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Solution of affine quadratic control problems ⋆

Ankita Sharma * **A. J. Shaiju** **

* *Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, INDIA(Tel: 044-2257-4600; e-mail: ankita.iitm22@gmail.com).*
** *Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, INDIA(Tel: 044-2257-4638; e-mail: ajshaiju@iitm.ac.in).*

Abstract: For affine control problems, it is known that the optimal controller for these classes of problems can be expressed in terms of the associated co-state. A method to obtain the co-state vector in terms of the state vector is given in this article. Using this method, the optimal control can be written as a function of state variable. Further the method is illustrated using examples.

Keywords: Optimal control; Affine control.

1. INTRODUCTION

Solution to the Linear Quadratic Regulator (LQR) problem is well known [1], [2]. There have been some works in the control literature in which the optimal control, for various classes of problems, is expressed as a function of the state [3], [4], [5]. Recently in [6] affine quadratic problems were studied and it was shown that the optimal control is a function of the associated co-state vector.
In this article, we show that the co-state vector for an affine quadratic control problem can be expressed in terms of the state vector. In Section 2, we introduce the affine quadratic control problem and show how the optimal control $u^*(\cdot)$ can be written in terms of the co-state. In Section 3, we explain how to obtain the co-state in terms of the state and hence establishing a way of getting the optimal control in terms of the state vector. This method is illustrated using an example in Section 4.

2. AFFINE QUADRATIC CONTROL PROBLEM

We consider the affine control system:

$$\dot{x}(t) = f(x(t),t) + g(x(t),t)u(t); \quad x(0) = x_0, \quad t \in [0,T]. \tag{1}$$

Here x is an n-vector, u is an m-vector, and f, g are C^1 functions. Corresponding to each control on the time interval [0, T], a cost is assigned via the cost or objective functional

$$J(x_0, u(\cdot)) = \frac{1}{2}\int_0^T (x'(t)Qx(t) + u'(t)Ru(t))\, dt, \tag{2}$$

where $'$ denotes transposition, $Q \in R^{n\times n}$ is a positive semidefinite matrix and $R \in R^{m\times m}$ is a positive definite matrix.

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The optimal control problem is to find a control $u^*(\cdot)$ which minimizes the cost functional $J(x_0, u(\cdot))$. The Hamiltonian associated with the optimal control problem (1), (2) is given by,

$$H(x,u,\lambda,t) = \frac{1}{2}(x'Qx + u'Ru) + \lambda'(f(x,t) + g(x,t)u), \tag{3}$$

where $\lambda \in R^n$ is the co-state variable.
To derive an expression for the optimal control, it is convenient to introduce the adjoint system:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}; \quad \lambda(T) = 0$$

$$\left.\begin{aligned} i.e.\ \ \dot{\lambda}(t) &= -Qx(t) - \left(\frac{\partial f}{\partial x}(x(t),t)\right)'\lambda(t) \\ &+ u'(t)\left(\frac{\partial g}{\partial x}(x(t),t)\right)'\lambda(t); \quad \lambda(T) = 0. \end{aligned}\right\} \tag{4}$$

By Pontryagin's Minimum Principle (PMP) [6], [7] the optimal control is

$$u^*(t) = -R^{-1}g'(x^*(t),t)\lambda^*(t). \tag{5}$$

Here $x^*(\cdot)$ and $\lambda^*(\cdot)$ are respectively the solutions of (1) and (4), corresponding to $u^*(\cdot)$. Now to obtain $\lambda^*(t)$, we solve the following 2n-dimensional system:

$$\left.\begin{aligned} \dot{x^*}(t) &= f(x^*(t),t) + g(x^*(t),t)(-R^{-1}g'(x^*(t),t)\lambda^*(t)); \\ &\quad x^*(0) = x_0^*, \\ \dot{\lambda^*}(t) &= -Qx^*(t) - \left(\frac{\partial f}{\partial x}(x^*(t),t)\right)'\lambda^*(t) + \Big(\Sigma_{i=1}^m \Sigma_{j=1}^m \\ &\quad \Sigma_{k=1}^n \lambda_k^*(t) g_{kj}(x^*(t),t) r_{ij} \frac{\partial g_i}{\partial x}(x^*(t),t)\Big)\lambda^*(t); \\ &\quad \lambda^*(0) = \lambda_0^*. \end{aligned}\right\} \tag{6}$$

Here $\lambda_k^*(t)$ denotes the k-th element of the vector $\lambda^*(t)$, $g_{kj}(x^*(t),t)$ denotes the (k, j)-th element of the matrix $g(x^*(t),t)$, r_{ij} denotes (i, j)-th element of the matrix

