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Global Output-Feedback Extremum Seeking Control for Nonlinear Systems with Arbitrary Relative Degree ⋆

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Abstract: An output-feedback sliding mode based extremum seeking controller was recently introduced for linear uncertain systems by using periodic switching functions. Nonlinear systems were also considered but restricted to relative degree one plants as well as the former linear case. Here, generalization is achieved to include more general dynamics with arbitrary relative degree. Global stability properties of the closed-loop system with convergence to a controlled neighborhood of the desired maximum point are also rigorously proved. Simulation results illustrate the performance of the proposed extremum seeking control algorithm.

Keywords: nonlinear systems, arbitrary relative degree, sliding mode control, extremum seeking, output-feedback, global stability.

1. INTRODUCTION

Extremum seeking is a real-time, non-model based adaptive control technique for tuning parameters to optimize an unknown nonlinear map. The most popular extremum seeking approach relies on persistence of excitation, usually a sinusoid, to perturb the parameters being tuned [1, 2, 3, 4]. This quantifies the effects of the parameters on the output of the nonlinear map, then uses that information to generate estimates of the optimal parameter values.

As an alternative, a novel output-feedback extremum seeking sliding mode control (SMC) for a class of linear plants with relative degree one and nonlinear output function was introduced in [5]. In lieu of the traditional sinusoidal dither perturbation technique [1, 2, 3, 4], the real-time optimization problem was solved through a periodic switching function [7]. Related results for more general dynamics including state dependent and unmatched nonlinearities which may provoke finite-time escape were explored in [8] and [9]. The latter by using another tool, named monitoring function. In both approaches, only relative degree one plants could be coped with.

In this paper, the arbitrary relative degree case is pursued. Relative degree compensation and the extremum searching are achieved by combining a high gain observer (with time varying gain) and a norm observer. We also remove some restrictions on the plant dynamics founded in the extremum seeking control literature [10]. This opens the possibility of effective application to autonomous vehicles navigation without position measurements [11, 12, 13].

Global asymptotic convergence with respect to a compact set is demonstrated and, in contrast with high gain observer based schemes [14], the control signal is free of peaking. The resulting approach guarantees convergence of the system output to a small neighborhood of the extremum point using only output-feedback. Numerical simulation examples corroborate the effectiveness of the proposed extremum seeking controller.

Remark 1. In what follows, control signals or inputs (disturbances) are assumed to be measurable locally essentially bounded functions $f : \mathbb{R}_+ \to \mathbb{R}$. The set of all such functions, endowed with the (essential) supremum norm $\|f\| = \operatorname{ess\,sup}\{|f(t)|, t \geq 0\}$ is denoted by L_∞. Moreover, for any pair of times $0 \leq t_1 \leq t_2$, $\|f_{[t_1,t_2]}\| = \sup_{t\in[t_1,t_2]}|f(t)|$. Note that, $|\cdot|$ stands for the Euclidean norm for vectors, or the induced matrix norm for matrices. For any measurable function Classes $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions are defined as usual ([14]). Here, Filippov's definition for the solution of discontinuous differential equations is assumed.For each initial condition and each control/inputs in L_∞, the Filippov's solution of discontinuous differential equations is defined on some maximal interval $[0, t_M)$, where t_M may be finite or infinite.

2. PROBLEM STATEMENT

Consider SISO nonlinear plants composed by a general subsystem

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$$\dot{x} = f(x) + g(x)u\,, \tag{1}$$
$$z = h(x)\,, \tag{2}$$

in cascade with a static subsystem

$$y = \Phi(z)\,, \tag{3}$$

where $u \in \mathbb{R}$ is the control input (discontinuous), $z \in \mathbb{R}$ is the unmeasured output of the first subsystem, $y \in \mathbb{R}$ is the measured output of the static subsystem, x is the state and the uncertain functions $f(\cdot), g(\cdot)$ and $h(\cdot)$ are locally Lipschitz continuous and sufficiently smooth (all required derivatives are continuous) to ensure local existence and uniqueness of the solution through every initial condition (x_0, t_0). For each solution of (1) there exists a maximal time interval of definition given by $[0, t_M)$, where t_M may be finite or infinite. Thus, finite-time escape is not precluded, *a priori*.

The function $\Phi : \mathbb{R} \to \mathbb{R}$ is regarded as an uncertain (unknown) and smooth cost function. We consider that there exists a unique point z^* (*unknow*) such that $y^* = \Phi(z^*)$ is the extremum (maximum) of Φ, which gradient is unknown for the control designer.

The global real-time optimization control problem, *i.e.*, maximization[1] of (3) under (1)–(2). We wish to find an output-feedback control law u so that, for any initial conditions, the system is steered to reach the extremum point and remain on such point thereafter, as close as possible.

Our output-feedback strategy relies on: (i) the implementation of a norm observer for the plant state x (1); (ii) representation of the plant in the normal form [14, pp. 516] and (iii) a HGO to estimate the time derivatives of the plant output.

