbb{R}^l \times \mathbb{R}^l \times \cdots \times \mathbb{R}^l}_{\tau \text{ terms}}\}_{n\in\mathbb{N}_0}$ defined by

$$\hat{\xi}(n) \triangleq \big(\xi(n\tau), \xi(n\tau+1), \ldots, \xi((n+1)\tau-1)\big), \tag{8}$$

for $n \in \mathbb{N}_0$. Our main results presented in Section 3 rely on Lemma 1 below, which shows that the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ defined in (8) is also *stationary* and *ergodic.*

Lemma 1. Suppose that $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is a discrete-time aperiodic, stationary, and ergodic Markov process. Then the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ that is defined in (8) for a given $\tau \in \mathbb{N}$ is stationary and ergodic.

Proof. The proof is omitted due to space limitations. □

In Section 4 we investigate *almost sure asymptotic stabilization* of a discrete-time linear stochastic parameter-varying system. The zero solution $x(k) \equiv 0$ of a stochastic system with a fixed initial condition $x_0(\cdot)$ is *asymptotically stable almost surely* if

$$\mathbb{P}[\lim_{k\to\infty} \|x(k)\|^2 = 0] = 1. \tag{9}$$

3. SAMPLED-PARAMETER FEEDBACK CONTROL OF DISCRETE-TIME LINEAR STOCHASTIC PARAMETER-VARYING SYSTEMS

In this section, we first provide the mathematical model for a discrete-time linear stochastic parameter-varying system. Then we explain the feedback control problem under periodically observed (sampled) parameter information and present our proposed sampled-parameter control framework for stabilizing discrete-time linear stochastic parameter-varying systems.

3.1 Mathematical Model

We consider the discrete-time linear stochastic dynamical system given by

$$x(k+1) = A(\xi(k))x(k) + B(\xi(k))u(k),\ k \in \mathbb{N}_0, \tag{10}$$

with the initial condition $x(0) = x_0$, where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input. Furthermore, $A : \mathbb{R}^l \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^l \to \mathbb{R}^{n\times m}$ denote the parameter-dependent system matrices. The parameter denoted by $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is assumed to be an aperiodic, stationary, and ergodic Markov process characterized by the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0,1]$ and the *initial stationary* distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0,1]$. Note that

$$\mathbb{P}[\xi(0) \in S] = \nu(S), \quad S \in \mathcal{B}(\mathbb{R}^l), \tag{11}$$

$$\int_{\mathbb{R}^l} P(s,S)\nu(\mathrm{d}s) = \nu(S), \quad S \in \mathcal{B}(\mathbb{R}^l). \tag{12}$$

Note that a class of *switched* stochastic systems can be modeled as stochastic parameter-varying systems of the form (10). For instance, the discrete-time switched linear stochastic system discussed in Cetinkaya and Hayakawa (2013a,b) is a special case of the dynamical system (10), where $\{\xi(k)\}_{k\in\mathbb{N}_0}$ is modeled as an aperiodic and irreducible finite-state Markov chain. Note that in Cetinkaya and Hayakawa (2013a,b), the parameter $\xi(\cdot)$ indicates the active mode (subsystem) that governs the overall dynamics of a switched system. Furthermore, note that linear systems with stationary and ergodic *autoregressive* parameters can also be characterized through (10), since vector autoregressions are Markov processes (see Durrett (2010)).

3.2 Control Under Periodic Parameter Observations

In this paper, we investigate feedback stabilization of the linear parameter-varying dynamical system (10) under the assumption that only a periodically-sampled version of the parameter process $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is available for control purposes. Specifically, we assume that the parameter $\xi(\cdot)$ is observed (sampled) periodically at time instants $0, \tau, 2\tau, \ldots$, where $\tau \in \mathbb{N}$ denotes the parameter observation period. The sampled parameter information that is available to the controller is characterized through the stochastic process $\{\phi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ defined by

$$\phi(k) = \xi(n\tau),\ k \in \{n\tau, n\tau+1, \ldots (n+1)\tau - 1\}, \tag{13}$$

for $n \in \mathbb{N}_0$.

