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Mean Field Estimation for Partially Observed LQG Systems with Major and Minor Agents

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Abstract: In the work of Huang (2010) and Nguyen and Huang (2012) the linear quadratic mean field systems and control problem is solved in the case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The new feature in this case is that the mean field becomes stochastic and by minor agent state extension Nguyen and Huang (2012) establish the existence of ϵ-Nash equilibria together with the individual agents' control laws that yield the equilibria. This paper presents results initially announced in Caines and Kizilkale (2013); Caines (2013) where it is shown that if the major agent's state is partially observed by the minor agents, and if the major agent completely observes its own state, all agents can recursively generate estimates (in general individually distinct) of the major agent's state and the mean field, and thence generate feedback controls yielding ϵ-Nash equilibria.

1. INTRODUCTION

Mean Field (MF) systems theory establishes the existence of approximate (aka ϵ-) Nash equilibria together with the corresponding individual strategies for stochastic dynamical system agents in games involving a large number of agents. The equilibria are generated by the local, limited information feedback control actions of each agent in the population, where the feedback control actions constitute the best responses of each agent with respect to the precomputed behaviour of the mass of agents and where the approximation error converges to zero as the population goes to infinity.

The determination of an approximate equilibrium and the corresponding individual agent control actions in the complex, arbitrarily large finite population case (i.e. the domain of application) is achieved by exploiting its relationship with the infinite population limit problem with its far simpler description and solution. Specifically, the solution to the infinite population problem is obtained via the MF Hamilton-Jacobi-Bellman PDE and the (McKean-Vlasov) Fokker-Planck-Kolmogorov PDE equations which are linked to each other by the state distribution of a generic agent, otherwise known as the system's mean field. This linked pair of HJB and FPK PDEs is referred to as the Mean Field Game (MFG) equations.

The analysis of this set of problems originated in Huang et al. (2003, 2006, 2007) and independently in Lasry and Lions (2006a,b). In the important work Huang (2010) and Nguyen and Huang (2012) analyse and solve the linear quadratic systems case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The new feature in this case is that the mean field becomes stochastic and by minor agent state extension the existence of ϵ-Nash equilibria is established together with the individual agents' control laws that yield the equilibria (Nguyen and Huang (2012)).

In the purely minor agent case the mean field is deterministic and this obviates the need for observations on other agents' states for the generation via recursive filtering of estimates of the global systems state or the mean field. (This is separate from an agent's need to estimate its own state (self-state, for short) if it is a partially observed (PO) system, see Huang et al. (2006).) However, a new and challenging problem for MF system theory is that since systems with major agents have stochastic mean fields, *systems with PO major agents have mean fields which must be recursively estimated together with the major agents' partially observed states.*

The main result of the present paper (first announced in Caines (2013); Caines and Kizilkale (2013)) is that if the major agent's state is partially observed by the minor agents they can recursively generate estimates (in general individually distinct) of the major agent's state and the mean field, and thence generate feedback controls yielding ϵ-Nash equilibria. An important assumption adopted here is that the major agent has complete observations of its own state (aka self-observation); this hypothesis is adopted as a sufficient condition to avoid problems arising from partially ordered major and minor agent information sets.

2. MAJOR-MINOR AGENT LQG SYSTEMS

In this section we give a succinct summary of the LQG major-minor agent MF framework together with the principal ϵ-Nash Equilibrium result.

Dynamics: Completely Observed Finite Population

Following Huang (2010), we consider a large population of N stochastic dynamic minor agents

$$\begin{aligned} dx_0 &= [A_0 x_0 + B_0 u_0]dt + D_0 dw_0, \\ dx_i &= [A(\theta_i)x_i + B(\theta_i)u_i + Gx_0]dt + Ddw_i, \end{aligned} \tag{1}$$