3. MAIN ASSUMPTIONS

In order to obtain the uncertainty bounds for control design, consider the following assumption:

(A0) (***On The Uncertainties***) All the uncertain plant parameters belong to a compact set Ω.■

3.1 Cost Function

Reminding Assumption **(A0)**, the further assumption here is that in Ω:

(A1) (***Cost Function***) The uncertain cost function $\Phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, sufficiently smooth and radially unbounded. Moreover, $y = \Phi(z)$ has a unique maximum point z^* and for any given $\Delta > 0$, there exists a constant $L_\Phi(\Delta) > 0$ such that

$$L_\Phi \leq \left| \frac{d\Phi}{dz} \right|, \quad \forall z \notin \mathcal{D}_\Delta := \{z : \; |z - z^*| < \Delta/2\}\,,$$

where $\mathcal{D}_\Delta$ is called Δ-vicinity of z^* and Δ can be made arbitrary small by allowing a smaller L_Φ.■

[1] Without lost of generality, we only address the maximum seeking problem.

3.2 Norm Plant State Estimation

According to the following assumption, it is possible to implement a norm observer for the plant state x (1) providing a norm bound for x by using only the available signals u and y. One possible class of norm observers is presented and discussed along the paper.

(A2) (***Norm Observability***) The plant (1)–(3) admits a norm observer, with state vector ω, such that

$$|x(t)| \leq \varphi_o(\omega(t)) + \pi_o(t)\,, \tag{4}$$

where $\varphi_o(\cdot)$ is a non-negative continuous function, $\pi_o := \beta_o(|\omega(0)|+|x(0)|)e^{-\lambda_o t}$ with some $\beta_o \in \mathcal{K}_\infty$ and positive constant λ_o.

It is well known that, if (1)–(3) is IOSS then it admits a norm observer.The class of norm observer considered here encompasses plants with linear growth condition in the unmeasured states and growth rate possibly depending on y. It should be stressed that strong polynomial nonlinearities in y are allowed.

Remark 2. **[Unboundedness Observability Property]** From Assumption **(A2)**, the system possesses an unboundedness observability property, i.e., if any closed loop system signal escapes in some finite time, then ω also escapes not latter than that. We will use this fact to design the control law so that finite time is avoided.

3.3 Normal Form

For time invariant plants, the uniform relative degree assumption [14] is a necessary and sufficient condition for the existence of a local change of coordinates (local diffeomorphism) which transforms (1)–(2) into the normal form [14].

Considering the output function $\bar{h}(\cdot) = \Phi(h(\cdot))$ and denoting the Lie derivative of $\bar{h}$ along a vector field f by $L_f\bar{h}$, it is well known that a sufficient condition to assure that the plant (1)–(3) is transformable to the normal form is given by [14, pp. 510]:

(A3) (***Normal Form***) Assume that $L_g[L_f^k\bar{h}] \equiv 0$ ($k \in \{0, \ldots, \rho-2\}$) and $L_g[L_f^{\rho-1}\bar{h}] \neq 0$.

From **(A3)**, system (1)–(3) is transformable into the normal form [14]:

$$\dot{\eta} = f_0(x)\,, \tag{5}$$
$$\dot{\xi} = A_\rho\xi + B_\rho\bar{k}_p(x)[u + d(x)]\,, \quad y = \xi_1\,, \tag{6}$$

where the transformed state is defined as

$$\bar{x} := [\eta^T \; \xi^T]^T = T(x) = [\eta^T \; T_\xi^T(x)]\,, \tag{7}$$

with $T_\xi := \left[\, L_f^0\bar{h} \; L_f\bar{h} \; \ldots \; L_f^{\rho-1}\bar{h} \, \right]^T$, $\eta \in \mathbb{R}^{n-\rho}$ and

$$\xi := [\, y \; \dot{y} \; \ldots \; y^{(\rho-1)} \,]^T\,. \tag{8}$$

The η-subsystem (ξ-subsystem) represents the inverse dynamics (external dynamics). The pair (A_ρ, B_ρ) is in Brunovsky's canonical controllable form. Note that, it is implicitly assumed that the plant (1)–(3) has a strong relative degree ρ.

The control signal coefficient k_p and the input disturbance d are such that

$$\bar{k}_p(x) = L_g[L_f^{\rho-1}\bar{h}] = L_g[L_f^{\rho-1}h]\frac{d\Phi}{dz} = k_p(x)\frac{d\Phi}{dz}\,,$$

and $\bar{k}_p(x)d(x) = (L_f^\rho \bar{h})$, respectively. In the following assumption we formulate the restrictions imposed on $T(x)$, $\bar{k}_p(x)$ and $d(x)$, where the dependence on $y = \bar{h}(x)$ is explicitly given.

3.4 Minimum-Phase

The following assumption assures that the inverse dynamics (5) has an ISS property with respect to an appropriate function of ξ.

(A4) (***Minimum-Phase]***) There exists a storage function $V(\eta)$ satisfying $\underline{\beta}(|\eta|) \leq V(\eta) \leq \bar{\beta}(|\eta|)$ with $\underline{\beta}, \bar{\beta} \in \mathcal{K}_\infty$, such that:

$$\frac{\partial V}{\partial \eta} f_0(x) \leq -\beta_0(|\eta|) + \varphi_0(|\xi|)\,,$$

$\forall x, y\,, \forall t \in [0, t_M)$, for some non-negative scalar function $\varphi_0(|\xi|)$, continuous in $|\xi|$ and some $\beta_0 \in \mathcal{K}_\infty$.

3.5 Bounding Functions

In the following assumption, let (for $i = 1, 2, 3$): (a) $\varphi_i(|x|, y)$ are non-negative functions continuous and increasing in $|x|$ and continuous in y; (b) $\bar{\varphi}_i(y)$ are non-negative functions continuous in y and (c) $\alpha_i(|x|)$ are locally Lipschitz class-$\mathcal{K}$ functions.