In order to achieve stabilization of the dynamical system (10), we propose the control law

$$u(k) = K(\phi(k))x(k), \quad k \in \mathbb{N}_0, \tag{14}$$

where $K : \mathbb{R}^l \to \mathbb{R}^{m\times n}$ denotes the sampled-parameter-dependent feedback gain. Note that the control law (14) requires *only* sampled parameter information.

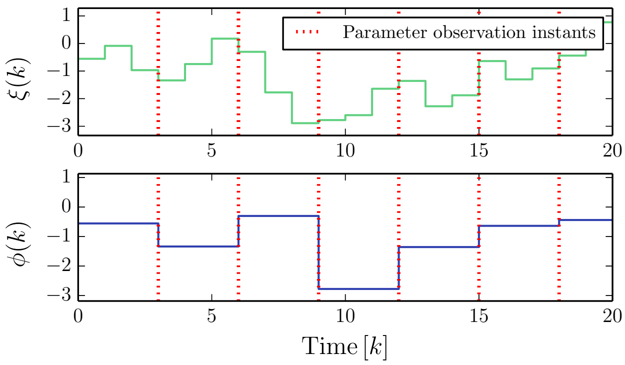

Fig. 1. Actual parameter $\xi(k)$ and its sampled version $\phi(k)$

Fig. 1 shows sample paths of a parameter process $\{\xi(k) \in \mathbb{R}\}_{k\in\mathbb{N}_0}$ (modeled as an autoregressive process), and its sampled version $\{\phi(k) \in \mathbb{R}\}_{k\in\mathbb{N}_0}$. In this example, the parameter $\xi(\cdot)$ is observed (sampled) at every $\tau = 3$ steps. At these parameter observation instants, actual parameter and its sampled version share the same value. However, at other time instants, actual parameter may differ from its sampled version. Hence, the perfect knowledge of the actual parameter is available to the controller only at the parameter observation instants.

Note that the system dynamics in (10) depend on the actual parameter $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$, whereas the feedback gain of the control law (14) depends on the sampled version of the parameter, $\{\phi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$. In the following lemma we present an ergodic theorem for the coupled stochastic process $\{(\xi(t), \phi(k)) \in \mathbb{R}^l \times \mathbb{R}^l\}_{k\in\mathbb{N}_0}$, which is composed of the original parameter process and its sampled version. Note that the result provided in Lemma 2 below is crucial for developing the main results of this paper presented in Theorems 3, 4, and Corollary 5.

Lemma 2. Suppose $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ is an aperiodic, stationary, and ergodic Markov process characterized by the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ and the *initial stationary* distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$. Furthermore, let $\{\phi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ defined as in (13) be the periodically sampled version of $\{\xi(k) \in \mathbb{R}^l\}_{k\in\mathbb{N}_0}$ for a given sampling period $\tau \in \mathbb{N}$. Then for any Borel measurable function $\gamma : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$, it follows that

$$\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = \frac{1}{\tau} \sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \gamma(\bar{\xi}, \bar{\phi}) P^{(i)}(\bar{\phi}, \mathrm{d}\bar{\xi}) \nu(\mathrm{d}\bar{\phi}), \tag{15}$$

almost surely.