$t \geq 0$, $1 \leq i \leq N < \infty$. Here $x_i \in \mathbb{R}^n$, $0 \leq i \leq N$, are the states, $u_i \in \mathbb{R}^m$, $0 \leq i \leq N$, are the control inputs, $\{w_i,\, 0 \leq i \leq N\}$ denotes $(N+1)$ independent standard Wiener processes in $\mathbb{R}^r$ on an underlying probability space $(\Omega, \mathcal{F}, P)$ which is sufficiently large that w is progressively measurable with respect to the filtration $\mathcal{F}^w \triangleq (\mathcal{F}^w_t; t \geq 0)$ on $\mathcal{F}$. Note that the common agent $\mathcal{A}_0$ affects each minor agent through its dynamics. The initial states are defined on $(\Omega, \mathcal{F}, P)$, and $\{x_i(0), 0 \leq i \leq N\}$ are mutually independent and also independent of $\mathcal{F}^w_\infty$; $\mathbb{E}w_i w_i^\top = \Sigma$, $0 \leq i \leq N$, and $\mathbb{E}\|x_i(0)\|^2 < \infty$, $0 \leq i \leq N$. We denote the minor agent population average state by $x^N = (1/N)\sum_{i=1}^N x_i$.

We now introduce two admissible control sets. The σ-field $\mathcal{F}_{i,t}, 1 \leq i \leq N$, is defined to be the increasing family of σ-fields generated by $(x_i(\tau); 0 \leq \tau \leq t)$, and by definition $\mathcal{F}_{0,t}$ is the increasing family of σ-fields generated by $(x_0(\tau); 0 \leq \tau \leq t)$. $\mathcal{F}^N_t$ is the increasing family of σ-fields generated by the set $\{x_j(\tau), x_0(\tau); 0 \leq \tau \leq t, 1 \leq j \leq N\}$. By definition the set $\mathcal{U}_0$ consists of the feedback controls adapted to the set $\{\mathcal{F}_{0,t};\, t \geq 0\}$. The set of control inputs $\mathcal{U}_i$, $1 \leq i \leq N$, based upon the local information set of the minor agent $\mathcal{A}_i$, $1 \leq i \leq N$, consists of the feedback controls adapted to the set $\{\mathcal{F}_{i,t},\, \mathcal{F}_{0,t};\, t \geq 0\}$ while $\mathcal{U}^N_g$ is adapted to $\{\mathcal{F}^N_t,\, t \geq 0\}$, $1 \leq N < \infty$.

Perfomance Functions

The individual infinite horizon performance, or cost, function for the major agent is then specified by

$$J_0(u_0, u_{-0}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_0 - \Phi(x^N)\right\|^2_{Q_0} + \|u_0\|^2_{R_0}\right\} dt,$$
$$\Phi(\cdot) := H_0 x^N + \eta_0,$$

and the individual infinite horizon cost for a minor agent $\mathcal{A}_i$, $1 \leq i \leq N$, is specified as

$$J_i(u_i, u_{-i}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_i - \Psi(x^N)\right\|^2_{Q} + \|u_i\|^2_{R}\right\} dt,$$
$$\Psi(\cdot) := H_1 x_0 + H_2 x^N + \eta.$$

Minor Agents' Types

The minor agents are not necessarily uniform in their parameters but are given in K types with $1 \leq K < \infty$:

$$\mathcal{I}_k = \{i : \theta_i = k,\, 1 \leq i \leq N\}, \quad N_k = |\mathcal{I}_k|, \quad 1 \leq k \leq K,$$

$\pi^N = (\pi^N_1, ..., \pi^N_K)$, $\pi^N_k = N_k/N$, $1 \leq k \leq K$, shall denote the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ sampled independently of the initial conditions and Wiener processes of the agents A_i, $1 \leq i \leq N$.

H1: There exists π such that $\lim_{N\to\infty} \pi^N = \pi$ a.s.

Introduce the (auxiliary) state averages:

$$x^N_k = \frac{1}{N_k}\sum_{i=1}^{N_k} x_{i,k}, \qquad 1 \leq k \leq K.$$

For each agent $\mathcal{A}_i$ of type k, $1 \leq k \leq K$, we consider uniform (with respect to i) feedback controls u_k depending upon: (i) time invariant linear functions of state $x_{i,k}$; (ii) bounded functions of time, and (iii) the major agent's state x_0.