(A5) (***Bounding Functions***) There exist *known* functions $\varphi_i, \bar{\varphi}_i, \alpha_i$ and a *known* positive constant c_p such that the following inequalities hold $\forall x, y\,, \forall t \in [0, t_M)$:

$$\beta_T(|x|) + \gamma_T(y) \leq |T(x)| \leq \varphi_1(|x|, y)\,,$$

$$0 < c_p \leq |\bar{k}_p(x)| \leq \varphi_2(|x|, y)\,,$$

$$|d(x)| \leq \varphi_3(|x|, y)\,,$$

where φ_i satisfies $\varphi_i(|x|, y) \leq \alpha_i(|x|) + \bar{\varphi}_i(y)$, β_T is some class-$\mathcal{K}_\infty$ function and γ_T is some scalar non-negative function continuous in y.

The lower bound for $|T|$ assures boundedness of x from boundedness of $\bar{x}$ and the lower bound for $|\bar{k}_p|$ guarantees that it is bounded away from zero. On the other hand, the upper bounding functions for $T, \bar{k}_p$ and d are used to obtain implementable norm bounds for $\xi, \bar{k}_p$ and d from the plant state norm estimator vector ω (16)–(17).

In general, the upper bounds given in Assumption **(A5)** impose no significant restriction since $T, \bar{k}_p$ and d are continuous in x.

4. HIGH GAIN OBSERVER

The estimate for ξ in (6) is provided by the following HGO:

$$\dot{\hat{\xi}} = A_\rho \hat{\xi} + B_\rho k_o u + H_\mu L_o (y - C_\rho \hat{\xi})\,, \tag{9}$$

where $C_\rho := [1\ 0\ \ldots\ 0]$ and L_o and H_μ are given by

$$L_o := [\, l_1\ \ldots\ l_\rho\,]^T \text{ and } H_\mu := \mathrm{diag}(\mu^{-1}, \ldots, \mu^{-\rho})\,. \tag{10}$$

The observer gain L_o is such that $s^\rho + l_1 s^{\rho-1} + \ldots + l_\rho$ is Hurwitz. The HGO parameter μ is a *variable* parameter $\mu = \mu(t) \neq 0, \forall t \in [0, t_M)$, of the form

$$\mu(\omega, t) := \frac{\bar{\mu}}{1 + \psi_\mu(\omega, t)}\,, \tag{11}$$

where ψ_μ, named **domination function**, is a non-negative function (to be designed later on) continuous in its arguments and $\bar{\mu} > 0$ is a design constant. For each system trajectory, μ is absolutely continuous and $\mu \leq \bar{\mu}$. Note that μ is bounded for t in any finite sub-interval of $[0, t_M)$. Therefore,

$$\mu(\omega, t) \in [\underline{\mu}, \bar{\mu}]\,, \quad \forall t \in [t_*, t_M)\,, \tag{12}$$

for some $t_* \in [0, t_M)$ and $\underline{\mu} \in (0, \bar{\mu})$.

4.1 High Gain Observer Error Dynamics

The transformation

$$\zeta := T_\mu \tilde{\xi}\,, \quad \tilde{\xi} := \xi - \hat{\xi}\,, \quad T_\mu := [\mu^\rho H_\mu]^{-1}\,, \tag{13}$$

is used to represent the $\tilde{\xi}$-dynamics in convenient coordinates to allow us show that $\tilde{\xi}$ is arbitrarily small, *modulo* exponentially decaying term. First, note that: $T_\mu(A_\rho - H_\mu L_o C_\rho)T_\mu^{-1} = \frac{1}{\mu} A_o$, $T_\mu B_\rho = B_\rho$ and $\dot{T}_\mu T_\mu^{-1} = \frac{\dot{\mu}}{\mu}\Delta$, where $A_o := A_\rho - L_o C_\rho$ and $\Delta := \mathrm{diag}(1-\rho, 2-\rho, \ldots, 0)$. Then, subtracting (9) from (6) and applying the above relationships (i), (ii) and (iii), the dynamics of $\tilde{\xi}$ in the new coordinates ζ (13) is given by:

$$\mu\dot{\zeta} = [A_o + \dot{\mu}(t)\Delta]\zeta + B_\rho[\mu\nu]\,, \tag{14}$$

where

$$\nu := (\bar{k}_p - k_o)u + \bar{k}_p d\,. \tag{15}$$

5. NORM OBSERVER

Our output-feedback strategy relies on the implementation of a norm observer for the plant state x (1). In the following definition let: (i) u be the plant input, (ii) y be the plant output, (iii) γ_o be a smooth function and (iv) $\varphi_o(\cdot, \cdot, t)$ and $\bar{\varphi}_o(\cdot, t)$ be non-negative functions, piecewise continuous and upperbounded in t and continuous in their other arguments.