$(R^{-1})'$, and
$g(x^*(t),t) = [g_1(x^*(t),t), g_2(x^*(t),t), ..., g_m(x^*(t),t)]$ with $g_i(x^*(t),t) \in R^n, i = 1,2,...,m$.
Now to obtain $\lambda^*(\cdot)$, it is enough to find λ_0^*. From the PMP, it follows that this λ_0^* has the property to minimize the map

$$\lambda_0 \mapsto |\lambda^*(T,\lambda_0)|^2 \tag{7}$$

In the next section, we explain how to minimize the map in (7) and obtained λ_0^*.
Note that this will help us in expressing $\lambda^*(\cdot)$ (and hence $u^*(\cdot)$ in (5)) in terms of the state $x^*(\cdot)$.

3. CO-STATE IN TERMS OF STATE

We begin by expressing $\lambda^*(t)$ in terms of an $n \times n$ matrix valued function $Z(t)$ and an $n \times 1$ matrix valued function $y(t)$, where $Z(t)$ and $y(t)$ satisfy differential equations simpler than (6).
(A) Let $x^*(t), \lambda^*(t)$ be the solution of (6). In this solution, the second component $\lambda^*(t)$ can be computed using the following relation:

$$\lambda^*(t) = Z^{-1}(t)y(t),$$

where $(y(t), Z(t))$ is the solution of

$$\left.\begin{aligned} \dot{y}(t) &= -\left(\frac{\partial f}{\partial x}(x^*(t),t)\right)' y(t) - Z(t)Qx^*(t) \\ &\quad ;\ y(0) = \lambda_0^*, \\ \dot{Z}(t) &= -\big(\Sigma_{i=1}^m \Sigma_{j=1}^m \Sigma_{k=1}^n y_k(t) g_{kj}(x^*(t),t) r_{ij} \\ &\quad \frac{\partial g_i}{\partial x}(x^*(t),t)\Big)\ ;\ Z(0) = I. \end{aligned}\right\} \tag{8}$$

Here $y_k(t)$ denotes the k-th element of the vector $y(t)$ and it is assumed that $Z(t)$ commutes with the matrices $\left(\frac{\partial f}{\partial x}(x^*(t),t)\right)'$ and $\left(\Sigma_{i=1}^m \Sigma_{j=1}^m \Sigma_{k=1}^n \lambda_k^*(t) g_{kj}(x^*(t),t) r_{ij} \frac{\partial g_i}{\partial x}(x^*(t),t)\right)$ for every t.
We now use (A) to find the derivative of the map given in (7). Note that this derivative, evaluated at $\lambda_0 = \lambda_0^*$, has to be zero because λ_0^* is the minimizer for the map given in (7).

Lemma 1. Under the assumption of (A), the derivative of the map in (7) is

$$2\left(\frac{\partial \lambda^*}{\partial \lambda_0}(T,\lambda_0)\right)' \lambda^*(T,\lambda_0),$$

where

$$\frac{\partial \lambda^*}{\partial \lambda_0}(T,\lambda_0) = exp\left\{\int_0^T \left[-\frac{\partial f}{\partial x}(x^*(t),t) + 2\Sigma_{i=1}^m \Sigma_{j=1}^m \Sigma_{k=1}^n \lambda_k^*(t,\lambda_0) g_{kj}(x^*(t),t) r_{ij} \frac{\partial g_i}{\partial x}(x^*(t),t)\right] dt\right\}.$$

Proof. We consider the map given in (7). It is clear that the derivative of this map is

$$2\left(\frac{\partial \lambda^*}{\partial \lambda_0}(T,\lambda_0)\right)' \lambda^*(T,\lambda_0).$$

Using corollary (Pg. 83 in [8]), we have

$$\frac{\partial \lambda^*}{\partial \lambda_0}(t,\lambda_0) = \Phi(t,\lambda_0), \tag{9}$$

where $\Phi(t,\lambda_0)$ is the fundamental matrix solution of

$$\begin{aligned} \dot{\Phi} &= D\left(-Qx^*(t) - \left(\frac{\partial f}{\partial x}(x^*(t),t)\right)' \lambda^*(t) + \Big(\Sigma_{i=1}^m \Sigma_{j=1}^m \Sigma_{k=1}^n \lambda_k^*(t) g_{kj}(x^*(t),t) r_{ij} \frac{\partial g_i}{\partial x}(x^*(t),t)\Big) \lambda^*(t)\right)\Phi; \\ &\Phi(0,\lambda_0) = I. \end{aligned}$$