Proof. Let $\{\hat{\xi}(n) \in \underbrace{\mathbb{R}^l \times \mathbb{R}^l \times \cdots \times \mathbb{R}^l}_{\tau \text{ terms}}\}_{n\in\mathbb{N}_0}$ be the stochastic process defined in (8). Note that $\hat{\xi}(n)$ denotes the sequence of values that the parameter $\xi(\cdot)$ takes between consecutive observation instants $n\tau$ and $(n+1)\tau$. Furthermore, let $N(k) \triangleq \lfloor k/\tau \rfloor$, $k \in \mathbb{N}_0$. The number of mode samples obtained up to time $k \in \mathbb{N}_0$ is given by $N(k) + 1$. Note that, for all $\bar{n} \in \mathbb{N}$ such that $\bar{n} > \tau$, we have

$$\sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = \sum_{n=0}^{N(\bar{n})-1} \sum_{i=0}^{\tau-1} \gamma(\xi(n\tau + i), \phi(n\tau + i)) + \sum_{k=N(\bar{n})\tau}^{\bar{n}-1} \gamma(\xi(k), \phi(k)). \tag{16}$$

Since $\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=N(\bar{n})\tau}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = 0$, it follows from (16) that

$$\begin{aligned}\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) &= \lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{n=0}^{N(\bar{n})-1} \sum_{i=0}^{\tau-1} \gamma(\xi(n\tau + i), \phi(n\tau + i)) \\ &= \lim_{\bar{n}\to\infty} \frac{N(\bar{n})}{\bar{n}} \frac{1}{N(\bar{n})} \sum_{n=0}^{N(\bar{n})-1} \sum_{k=0}^{\tau-1} \gamma(\xi(n\tau + i), \phi(n\tau + i)) \\ &= \lim_{\bar{n}\to\infty} \frac{N(\bar{n})}{\bar{n}} \frac{1}{N(\bar{n})} \sum_{n=0}^{N(\bar{n})-1} \hat{\gamma}(\hat{\xi}(n)),\end{aligned} \tag{17}$$

where $\hat{\gamma}(\hat{\xi}(n)) \triangleq \sum_{k=0}^{\tau-1} \gamma(\xi(n\tau + k), \phi(n\tau + k))$. Now, by using the definition of $N(\cdot)$, we obtain $\lim_{\bar{n}\to\infty} \frac{N(\bar{n})}{\bar{n}} = \frac{1}{\tau}$. Furthermore, it follows from Lemma 1 that the stochastic process $\{\hat{\xi}(n)\}_{n\in\mathbb{N}_0}$ is stationary and ergodic. Thus, by the ergodic theorem for stationary and ergodic stochastic processes, we obtain $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \bar{\gamma}(\hat{\xi}(n)) = \mathbb{E}[\hat{\gamma}(\hat{\xi}(0))]$. Therefore,

$$\begin{aligned}\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) &= \frac{1}{\tau} \mathbb{E}[\hat{\gamma}(\hat{\xi}(0))] \\ &= \frac{1}{\tau} \mathbb{E}[\sum_{i=0}^{\tau-1} \gamma(\xi(i), \phi(i))] \\ &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} \mathbb{E}[\gamma(\xi(i), \phi(i))].\end{aligned} \tag{18}$$

Note that since the value of sampled parameter process $\phi(\cdot)$ does not change between parameter observation instants, we have $\phi(i) = \phi(0) = \xi(0)$, $i \in \{0, 1, \ldots, \tau - 1\}$. It then follows that

$$\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) = \frac{1}{\tau} \sum_{i=0}^{\tau-1} \mathbb{E}[\gamma(\xi(i), \xi(0))]. \tag{19}$$

Now by using the transition probability function $P : \mathbb{R}^l \times \mathcal{B}(\mathbb{R}^l) \to [0, 1]$ and the initial stationary distribution $\nu : \mathcal{B}(\mathbb{R}^l) \to [0, 1]$, we obtain

$$\begin{aligned}\lim_{\bar{n}\to\infty} \frac{1}{\bar{n}} \sum_{k=0}^{\bar{n}-1} \gamma(\xi(k), \phi(k)) &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \gamma(\bar{\xi}, \bar{\phi}) \mathbb{P}[\xi(i) \in \mathrm{d}\bar{\xi}, \xi(0) \in \mathrm{d}\bar{\phi}] \\ &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \gamma(\bar{\xi}, \bar{\phi}) P^{(i)}(\bar{\phi}, \mathrm{d}\bar{\xi}) \nu(\mathrm{d}\bar{\phi}),\end{aligned} \tag{20}$$

which completes the proof. □

4. SUFFICIENT CONDITIONS FOR ALMOST SURE ASYMPTOTIC STABILIZATION

In this section, we utilize the result presented in Lemma 2 and obtain sufficient almost sure asymptotic stabilization conditions for the closed-loop stochastic parameter-varying system (10), (14).