Definition 1. A norm observer for system (1)–(3) is a m-order dynamic system of the form:

$$\tau_1 \dot{\omega}_1 = -\omega_1 + u\,, \tag{16}$$

$$\tau_2 \dot{\omega}_2 = \gamma_o(\omega_2) + \tau_2 \varphi_o(\omega_1, y, t)\,, \tag{17}$$

with states $\omega_1 \in \mathrm{I\!R}$, $\omega_2 \in \mathrm{I\!R}^{m-1}$ and positive constants τ_1, τ_2 such that for $t \in [0, t_M)$: (i) if $|\varphi_o|$ is uniformly bounded by a constant $c_o > 0$, then $|\omega_2|$ can escape at most exponentially and there exists $\tau_2^*(c_o)$ such that the ω_2-dynamics is BIBS (Bounded-Input-Bounded-State) stable w.r.t. φ_o for $\tau_2 \leq \tau_2^*$; (ii) for each $x(0), \omega_1(0), \omega_2(0)$, there exists $\bar{\varphi}_o$ such that

$$|x(t)| \leq \bar{\varphi}_o(\omega(t), t) + \pi_o(t)\,, \quad \omega := [\omega_1\ \omega_2^T\ y]^T\,, \tag{18}$$

where $\pi_o := \beta_o(|\omega_1(0)| + |\omega_2(0)| + |x(0)|)e^{-\lambda_o t}$ with some $\beta_o \in \mathcal{K}_\infty$ and positive constant λ_o.

Remark 3. **(On the Availability of the Signal z)** If one can obtain a norm bound for z by using only the measured output y and/or the control signal u, the output z is not required to be measured. Indeed, first note that the radially unbounded condition in **(A1)** and the nonsingularity of $\frac{\partial \Phi}{\partial z}$ assure that $\Phi(\cdot)$ has a piecewise continuous inverse. Thus, it is reasonable to assume that one can obtain a known function $\varphi \in \mathcal{K}$ and a known constant $k_\phi \geq 0$ such that

$$|z| \leq \bar{z} := \varphi(|y|) + k_\phi\,.$$

Moreover, when (1)–(2) is strictly stable, $\bar{z}$ can be generated by a proper first order linear filter driven by the norm of the average control u_{av} which satisfies $\tau_{av}\dot{u}_{av} = -u_{av} + u$, with an appropriate constant $\tau_{av} > 0$. ■

6. OUTPUT-FEEDBACK EXTREMUM-SEEKING CONTROLLER

6.1 Periodic Switching Function

Consider the unmeasured signal e given by

$$e(t) = \sigma(t) - \sigma_m(t) \tag{19}$$

where σ_m is a simple ramp time function. For analysis convenience such a ramp is generated by

$$\dot{\sigma}_m = k_m\,, \quad \sigma_m(0) = \sigma_{m0}\,, \tag{20}$$

with $k_m > 0$ and σ_{m0} being design constants. Regarding the plant (1)–(3), σ is the relative degree one (unmeasured) output

$$\sigma = S\xi\,,$$

where

$$S := a_\rho C_\rho + a_{\rho-1} C_\rho A_\rho + \ldots + a_1 C_\rho A_\rho^{\rho-1}\,,$$

and a_i $(i = 1, \ldots, \rho)$ is such that

$$L(s) := a_1 s^{\rho-1} + a_2 s^{\rho-2} + \ldots + a_\rho$$

is Hurwitz. Therefore, one has

$$\sigma = L(s)y\,,$$

and, equivalently,

$$y = \frac{1}{L(s)}\sigma\,.$$

Moreover, the e-dynamics is given by

$$\dot{e} = \dot{\sigma} - k_m = S\dot{\xi} - k_m = SA_\rho\xi + SB_\rho\bar{k}_p[u + d] - k_m\,,$$

where $SB_\rho \neq 0$ and can be rewritten as

$$\dot{e} = \kappa u + d_e\,, \tag{21}$$

where $\kappa = SB_\rho\bar{k}_p$ and

$$d_e := SA_\rho\xi + SB_\rho\bar{k}_p d - k_m\,. \tag{22}$$

Now, considering the estimate for σ defined as

$$\hat{\sigma} = S\hat{\xi}\,, \tag{23}$$

and the corresponding estimate $\hat{e}$ for e,

$$\hat{e} = \hat{\sigma} - \sigma_m\,,$$

the proposed output-feedback ESC with *periodic switching function* is given by

$$u = \varrho(t)\,\mathrm{sgn}\left(\sin\left[\frac{\pi}{\varepsilon}\hat{e}(t)\right]\right)\,, \tag{24}$$

where $\varrho(t)$ is a designed *modulation function* (continuous in t) to be defined later on and $\varepsilon > 0$ is an appropriate constant. Note that, the estimate $\hat{e}$ satisfies

$$\hat{e} = e - \tilde{e}\,,$$

where

$$\tilde{e} = \sigma - \hat{\sigma} = S\tilde{\xi}\,.$$

Hence, the signal $e - \tilde{e}$ is available and the control signal can be written as

$$u = \varrho(t)\,\mathrm{sgn}\left(\sin\left[\frac{\pi}{\varepsilon}e(t) - \frac{\pi}{\varepsilon}\tilde{e}(t)\right]\right)\,. \tag{25}$$

Remark 4. Note that, for the case where the HGO has a constant parameter μ and $k_o = 0$, the estimate $\hat{\sigma}$ is obtained by the following linear lead filter:

$$\hat{\sigma} = S[sI - (A_\rho - H_\mu L_o C_\rho)]^{-1} H_\mu L_o\; y\,,$$

which approximates the *unmeasured* signal

$$\sigma = L(s)y\,.$$

6.2 Available Bounding Functions

The following available norm bounds for ξ, $\bar{k}_p$ and d are obtained, *modulo* exponentially decaying term, by using the bounding functions given in **(A5)** and the norm observer state vector in **(A2)**:

$$|\xi| \leq \psi_1(\omega, t) + \pi_1\,, \tag{26}$$

$$\bar{k}_p(x) \leq \psi_2(\omega, t) + \pi_1\,, \tag{27}$$

$$|d(x)| \leq \psi_3(\omega, t) + \pi_1\,, \tag{28}$$

where $\psi_i(\omega, t) := \varphi_i(2\bar{\varphi}_o, y, t) + \bar{\varphi}_i(y, t)$ $(i = 1, 2, 3)$ and $\pi_1 = \beta_1(|\omega(0)| + |x(0)|)e^{-\lambda_o t}$ with some $\beta_1 \in \mathcal{K}_\infty$ and λ_o in **(A2)**.