Here D denotes the Jacobian evaluated at $\lambda^*(t) = Z^{-1}(t)y(t)$ and its value is given as

$$\left(\frac{\partial f}{\partial x}(x^*(t),t) + 2\Sigma_{i=1}^m \Sigma_{j=1}^m \Sigma_{k=1}^n \lambda_k^*(t) g_{kj}(x^*(t),t) r_{ij} \frac{\partial g_i}{\partial x}(x^*(t),t)\right).$$

Now using the above fact, we have

$$\Phi(t,\lambda_0) = exp\left\{\int_0^t \left[-\frac{\partial f}{\partial x}(x^*(s),s) + 2\Sigma_{i=1}^m \Sigma_{j=1}^m \Sigma_{k=1}^n \lambda_k^*(s,\lambda_0) g_{kj}(x^*(s),s) r_{ij} \frac{\partial g_i}{\partial x}(x^*(s),s)\right] ds\right\}. \tag{10}$$

Hence the result follows by using (9) and (10).

We now recall the fact that $u^*(t)$ can be written in terms of $\lambda^*(t)$ (see equation (5)). From Lemma 1, we obtain a necessary condition that should be satisfied by $\lambda_0^*(t)$, as given in the theorem below.

Theorem 2. Under the assumption of (A), the derivative

$$\left(\frac{\partial \lambda^*}{\partial \lambda_0}(T,\lambda_0)\right)' \lambda^*(T,\lambda_0) = 0 \text{ at } \lambda_0 = \lambda_0^*.$$

Proof. From PMP, it is clear that λ_0^* minimizes the map in (7). Therefore the derivative of the map in (7), evaluated at $\lambda_0 = \lambda_0^*$, has to be zero. This proves the result.

Remark 3. Theorem 2 helps us to express λ_0^* in terms of $x^*(\cdot)$. This allows us to express $\lambda^*(\cdot)$ (and hence $u^*(\cdot)$) as a function of $x^*(\cdot)$.

4. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the theoretical development of Section 3 using some examples.

Example 4. We consider the one-dimensional nonlinear affine control problem (as in (1), (2)) with

$$f(x(t),t) = x^2, g(x(t),t) = x, Q = -1, \text{ and } R = 1.$$

With this choice, (1), (2) reduces to

$$\dot{x}(t) = x^2 + xu;\ x(0) = x_0,\ t \in [0,T], \tag{11}$$

$$J(x_0, u(\cdot)) = \frac{1}{2}\int_0^T (-x^2(t) + u^2(t))\ dt\ . \tag{12}$$

Hamiltonian for this system is

$$H(x,\lambda,u,t) = \frac{1}{2}(-x^2 + u^2) + \lambda(x^2 + xu), \tag{13}$$

and the adjoint equation is

$$\dot{\lambda}(t) = x - 2x\lambda + x\lambda^2;\ \lambda(T) = 0. \tag{14}$$

Now by PMP, the optimal control is

$$u^*(t) = -x^*(t)\lambda^*(t). \tag{15}$$

Here $x^*(\cdot)$ and $\lambda^*(\cdot)$ are respectively the solutions of (11) and (14), corresponding to $u^*(\cdot)$. Now to obtain $\lambda^*(t)$, we solve the following system:

$$\left.\begin{aligned} \dot{x^*}(t) &= {x^*}^2(t) - {x^*}^2(t)\lambda^*(t);\ x^*(0) = x_0^*, \\ \dot{\lambda^*}(t) &= x^*(t) - 2x^*(t)\lambda^*(t) + x^*(t){\lambda^*}^2(t); \\ &\lambda^*(0) = \lambda_0^*. \end{aligned}\right\} \tag{16}$$

In the solution $x^*(t), \lambda^*(t)$ of the initial value problem (16), the second component $\lambda^*(t)$ can be computed as

$$\lambda^*(t) = \frac{y(t)}{z(t)}, \tag{17}$$

where y(t) and z(t) satisfy the differential equations:

$$\left.\begin{aligned} \dot{y}(t) &= -2x^*(t)y(t) + x^*(t)z(t);\ y(0) = \lambda_0^*, \\ \dot{z}(t) &= -x^*(t)y(t);\ z(0) = 1. \end{aligned}\right\} \tag{18}$$

Solving (18) and then substituting $y(t), z(t)$ into (17), we get

$$\lambda^*(t) = 1 + \frac{\lambda_0^* - 1}{(1 - \lambda_0^*)tx^*(t) + 1}\ . \tag{19}$$

Now using the results given in Section 3, we have

$$\lambda_0^* = \frac{x^*(t)T}{x^*(t)T - 1}\ . \tag{20}$$

Now substituting (20) into (19), we get $\lambda^*(t)$ (and hence $u^*(t)$ in (15)) in terms of the state variable $x^*(t)$ i.e.

$$\lambda^*(t) = \frac{(T - t)x^*(t)}{(T - t)x^*(t) - 1}\ .$$

Example 5. We consider the two-dimensional nonlinear affine control problem (as in (1), (2)) with

$$f(x(t), t) = \begin{bmatrix} 0 \\ sinx_2 \end{bmatrix},\ g(x(t), t) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix},\ Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and $R = 1$.