Then, conditioned on $\mathcal{F}_{0,t}$, for all k, $1 \leq k \leq K$,

$$\mathbb{E}x^N_k(t) = \frac{1}{N_k}\sum_{i=1}^{N_k}\mathbb{E}x_{i,k}(t) =: \bar{x}_k(t), \quad 0 \leq t < \infty,$$

satisfies the mean generic agent's dynamical equation

$$d\bar{x}_k = \sum_{j=1}^K A_{k,j}\bar{x}_j dt + B(\theta_k)u_k dt + Gx_0 dt, \quad 1 \leq k \leq K,$$

or

$$d\bar{x}(t) = \bar{A}\bar{x}(t)dt + \bar{G}x_0(t)dt + \bar{m}(t)dt,$$

where $\bar{A}$, $\bar{G}$, $\bar{m}$, are to be solved for in the tracking solution.

When it exists at any t, $0 \leq t \leq \infty$,

$$\bar{x}(t) := [\bar{x}_1(t), ..., \bar{x}_t(t)],$$

will be termed the system's *mean field*.

We now consider the major agent's state extension $[x_0, \bar{x}]$ by the mean field and obtain the major agent's dynamics in the infinite population case as

$$\begin{bmatrix} dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_0 & 0_{nK\times n} \\ \bar{G} & \bar{A} \end{bmatrix}\begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix} u_0 dt + \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix} dt + \begin{bmatrix} D_0 dw_0 \\ 0_{nK\times 1} \end{bmatrix},$$

where

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & 0_{nK\times n} \\ \bar{G} & \bar{A} \end{bmatrix}, \qquad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix},$$
$$\mathbb{M}_0 = \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix}, \qquad Q^\pi_0 = \begin{bmatrix} Q_0 & -Q_0 H^\pi_0 \\ -H^{\pi\top}_0 Q_0 & H^{\pi\top}_0 Q_0 H^\pi_0 \end{bmatrix},$$
$$\bar{\eta}_0 = [I_{n\times n}, -H^\pi_0]^\top Q_0 \eta_0,$$
$$H^\pi_0 = \pi \otimes H_0 \triangleq [\pi_1 H_0\ \pi_2 H_0\ ...\ \pi_K H_0].$$

Similarly we introduce the minor agent's state extended by the major agent's state and the mean field to obtain $[x_i, x_0, \bar{x}]^\top$. Then each minor agent's dynamics in the infinite population case is given by

$$\begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_k & [G\ \ 0_{n\times nK}] \\ 0_{(nK+n)\times n} & \mathbb{A}_0 \end{bmatrix}\begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} B_k \\ 0_{(nK+n)\times m} \end{bmatrix} u_i dt + \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 \end{bmatrix} dt$$
$$\begin{bmatrix} 0_{n\times m} \\ B_0 \\ 0_{nK\times m} \end{bmatrix} u_0 dt + \begin{bmatrix} Ddw_i \\ D_0 dw_0 \\ 0_{nK\times 1} \end{bmatrix},$$

with the matrices above defined as follows:

$$\mathbb{A}_k = \begin{bmatrix} A_k & [G \ 0_{n\times nK}] \\ 0_{(nK+n)\times n} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^\top \Pi_0 \end{bmatrix}$$

$$\mathbb{B}_k = \begin{bmatrix} B_k \\ 0_{(nK+n)\times m} \end{bmatrix} \quad \mathbb{M} = \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^\top s_0 \end{bmatrix}$$

$$\bar{\eta} = [I_{n\times n}, -H, -H_2^\pi]^\top Q\eta \quad H_2^\pi = \pi \otimes H_2.$$

In the infinite population case, the individual cost for the major agent is given by

$$J_0^\infty(u_0, u_{-0}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_0 - \Phi(\bar{x})\right\|_{Q_0}^2 + \|u_0\|_{R_0}^2\right\}dt,$$

$$\Phi(\cdot) = H_0^\pi \bar{x} + \eta_0,$$

and the individual cost for a minor agent i, $i \in \mathbf{N}$ is given by

$$J_i^\infty(u_i, u_{-i}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_i - \Psi(\bar{x})\right\|_{Q}^2 + \|u_i\|_{R}^2\right\}dt,$$

$$\Psi(\cdot) = H_1 x_0 + H_2^\pi \bar{x} + \eta.$$

We then have the major agent tracking problem solution:

$$\rho\Pi_0 = \Pi_0\mathbb{A}_0 + \mathbb{A}_0^\top\Pi_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^\top\Pi_0 + \mathbb{Q}_0^\pi,$$