6.3 Modulation Function Design

The modulation function is designed to overcome the disturbance d_e, in (21), outside the Δ-vicinity $\mathcal{D}_\Delta$, i.e., when κ in (21) is bounded away from zero. So, a norm bound for d_e must be implemented by using only available signals (ω). First, from (22) one can verify that $|d_e| \leq |k_p d| + (|SA_\rho||\xi| + k_m)$. Moreover, with c_p defined in Assumption **(A5)** and (26) and (28) the following upper bound holds:

$$|d_e(x, \xi)| + \delta \leq \bar{\varrho}(\omega, t) + \pi_2\,, \tag{29}$$

where δ is an arbitrary non-negative constant,

$$\bar{\varrho}(\omega, t) := \psi_3 + (|K_m|\psi_1 + k_m)/c_p + \delta\,, \tag{30}$$

and $\pi_2 := |K_m|\pi_1/c_p + \pi_1$. Now, choose a polynomial $\bar{p}_\varrho(|\omega|)$ in $|\omega|$, with positive real coefficients, such that $\bar{p}_\varrho(|\omega|) \geq \bar{\varrho}(\omega, t)$ and implement the modulation functions as

$$\varrho(\omega) := \bar{p}_\varrho(|\omega|) + \|\omega_t\| e^{-\beta_\varrho t}\,, \tag{31}$$

where $\beta_\varrho > 0$ is a design constant.

6.4 Variable Gain (μ) Design

Choose a polynomial $\bar{p}_\mu(|\omega|)$ in $|\omega|$, with positive real coefficients, such that the functions $\varphi_o, \bar{\varphi}_o$ (Definition 1) and the bounding functions $\varphi_i, \bar{\varphi}_i$ in (A5) satisfy $(i = 1, 2, 3)$:

$$|\gamma_o(\omega_2)|\,,\; |\varphi_o(\omega_1, y)| \leq \bar{p}_\mu(|\omega|)\,,$$

$$\varphi_i(2\bar{\varphi}_o(\omega, t), y)\,,\; \bar{\varphi}_i(y) \leq \bar{p}_\mu(|\omega|)\,.$$

This is not so restrictive since only polynomial growth condition is imposed on φ_o, $\bar{\varphi}_o$, γ_o, φ_i, $\bar{\varphi}_i$. We choose ψ_μ as:

$$\psi_\mu(\omega, t) := p_\mu(|\omega|) + \|\omega_t\| e^{-\beta_\mu t}\,, \tag{32}$$

where $\beta_\mu > 0$ is a design constant.

Remark 5. The exponential terms with rates β_ϱ and β_μ in (31) and (32) act like a forgetting factor which allows a less conservative design. Note that the functional norm term is fundamental to avoid finite-time escape of the system signals.

7. MAIN CONVERGENCE RESULTS

7.1 Preliminaries

Let $\mathcal{S}$ be the set of real numbers x such that $\sin \pi x/\varepsilon = 0$, i.e., real numbers of the form $x = k\varepsilon$, $k \in Z$. The set $\mathcal{S}$ is given by $\mathcal{S} = \{\ldots, -2\varepsilon, -\varepsilon, 0, \varepsilon, 2\varepsilon, \ldots\}$. Moreover, given an arbitrary constant $r > 0$, let $\mathcal{S}_r$ be the union of neighborhoods of radius r centered at $x = k\varepsilon$, i.e., the set defined by $\mathcal{S}_r = \{x \in \mathbb{R} \mid x \in (k\varepsilon - r, k\varepsilon + r)\,,\ \forall k \in Z\} = \ldots \cup (-r-2\varepsilon, r-2\varepsilon) \cup (-r-\varepsilon, r-\varepsilon) \cup (-r, r) \cup (-r+\varepsilon, r+\varepsilon) \cup (-r+2\varepsilon, r+2\varepsilon) \cup \ldots$. Note that, $\mathcal{S}_r = \mathbb{R}$, for $r \geq \varepsilon/2$.

Lemma 1. For any $x \notin \mathcal{S}_r$ one has that

$$\operatorname{sgn}(\sin(\pi x/\varepsilon + \beta)) = \operatorname{sgn}(\sin(\pi x/\varepsilon))\,,$$

provided that $|\beta| < r < \varepsilon/2$.