With this choice, (1), (2) reduces to

$$\dot{x}(t) = \begin{bmatrix} x_1 u \\ sinx_2 \end{bmatrix};\ x(0) = x_0,\ t \in [0, T], \tag{21}$$

$$J(x_0, u(\cdot)) = \frac{1}{2}\int_0^T (x_1^2(t) + u^2(t))\ dt\ . \tag{22}$$

Hamiltonian for this system is

$$H(x, \lambda, u, t) = \frac{1}{2}(x_1^2 + u^2) + \lambda' \begin{bmatrix} x_1 u \\ sinx_2 \end{bmatrix}, \tag{23}$$

and the adjoint equation is

$$\dot{\lambda}(t) = -\begin{bmatrix} x_1 + \lambda_1 u \\ \lambda_2 cosx_2 \end{bmatrix};\ \lambda(T) = 0. \tag{24}$$

Now by PMP, the optimal control is

$$u^*(t) = -x_1^*(t)\lambda_1^*(t). \tag{25}$$

Here $x_1^*(\cdot)$ and $\lambda_1^*(\cdot)$ are respectively the solutions of (21) and (24), corresponding to $u^*(\cdot)$. Now to obtain $\lambda_1^*(t)$, we solve the following system:

$$\left.\begin{aligned} \dot{x^*}(t) &= \begin{bmatrix} -{x_1^*}^2(t)\lambda_1^*(t) \\ sinx_2^*(t) \end{bmatrix};\ x^*(0) = \begin{bmatrix} x_{01}^* \\ x_{02}^* \end{bmatrix}, \\ \dot{\lambda^*}(t) &= -\begin{bmatrix} x_1^*(t) - x_1^*(t){\lambda_1^*}^2(t) \\ \lambda_2^*(t)cosx_2^*(t) \end{bmatrix};\ \lambda^*(0) = \begin{bmatrix} \lambda_{01}^* \\ \lambda_{02}^* \end{bmatrix}. \end{aligned}\right\} \tag{26}$$

In the solution $x^*(t), \lambda^*(t)$ of the initial value problem (26), the second component $\lambda^*(t)$ can be computed as

$$\lambda^*(t) = Z^{-1}(t)y(t), \tag{27}$$

where y(t) and Z(t) satisfy the differential equations:

$$\left.\begin{aligned} \dot{y}(t) &= \begin{bmatrix} -z_1(t)x_1^*(t) \\ -y_2(t)cosx_2^*(t) - z_3(t)x_1^*(t) \end{bmatrix}; \\ & y(0) = \begin{bmatrix} \lambda_{01}^* \\ \lambda_{02}^* \end{bmatrix}, \\ \dot{Z}(t) &= -\begin{bmatrix} y_1(t)x_1^*(t) & 0 \\ 0 & 0 \end{bmatrix};\ Z(0) = I. \end{aligned}\right\} \tag{28}$$

Solving (28) and then substituting $y(t), z(t)$ into (27), we get

$$\lambda^*(t) = \begin{bmatrix} \dfrac{(\lambda_{01}^* - 1)e^{tx_1^*(t)} + (\lambda_{01}^* + 1)e^{-tx_1^*(t)}}{-(\lambda_{01}^* - 1)e^{tx_1^*(t)} + (\lambda_{01}^* + 1)e^{-tx_1^*(t)}} \\ \\ \lambda_{02}^* e^{-tcosx_2^*(t)} \end{bmatrix}. \tag{29}$$

Now using the results given in Section 3, we have

$$\lambda_0^* = \begin{bmatrix} \dfrac{e^{Tx_1^*(t)} - e^{-Tx_1^*(t)}}{e^{Tx_1^*(t)} + e^{-Tx_1^*(t)}} \\ \\ 0 \end{bmatrix}. \tag{30}$$

Now substituting (30) into (29), we get $\lambda^*(t)$ (and hence $u^*(t)$ in (25)) in terms of the state variable $x^*(t)$ i.e.

$$\lambda^*(t) = \begin{bmatrix} \dfrac{-e^{-(T-t)x_1^*(t)} + e^{(T-t)x_1^*(t)}}{e^{-(T-t)x_1^*(t)} + e^{(T-t)x_1^*(t)}} \\ \\ 0 \end{bmatrix}.$$

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