Theorem 3. Consider the linear parameter-varying control system (10), (14). If there exist a matrix $R > 0$ and a measurable function $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to (0, \infty)$ such that

$$0 \geq (A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi}))^{\mathrm{T}} R \cdot (A(\bar{\xi}) + B(\bar{\xi})K(\bar{\phi})) - \lambda(\bar{\xi}, \bar{\phi})R, \quad \bar{\xi}, \bar{\phi} \in \mathbb{R}^l, \tag{21}$$

$$\sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \ln(\lambda(\bar{\xi}, \bar{\phi})) P^{(i)}(\bar{\phi}, \mathrm{d}\bar{\xi}) \nu(\mathrm{d}\bar{\phi}) < 0, \tag{22}$$

then the zero solution $x(k) \equiv 0$ of the closed-loop system (10), (14) is asymptotically stable almost surely.

Proof. First, let $V : \mathbb{R}^n \to [0, \infty)$ be the positive-definite function defined by $V(x) \triangleq x^{\mathrm{T}} R x$. It follows from (10) and (14) that for $k \in \mathbb{N}_0$,

$$V(x(k+1)) = x^{\mathrm{T}}(k)\big(A(\xi(k)) + B(\xi(k))K(\phi(k))\big)^{\mathrm{T}} R \cdot \big(A(\xi(k)) + B(\xi(k))K(\phi(k))\big)x(k). \tag{23}$$

We now use (21), (23) and definition of $V(\cdot)$ to obtain

$$V(x(k+1)) \leq \lambda(\xi(k), \phi(k))V(x(k)) \leq \theta(k)V(x(0)), \quad k \in \mathbb{N}_0, \tag{24}$$

where $\theta(k) \triangleq \prod_{n=0}^{k} \lambda(\xi(n), \phi(n))$, $k \in \mathbb{N}_0$. It follows that

$$\ln(\theta(k)) = \sum_{n=0}^{k} \ln(\lambda(\xi(n), \phi(n))), \quad k \in \mathbb{N}_0. \tag{25}$$

Furthermore, as a consequence of Lemma 2,

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{k} \ln(\theta(k)) \\ &= \lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k} \ln(\lambda(\xi(n), \phi(n))) \\ &= \frac{1}{\tau} \sum_{i=0}^{\tau-1} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \ln(\lambda(\bar{\xi}, \bar{\phi})) P^{(i)}(\bar{\phi}, \mathrm{d}\bar{\xi}) \nu(\mathrm{d}\bar{\phi}). \end{aligned} \tag{26}$$

It then follows from (22) and (26) that $\lim_{k\to\infty} \frac{1}{k} \ln(\theta(k)) < 0$, almost surely. Hence, $\lim_{k\to\infty} \ln \theta(k) = -\infty$, almost surely, and therefore, $\mathbb{P}[\lim_{k\to\infty} \theta(k) = 0] = 1$. Now, as a result of (24), we obtain $\mathbb{P}[\lim_{k\to\infty} V(x(k)) = 0] = 1$, which implies that the zero solution $x(k) \equiv 0$ of the closed-loop system (10), (14) is asymptotically stable almost surely. □

Theorem 3 provides sufficient conditions for almost sure asymptotic stability of the zero solution of the closed-loop system (10) under the control law (14). Conditions (21) and (22) of Theorem 3 reflect that the stabilization performance depend not only on the system dynamics but also on the probabilistic dynamics of parameter transitions as well as the parameter observation period $\tau \in \mathbb{N}$.