Proof: If $x \notin \mathcal{S}_r$ then the following inequality holds for some integer k^*

$$k^*\varepsilon + r \leq x \leq (k^*+1)\varepsilon - r\,,$$

and since $r > 0$ one has that $x \in (k^*\varepsilon, k^*\varepsilon + \varepsilon)$. Moreover, by adding γ one has that

$$k^*\varepsilon + \gamma + r \leq x + \gamma \leq (k^*+1)\varepsilon + \gamma - r\,,$$

and since $-r < \gamma < r$, one can further conclude that

$$k^*\varepsilon < k^*\varepsilon + \gamma + r \leq x + \gamma \leq (k^*+1)\varepsilon + \gamma - r < (k^*+1)\varepsilon\,.$$

Therefore, $x\,, x + \gamma \in (k^*\varepsilon, k^*\varepsilon + \varepsilon)$. Consequently, by setting $\gamma = \varepsilon\beta/\pi$, one has $\pi x/\varepsilon\,, \pi x/\varepsilon + \beta \in (k^*\pi, k^*\pi + \pi)$ and $\operatorname{sgn}(\sin(\pi x/\varepsilon + \beta)) = \operatorname{sgn}(\sin(\pi x/\varepsilon))$.■

7.2 Auxiliary Lemmas

In what follows, let $\mathcal{X}^T := \begin{bmatrix} x^T & \omega^T & \zeta^T \end{bmatrix}$. The following lemma assures that the modulation function overcome the disturbance in the error dynamics after some finite time.

Lemma 2. If ϱ is designed as in (31), then there exists a finite $t_\varrho \in [0, t_M)$ such that:

$$|\omega|\,, |\mathcal{X}| \leq \alpha(|\mathcal{X}(0)|)\,, \quad \forall t \in [0, t_\varrho)\,, \tag{33}$$

$$|\varrho| \geq |d_e|\,, \quad \forall t \in [t_\varrho, t_M)\,. \tag{34}$$

Proof: If $\pi_2 \leq 1$ in (29) or t_M is infinite it is trivial due to the vanishing exponential π_2. Now, consider that $\pi_2 > 1$ and t_M is finite. Then, one has: (i) $e^{-\beta_\varrho t} \geq e^{-\beta_\varrho t_M}$, $\forall t \in [0, t_M)$; (ii) $\exists t_1 \in [0, t_M)$ such that $\|\omega_t\| \geq \delta$, $\forall t \in [t_1, t_M)$, where δ is an arbitrary constant. Hence, from (i) and (ii) and taking δ larger enough, one also has that $|\varrho| \geq |d_e|$, $\forall t \in [t_1, t_M)$. The upper bound (33) is a direct consequence of the unboundedness observability property of the closed loop control system, see Remark 2.■

The following lemma states the HGO estimation error convergence.

Lemma 3. If ψ_μ is designed as in (32), then there exists a finite $t_\mu \in [0, t_M)$ such that:

$$|\omega|\,, |\zeta| \leq \alpha(|\mathcal{X}(0)|)\,, \quad \forall t \in [0, t_\mu)\,, \tag{35}$$

$$|\dot{\mu}(t)|\,,\ \mu|\nu|\,,\ |\tilde{\xi}| \leq \mathcal{O}(\bar{\mu})\,, \quad \forall t \in [t_\mu, t_M)\,, \tag{36}$$

with some $\alpha \in \mathcal{K}_\infty$.

Proof: To see that (35) and (36) hold, refer to [6].■

Now, let $t_0 = \max\{t_\varrho, t_\mu\}$. From Lemma 3, since $\tilde{\xi}$ is of order $\mathcal{O}(\bar{\mu})$, $\forall t \in [t_0, t_M)$, it is easy to conclude that $\pi\tilde{e}(t)/\varepsilon$ is also of order $\mathcal{O}(\bar{\mu})$, $\forall t \in [t_0, t_M)$, once $\tilde{e} = S\tilde{\xi}$.

From Lemma 1, if $e(t) \notin \mathcal{S}_r$, then $\operatorname{sgn}(\sin(\pi e/\varepsilon + \pi\tilde{e}/\varepsilon)) = \operatorname{sgn}(\sin(\pi e/\varepsilon))$, since $|\pi\tilde{e}/\varepsilon| < r = \mathcal{O}(\bar{\mu}) < \varepsilon/2$. Therefore, $u = \varrho(t)\operatorname{sgn}(\sin(\pi e(t)/\varepsilon))$ while $e(t) \notin \mathcal{S}_r$ with $t > t_0$.

The following lemma holds while z stays outside the Δ-vicinity. It assured that no finite-time escape occurs for the system signals and the error $e(t)$ reach the set $\mathcal{S}_r$ and remains there in after some finite time for a constant r of order $\mathcal{O}(\bar{\mu})$.

Lemma 4. Consider the error dynamics (21) with control law (23), (24) and (31). Outside the Δ-vicinity $\mathcal{D}_\Delta$, assume that **(A0)-(A5)** hold. Then, for any constant $0 < r < \bar{\mu}$ and sufficiently small $\bar{\mu}, \varepsilon$, independently of $\operatorname{sgn}(\kappa(t))$: (a) the control signal drives $e(t)$ to the set $\mathcal{S}_r$ after some finite time. Moreover, $e(t)$ remains in $\mathcal{S}_r$ thereafter, i.e., there exists $t_r > t_0$ such that $e(t) \in \mathcal{S}_r$, $\forall t \in [t_r, t_M)$ and (b) $t_M \to \infty$.

Proof: Outside the Δ-vicinity, the derivative of the cost function $\Phi(z)$ does not vanish ($d\Phi(z)/dz \neq \underline{0}\,, \forall z \notin \mathcal{D}_\Delta$). Thus, by using the lower norm bound c_p for $\underline{k}_p$ one can be obtained a lower bound for $|\kappa|$ from the lower bound L_Φ given in **(A2)**. Without lost of generality, consider that $\operatorname{sgn}(\kappa(t)) < 0$, $\forall t \in [t_0, t_M)$.