$$\rho s_0^* = \frac{ds_0^*}{dt} + (\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^\top\Pi_0)^\top s_0^* + \Pi_0\mathbb{M}_0 - \bar{\eta}_0,$$

$$u_i^0 = -R_0^{-1}\mathbb{B}_0^\top\left[\Pi_0(x_0^\top, \bar{z}^\top)^\top + s_0^*\right],$$

and analogously the minor agent tracking problem solution:

$$\rho\Pi_k = \Pi_k\mathbb{A}_k + \mathbb{A}_k^\top\Pi_k - \Pi_k\mathbb{B}_kR^{-1}\mathbb{B}_k^\top\Pi_k + \mathbb{Q},$$

$$\rho s_k^* = \frac{ds_k^*}{dt} + (\mathbb{A}_k - \mathbb{B}_kR^{-1}\mathbb{B}_k^\top\Pi_k)^\top s_k^* + \Pi_k\mathbb{M} - \bar{\eta},$$

$$u_i^0 = -R^{-1}\mathbb{B}_k^\top\left[\Pi_k(x_i^\top, x_0^\top, \bar{z}^\top)^\top + s_k^*\right].$$

$$\text{Now define: } \Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} \\ \Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} \\ \Pi_{k,31} & \Pi_{k,32} & \Pi_{k,33} \end{bmatrix}, \quad 1 \le k \le K,$$

and $\mathbf{e}_k = [0_{n\times n}, ..., 0_{n\times n}, I_n, 0_{n\times n}, ..., 0_{n\times n}]$, where the $n \times n$ identity matrix I_n is at the kth block.

This notation permits a compact description of the Major-Minor MF Equations determining $\bar{A}$, $\bar{G}$, $\bar{m}$: via the consistency requirements

$$\begin{aligned}
\rho\Pi_0 &= \Pi_0\mathbb{A}_0 + \mathbb{A}_0^\top\Pi_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^\top\Pi_0 + Q_0^\pi, \\
\rho\Pi_k &= \Pi_k\mathbb{A}_k + \mathbb{A}_k^\top\Pi_k - \Pi_k\mathbb{B}_kR^{-1}\mathbb{B}_k^\top\Pi_k + Q^\pi, \ \forall k, \\
\bar{A}_k &= [A_k - B_kR^{-1}B_k^\top\Pi_{k,11}]\mathbf{e}_k - B_kR^{-1}B_k^\top\Pi_{k,13}, \ \forall k, \\
\bar{G}_k &= -B_kR^{-1}B_k^\top\Pi_{k,12}, \ \forall k, \qquad (2) \\
\rho s_0^* &= \frac{ds_0^*}{dt} + (\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^\top\Pi_0)^\top s_0^* + \Pi_0\mathbb{M}_0 - \bar{\eta}_0, \\
\rho s_k^* &= \frac{ds_k^*}{dt} + (\mathbb{A}_k - \mathbb{B}_kR^{-1}\mathbb{B}_k^\top\Pi_k)^\top s_k^* + \Pi_k\mathbb{M} - \bar{\eta}, \ \forall k, \\
\bar{m}_k &= -B_kR^{-1}B_k^\top s_k^*, \ \forall k.
\end{aligned}$$

Finally one defines:

$$M_1 = \begin{bmatrix} A_1 - B_1R^{-1}B_1^\top\Pi_{1,11} & & \\ & \ddots & \\ & & A_K - B_KR^{-1}B_K^\top\Pi_{K,11} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} B_1R^{-1}B_1^\top\Pi_{1,13} \\ \vdots \\ B_KR^{-1}B_K^\top\Pi_{K,13} \end{bmatrix},$$

$$M_3 = \begin{bmatrix} A_0 & 0 & 0 \\ \bar{G} & \bar{A} & 0 \\ \bar{G} & -M_2 & M_1 \end{bmatrix},$$

$$L_{0,H} = Q_0^{1/2}[I, 0, -H_0^\pi].$$

The final set of hypotheses is as follows:

- H2: The initial states are independent, $\mathbb{E}x_i(0) = 0$ for each $i \ge 1$, with $\sup_{j\ge 0}\mathbb{E}|x_j(0)|^2 \le c$.
- H3: The pair $(L_{0,H}, M_3)$ is observable.
- H4: The pair $(L_a, \mathbb{A}_0 - (\rho/2)I)$ is detectable, and for each $k = 1, ..., K$, the pair $(L_b, \mathbb{A}_k - (\rho/2)I)$ is detectable, where $L_a = Q_0^{1/2}[I, -H_0^\pi]$ and $L_b = Q^{1/2}[I, -H, -\hat{H}^\pi]$. The pair $(\mathbb{A}_0 - (\rho/2)I, \mathbb{B}_0)$ is stabilizable and $(\mathbb{A}_k - (\rho/2)I, \mathbb{B}_k)$ is stabilizable for each $k = 1, ..., K$.
- H5: There exists a unique stabilizing solution Π_0, s_0^*, Π_k, $\bar{A}_k$, $\bar{G}_k$, s_k^*, $\bar{m}_k$ to Major-Minor MF equations (2).

Theorem 1. (Huang, 2010) Nash Equilibria for Major-Minor Agent MF Systems

Subject to H1-H5 the MF equations generate a set of stochastic control laws $\mathcal{U}_{MF}^N \triangleq \{u_i^0; 0 \le i \le N\}$, $1 \le N < \infty$, such that

(i) All agent systems $S(A_i)$, $0 \le i \le N$, are second order stable.

(ii) $\{\mathcal{U}_{MF}^N; 1 \le N < \infty\}$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \le \inf_{u_i \in \mathcal{U}_g^N} J_i^N(u_i, u_{-i}^0) \le J_i^N(u_i^0, u_{-i}^0).$$

□

3. PARTIALLY OBSERVED MAJOR-MINOR AGENT LQG SYSTEMS

We now formulate the partial observations problem for the Major-Minor LQG MF problem by specifying the partial observation equation for a generic Minor agent with respect to its extended state for the fully observerd Major-Minor LQG MF problem above, and then similarly for the Major agent:

$$dx_i = [A(\theta_i)x_i + B(\theta_i)u_i + Gx_0]dt + Ddw_i,$$
$$dx_0 = [A_0x_0 + B_0u_0]dt + D_0dw_0.$$

The observation process for any minor agent A_i is defined to be:

$$dy_i(t) = \mathbb{L}x_i^{0,\bar{x}}dt + dv_i(t) \equiv \mathbb{L}\begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix}dt + dv_i(t), \qquad (3)$$

where

$$\mathbb{L} = [L_1 \ L_2 \ 0],$$

while the complete observations process for the major agent A_0 is defined to be

$$dy_0(t) = dx_0(t).$$

We now introduce the family of partial observation information sets $\mathcal{F}_i^y$, $1 \le i \le N$, defined to be the increasing σ-fields generated by agent $\mathcal{A}_i$'s partial observations $(y_i(\tau);\, 0 \le \tau \le t)$, $1 \le i \le N$, on its own state and the major agent's state, as given in (3).

For each minor agent $\mathcal{A}_i$, $1 \le i \le N$, the set of control inputs $\mathcal{U}_i^y$ based upon the local partial information set of that minor agent is defined to be the collection of feedback controls adapted to the increasing σ-field $\{\mathcal{F}_{i,t}^y,\, t \ge 0\}$.

The major agent is assumed to have complete observations of its own state; namely, as in Section 2, its control set $\mathcal{U}_0$ consists of feedback controls adapted to the set $\{\mathcal{F}_{0,t};\, t \ge 0\}$.

The following important observation is to be made concerning this formulation. Since the major agent is assumed to have complete observations of its own state, the minor agents are able to form conditional expectations of the major agent's MFG control action u_0 since it is a (linear) function of the major agent's state.

To show why this is assumed, consider the situation where the major agent's controls were measurable with respect to the σ-field $\tilde{\mathcal{F}}_0$ which was generated by partial observations on the major agent's state which are available only to the major agent. Then conditional expectations of the major agent's control action could not be generated by the minor agents since these would be conditional expectations (with respect to $\mathcal{F}_i^y$) of a conditional expectation $\hat{x}_{0|\tilde{\mathcal{F}}_0}$, computed with respect to $\tilde{\mathcal{F}}_0$.