In the next section, we explore the sampled-parameter control problem for a linear parameter-varying system with a state matrix that depend affinely on the stochastic parameter $\{\xi(k)\}_{k\in\mathbb{N}_0}$.

4.1 Stabilization of Linear Parameter-Varying Systems with Affine Parameter Dependence

We now consider a special case of the parameter-varying dynamical system (10) where the state matrix $A(\cdot)$ is defined as an affine function of the entries of the parameter vector $\xi(\cdot) \in \mathbb{R}^l$; moreover, the input matrix $B(\cdot)$ is defined as a constant matrix. Specifically, we consider the linear parameter-varying system (10) with

$$A(\bar{\xi}) \triangleq \big(\bar{A}_0 + \sum_{i=1}^{l} \bar{\xi}_i \bar{A}_i\big), \quad \bar{\xi} \in \mathbb{R}^l, \tag{27}$$

$$B(\bar{\xi}) \triangleq \bar{B}, \tag{28}$$

where $\bar{A}_i \in \mathbb{R}^{m\times n}$, $i \in \{0, 1, \ldots, l\}$, and $\bar{B} \in \mathbb{R}^{n\times m}$ are constant matrices. In order to achieve stabilization of the zero solution of dynamical system (10) with state and input matrices given by (27) and (28), we employ the control law (14) with the sampled-parameter-dependent feedback gain function

$$K(\bar{\phi}) \triangleq \bar{K}_0 + \sum_{i=1}^{l} \bar{\phi}_i \bar{K}_i, \quad \bar{\phi} \in \mathbb{R}^l, \tag{29}$$

where $\bar{K}_i \in \mathbb{R}^{m\times n}$, $i \in \{0, 1, \ldots, l\}$, are constant matrices. Note that the feedback gain (29) is an affine function of the entries of the sampled parameter vector $\phi(\cdot)$.

In Theorem 4 below, we present sufficient conditions under which the proposed control law (14) with the feedback gain (29) guarantees almost sure asymptotic stabilization of the linear stochastic parameter-varying system (10) with state and input matrices given by (27) and (28).

Theorem 4. Consider the linear parameter-varying system (10) with state and input matrices given by (27) and (28). If there exist a matrix $R > 0$ and scalars $\alpha_i \in (0, \infty)$, $i \in \{1, 2, \ldots, l\}$, $\beta_i \in (0, \infty)$, $i \in \{0, 1, \ldots, l\}$, such that

$$0 \geq \bar{A}_i^{\mathrm{T}} R \bar{A}_i - \alpha_i R, \quad i \in \{1, \ldots, l\}, \tag{30}$$

$$0 \geq (\bar{A}_i + \bar{B}\bar{K}_i)^{\mathrm{T}} R \cdot (\bar{A}_i + \bar{B}\bar{K}_i) - \beta_i R, \quad i \in \{0, 1, \ldots, l\}, \tag{31}$$

and (22) hold with $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to (0, \infty)$ given by

$$\lambda(\bar{\xi}, \bar{\phi}) \triangleq (2l+1)\big(\beta_0^2 + \sum_{i=1}^{l} \big((\bar{\xi}_i - \bar{\phi}_i)^2 \alpha_i^2 + \bar{\phi}_i^2 \beta_i^2\big)\big), \tag{32}$$

then the control law (14) with the feedback gain (29) guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (10), (14) is asymptotically stable almost surely.

Proof. The proof is omitted due to space limitations. □

Note that the conditions presented in Theorem 4 can be used for assessing almost sure asymptotic stability of the closed-loop system (10), (14) with the system matrices (27), (28) when the gain matrices $\bar{K}_i \in \mathbb{R}^{m\times n}$, $i \in \{0, 1, \ldots, l\}$, for the control law (14), (29) are already known. In practice, we often need to employ numerical methods for finding gain matrices so that the proposed control law (14) with those gains achieves almost sure asymptotic stabilization. In Corollary 5 below, we present an alternative set of sufficient almost sure asymptotic stabilization conditions, which are well suited for finding *stabilizing gain matrices* $\bar{K}_i \in \mathbb{R}^{m\times n}$, $i \in \{0,1,\ldots,l\}$, through numerical methods.