In view of the Lyapunov stability theory of nonsmooth systems, consider the following nonnegative Lure-type function [14]

$$V(e) := \int_0^e \operatorname{sgn}\left[\sin\left(\frac{\pi}{\varepsilon} e(\tau)\right)\right] d\tau\,. \tag{37}$$

It is easy to verify that:

$$e \in \mathcal{S}_r \;\Leftrightarrow\; V(e) < V(r)\,,$$

and thus, $\mathcal{S}_r = \mathcal{S} := \{\nu : V(\nu) < V(r)\}$. In addition, one can conclude that, if there exists $t_a \geq t_0$ such that $e(t_a) \in \mathcal{S}_r$, then $e(t) \in \mathcal{S}_r$, $\forall t \in [t_a, t_M)$. Indeed, by contradiction, assume that there exists some $t > t_a$ and some $\epsilon > 0$ such that $V(e(t)) > V(r) + \epsilon$. Let, $t_b \geq t_a$ the first time instant such that $V(e(t)) \geq V(r) + \epsilon$. Therefore, $e(t_b) \notin \mathcal{S}_r$ and, from Lemma 1, $u = \varrho(t)\operatorname{sgn}(\sin(\pi e(t)/\varepsilon))$.

At a point where $e(t)$ and $V(e(t))$ are both differentiable (almost everywhere), the time derivative of V along the trajectories of the e-dynamics, $\dot{V} = \frac{\partial V}{\partial e}\dot{e}$, is given by

$$\dot{V} = \kappa\left[\varrho + d_e \operatorname{sgn}(\sin(\pi e(t)/\varepsilon))\right]\,, \textbf{if } e \notin \mathcal{S}_r\,. \tag{38}$$

Clearly, in this case, $\dot{V} \leq \kappa\varrho + |\kappa||d_e|$. Hence, one has $\dot{V} \leq -|\kappa|(\varrho - |d_e|)$, **since sgn**$(\kappa) < 0$. Moreover, reminding that for $t > t_0 \geq t_\varrho$, the modulation function overcome the disturbance d_e, then it is easy to conclude that

$$\dot{V} \leq -\delta_1\,, \textbf{if } e \notin \mathcal{S}_r\,, \tag{39}$$

holds almost everywhere while $e \notin \mathcal{S}_r$, with an arbitrary constant $\delta_1 > 0$. Hence, $\dot{V}(e(t))\,|_{t=t_b} < 0$ and, thus, $V(e(t)) > V(e(t_b))$, for some $t \in (t_a, t_b)$, which contradicts the minimality of t_b. In addition, the existence of t_a is assured by noting that: (i) for t_M infinite, the inequality $\dot{V} \leq -\delta_1$ assures that there exists $t_1 \in [t_0, t_M)$ such that $e(t_1) \in \mathcal{S}_r$ and (ii) assume by contradiction that $|e(t)|$ escapes in some finite time $t_0 < t_M$. Thus, there exists $t_2 \in [t_0, t_M)$, such that $e(t_2) \in \mathcal{S}_r$. Let $t_a = \max\{t_1, t_2\}$. Finally, during the time interval $[t_a, t_M)$, the error signal e remains bounded, which is a contradiction. Thus, $t_M \to \infty$.■

7.3 Main Result

In the next theorem, we show that the proposed output-feedback control law drives z to the Δ-vicinity of the unknown maximizer z^* defined in (A1). It does not imply that $z(t)$ remains in $\mathcal{D}_\Delta$, $\forall t$. However, the oscillations around y^* can be made of order $\mathcal{O}(\varepsilon + \bar{\mu})$.

Theorem 1. **(Global ESC)** Consider the plant (1), (2), with output or cost function in (3), control law (24). Assume that (A0)-(A5) hold. Then: (i) the Δ-vicinity $\mathcal{D}_\Delta$ in (A1) is globally attractive being reached in finite time and (ii) for L_Φ sufficiently small, the oscillations around the maximum value y^* of y can be made of order $\mathcal{O}(\varepsilon + \bar{\mu})$, with ε from (24) and $\bar{\mu}$ from (11). Moreover, all signals in the closed-loop system remain uniformly bounded except for $\hat{e}(t)$ which is only an argument of a sine function in (24).

Proof: Please refer to [www.coep.ufrj.br/˜jacoud/IFAC14] for the detailed proofs of stability.■

8. SIMULATION RESULTS

The following academic but nontrivial example illustrates the performance of the proposed controller. Consider the simple case where the nonlinear plant is reduced to a linear plant ($\rho = 2$) with transfer function (from u to z)

$$G(s) := \frac{k_p}{s(s + \delta_1)}\,,$$

in cascade with the output cost function $y = \Phi(z) = -(z-3)^2 + 1$. The plant can be trivially transformed to the normal form. It is assumed uncertain and only norm bounds are known. The uncertain parameters are: $1 \leq k_p \leq 2$ and $0 \leq \delta_1 < 1$. The zero dynamics is dropped.