The Riccati equation associated with the Kalman filtering equations for $x_i^{0,\bar{x}} \triangleq [x_i, x_0, \bar{x}]$ is now given by:

$$\dot{V}(t) = \mathbb{A}_k V(t) + V(t)\mathbb{A}_k^\top - K(t) R_v K^\top(t) + Q_w,$$

where

$$Q_w = \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & \Sigma_0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbb{A}_k = \begin{bmatrix} A_k & [G \;\; 0_{n\times nK}] \\ 0_{(nK+n)\times n} & \begin{bmatrix} A_0 & 0_{(nK\times n)} \\ \bar{G} & \bar{A} \end{bmatrix} \end{bmatrix},$$

and

$$V(0) = \mathbb{E}\left[x_i^{0,\bar{x}}(0) - \hat{x}_i^{0,\bar{x}}(0)\right]\left[x_i^{0,\bar{x}}(0) - \hat{x}_i^{0,\bar{x}}(0)\right]^\top.$$

The innovation process is evidently

$$d\nu_i = dy_i - \mathbb{L}\begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix},$$

and the Kalman filter gain is given by

$$K(t) = V(t)\mathbb{L}^\top R_v^{-1}.$$

Finally, adopting the additional assumption below yields the infinite population minor agent filtering equations.

H6: The system parameter set Θ satisfies $[\mathbb{A}_k, Q_w]$ controllable and $[\mathbb{L}, \mathbb{A}_k]$ observable for $1 \le k \le K$.

Minor Agent Filtering Equations:

$$\begin{bmatrix} d\hat{x}_{i|\mathcal{F}_i^y} \\ d\hat{x}_{0|\mathcal{F}_i^y} \\ d\hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix} = \begin{bmatrix} A_k & [G \;\; 0_{n\times nK}] \\ \begin{bmatrix} 0_{n\times n} \\ 0_{nK\times n} \end{bmatrix} & \begin{bmatrix} A_0 & 0_{nK\times n} \\ \bar{G} & \bar{A} \end{bmatrix} \end{bmatrix} \times \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix} dt + \begin{bmatrix} B_k \\ 0_{n\times m} \\ 0_{nK\times m} \end{bmatrix} u_i dt + \begin{bmatrix} 0_{n\times m} \\ B_0 \\ 0_{nK\times m} \end{bmatrix} \hat{u}_{0|\mathcal{F}_i^y} dt + \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \bar{m} \end{bmatrix} dt + K d\nu_i.$$

Theorem 2. ϵ-Nash Equilibria for PO MM-MF Systems Subject to H1-H6, the KF-MF state estimation scheme plus MM-MFG equations generate the set of control laws $\hat{\mathcal{U}}_{MF}^N \triangleq \{\hat{u}_i^0; 0 \le i \le N\}$, $1 \le N < \infty$, given by

$$u_0^0 = -R_0^{-1}\mathbb{B}_0^\top \left[\Pi_0 (x_0^\top, \bar{x}^\top)^\top + s_0^*\right],$$

$$\hat{u}_i^0 = -R^{-1}\mathbb{B}_k^\top \left[\Pi_k (\hat{x}_{i|\mathcal{F}_i^y}^\top, \hat{x}_{0|\mathcal{F}_i^y}^\top, \hat{\bar{x}}_{|\mathcal{F}_i^y}^\top)^\top + s_k^*\right]$$

$\{1 \le i \le N\}$, such that

(i) All agent systems $S(A_i)$, $0 \le i \le N$, are second order stable.
(ii) $\{\hat{\mathcal{U}}_{MF}^N; 1 \le N < \infty\}$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$

$$J_i^N(\hat{u}_i^0, \hat{u}_{-i}^0) - \epsilon \le \inf_{u_i \in \mathcal{U}_i^y} J_i^N(u_i, \hat{u}_{-i}^0) \le J_i^N(\hat{u}_i^0, \hat{u}_{-i}^0).$$

Proof.