Corollary 5. Consider the linear parameter-varying system (10) with state and input matrices given by (27) and (28). If there exist matrices $\tilde{R} > 0$, $\bar{L}_i \in \mathbb{R}^{m\times n}$, $i \in \{0,1,\ldots,l\}$, and scalars $\alpha_i \in (0,\infty), i \in \{1,2,\ldots,l\}$, $\beta_i \in (0,\infty)$, $i \in \{0,1,\ldots,l\}$, such that

$$0 \geq (\bar{A}_i\tilde{R})^{\mathrm{T}}\tilde{R}^{-1}(\bar{A}_i\tilde{R}) - \alpha_i\tilde{R}, \quad i \in \{1,2,\ldots,l\}, \quad (33)$$

$$0 \geq (\bar{A}_i\tilde{R} + \bar{B}\bar{L}_i)^{\mathrm{T}}\tilde{R}^{-1} \cdot (\bar{A}_i\tilde{R} + \bar{B}\bar{L}_i) - \beta_i\tilde{R}, \quad i \in \{0,1,\ldots,l\}, \quad (34)$$

and (22) hold with $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to (0,\infty)$ given in (32), then the control law (14), (29) with gain matrices $\bar{K}_i = \bar{L}_i\tilde{R}^{-1}$, $i \in \{0,1,\ldots,l\}$, guarantees that the zero solution $x(k) \equiv 0$ of the closed-loop system (10), (14) is asymptotically stable almost surely.

Proof. The result is a direct consequence of Theorem 4 with $R = \tilde{R}^{-1}$. □

Remark 6. Conditions (22), (33), and (34) of Corollary 5 can be verified using a numerical technique. Specifically, following the approach presented in Cetinkaya and Hayakawa (2013a,b), we transform conditions (33) and (34) into the matrix inequalities

$$0 \leq \begin{bmatrix} \alpha_i\tilde{R} & (\bar{A}_i\tilde{R})^{\mathrm{T}} \\ (\bar{A}_i\tilde{R}) & \tilde{R} \end{bmatrix}, \quad i \in \{1,\ldots,l\}, \quad (35)$$

$$0 \leq \begin{bmatrix} \beta_i\tilde{R} & (\bar{A}_i\tilde{R} + \bar{B}\bar{L}_i)^{\mathrm{T}} \\ (\bar{A}_i\tilde{R} + \bar{B}\bar{L}_i) & \tilde{R} \end{bmatrix}, i \in \{0,1,\ldots,l\}, (36)$$

by using Schur complements (see Bernstein (2009)). Note that given $\alpha_i, i \in \{1,2,\ldots,l\}$, and β_i, $i \in \{0,1,\ldots,l\}$, the inequalities (35) and (36) are linear in $\tilde{R}$ and $\bar{L}_i$, $i \in \{0,1,\ldots,l\}$. We first find a set of values of $\alpha_i, i \in \{1,2,\ldots,l\}$, and β_i, $i \in \{0,1,\ldots,l\}$, that satisfy (22) with $\lambda(\cdot)$ calculated according to (32). We then iterate over the values of $\alpha_i, i \in \{1,2,\ldots,l\}$, and β_i, $i \in \{0,1,\ldots,l\}$, in this set, and look for feasible solutions to the linear matrix inequalities (35) and (36).

5. CONCLUSION

We investigated feedback control of discrete-time linear stochastic parameter-varying systems under sampled parameter information. Specifically, we considered the case where the parameter of the system is observed (sampled) periodically; furthermore, we proposed a control law that depends only on the sampled version of the parameter. We obtained sufficient conditions under which our control framework guarantees almost sure asymptotic stabilization of the zero solution.

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