In order to simplify the control implementation, the norm observer was disregarded. In fact, for this simple linear plant and initial conditions $z(0) = 2$ and $\dot{z}(0) = 0$, the norm observer was not needed. The control law (24) can be implemented with modulation function $\varrho = |y| + \delta$. Moreover, the HGO and the sliding surface are implemented with $l_1 = 2$, $l_2 = 1$ and $S = [\,2\ 1\,]$, corresponding to $L(s) = s + 1$. The time varying HGO parameter is given by $\mu = \bar{\mu}/(1 + |y|)$. The other parameters are: $\bar{\mu} = 0.01$, $\delta = 0.1$, $\varepsilon = 2$, $c_d = 1$, $c_1 = \underline{k}_p = 0.5$, $k_m = 0.6$. Moreover, in this example, the term $\|\omega_t\| e^{-\beta_e t}$ in is dropped. The Euler Method with step-size $h = 10^{-4}$s is used for numerical integration. Fig. 1 gives the performance of the control using $\varepsilon = 2$. We can note that the vicinity of the extremum point was achieved. Fig. 1 (b) and (c) present the behavior of the variable $\hat{e}$ and $\hat{\sigma}$, respectivelly.

9. CONCLUSIONS

An extremum seeking sliding mode controller via output-feedback was developed for possibly unstable and uncertain nonlinear systems with arbitrary relative degree, generalizing the controllers in [5] and [8]. The combination of high-gain observers with time varying gain, a norm observer and a periodic switching function leads to global asymptotic stability and ultimate convergence of the system output to an arbitrarily small neighborhood of the extremum point. The proposed control strategy was successfully tested with a numerical simulation example. The application of the presented approach to compensate the relative degree obstacle in the extremum seeking controller based on monitoring functions given in [9] seems to be

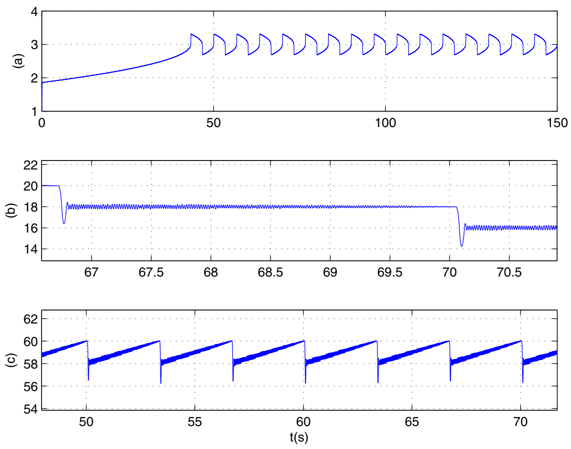

Fig. 1. (a) plant output z, (b) variable $\hat{e}$ and (c) variable $\hat{\sigma}$.

straightforward. The extension of the theoretical results to the multivariable problem is under development.

REFERENCES

[1] M. Krstić, H. H. Wang, Stability of extremum seeking feedback for general nonlinear dynamic systems, *Automatica*, pp. 595–601, 2000.

[2] K. B. Ariyur, M. Krstić, *Real-Time Optimization by Extremum-Seeking Control*, John Wiley & Sons; 2003.

[3] Y. Tan, D. Nesić and I. Mareels. On non-local stability properties of extremum seeking control. *Automatica*, 42(6): pp. 889–903, 2006.

[4] Y. Tan, D. Nesić, I. M. Y. Mareels and A. Astolfi. On global extremum seeking in the presence of local extrema. *Automatica*, 45(1): 245–251, 2009.

[5] T. R. Oliveira, L. Hsu, A. J. Peixoto. Output-feedback global tracking for unknown control direction plants with application to extremum-seeking control. *Automatica*, pp. 2029–2038, 2011.

[6] A. J. Peixoto, T. R. Oliveira, L. Hsu. Global tracking sliding mode control for a class of nonlinear systems via variable gain observer. *International Journal of Robust and Nonlinear Control*, 21, pp. 177–196, 2011.

[7] Y. Pan, Ü. Özgüner, T. Acarman. Stability and performance improvement of extremum seeking control with sliding mode. *International Journal of Control*, 76(9), pp. 968–985, 2003.

[8] T. R. Oliveira, A. J. Peixoto, L. Hsu. Global real-time optimization by output-feedback extremum-seeking control with sliding modes. *Journal of The Franklin Institute*, pp. 1397–1415, 2012.

[9] N. O. Aminde, T. R. Oliveira, L. Hsu, Global output-feedback extremum seeking control via monitoring functions, *52nd IEEE Conference on Decision and Control*, to appear, 2013.

[10] C. Zhang, A. Siranosian, M. Krstić, Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking without position measurements, *Automatica*, 43, pp. 1832–1839, 2007.

[11] C. Zhang, D. Arnold, N. Ghods, A. Siranosian, M. Krstić, Source seeking with nonholonomic unicycle without position measurement and with tuning of forward velocity, *Systems & Contr. Letters*, 56, pp. 245–252, 2007.

[12] J. Cochran, N. Ghods, A. Siranosian, M. Krstić, 3D source seeking for underactuated vehicles without position measurement, *IEEE Trans. on Robotics*, 25, pp. 245–252, 2009.

[13] L. Fu, Ü. Özgüner, Extremum-seeking control in constrained source tracing with nonholonomic vehicles, *IEEE Trans. on Ind. Electronics*, 56, pp. 3602–3608, 2009.

[14] H. K. Khalil, Nonlinear Systems. *Prentice Hall*, 3^{rd} ed., 2002.