The sequence of steps to this result are now a combination of standard Separation Theorem methods and the LQG Major-Minor agent LQG MF method applied to the controlled estimated state equations.The sequence of steps is the following:

(1) The major agent and individual minor agent state estimation recursive equations schemes are given by the MM KF-MF Equations above (for size N finite populations and infinite populations).
(2) One next applies the Separation Theorem strategy for reducing a partially observed SOC problem to a completely observed SOC problem for the controlled state estimate processes beginning with the re-expression of the perfomance functions in terms of the state estimation proceses.
(3) The control law dependent summand of the individual cost for the major agent A_0:

$$J_0^N(u_0, u_{-0}) = \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\big\|x_0 - H_0 \hat{x}_{|\mathcal{F}_0}^N - \eta_0\big\|_{Q_0}^2 + \|u_0\|_{R_0}^2\Big\} dt,$$

where $\hat{x}_{|\mathcal{F}_0}^N = (1/N)\sum_{i=1}^N \hat{x}_{i|\mathcal{F}_0}$.

(4) The control law dependent summand of the individual cost for a minor agent A_i, $i \in \mathbf{N}$:

$$J_i^N(u_i, u_{-i}) = \mathbb{E}\int_0^\infty e^{-\rho t}\Big\{\big\|\hat{x}_{i|\mathcal{F}_i^y} - H_1 \hat{x}_{0|\mathcal{F}_i^y} - H_2 \hat{x}_{|\mathcal{F}_i^y}^N - \eta\big\|_Q^2 + \|u_i\|_R^2\Big\} dt.$$

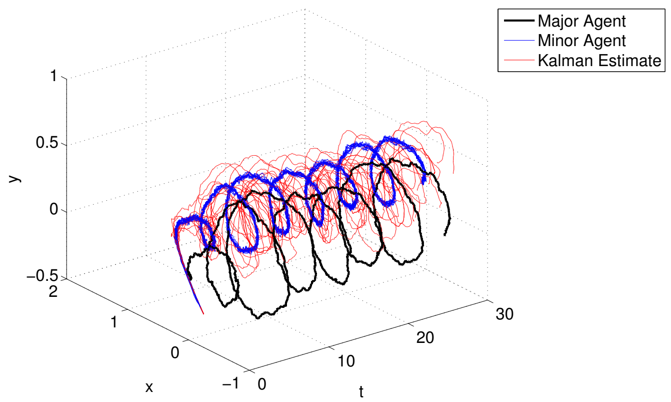

Fig. 1. State trajectories

It is to be noted that this step transforms the J_0 and J_i MM performance functions into LQG MM tracking performance functions on the controlled state estimate processes in both the infinite and finite population cases.

(5) The problem in (2) for the state estimate processes is next solved using the completely observed LQG MM-MFG methodology (Huang (2010)) which yields the $\hat{u}_0$ and $\hat{u}_i$ control laws and the J_0^∞ and J_i^∞ performance function values.

(6) The major agent's performance value J_0^∞ and the minor agents' performance value functions J_i^∞ necessarily correspond to an infinite population Nash equilibrium.

(7) We conclude by applying the infinite population controls in the finite population game case where the standard approximation analysis (Huang et al. (2007); Huang (2010)) gives ϵ-Nash equilibria with respect to J_0^∞ and J_i^∞ for J_0^N and J_i^N in finite N populations.

4. SIMULATION

Consider a system of 100 minor agents and a single major agent. The system matrices $\{A_k, B_k, 1 \leq k \leq 100\}$ for the minor agents are uniformly defined as

$$A \triangleq \begin{bmatrix} -0.05 & -2 \\ 1 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and for the major agent we have

$$A_0 \triangleq \begin{bmatrix} -0.05 & -2 \\ 1 & 0 \end{bmatrix}, \quad B_0 \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The parameters used in the simulation are: $t_{final} = 30s$, $\Delta t = 0.01s$, $\sigma_w = 0.05$, $\sigma_v = 0.05$, $\rho = 0.01$, $\eta = [0.25, 0.25]^\top$, $\eta_0 = [0.25, 0.25]^\top$, $Q = 100 \times I_{2\times2}$, $Q_0 = 100 \times I_{2\times2}$, $R = 1$, $R_0 = 1$, $H = 0.6 \times I_{2\times2}$, $H_0 = 0.6 \times I_{2\times2}$, $\hat{H} = 0_{2\times2}$, $G = 0_{2\times2}$, and the mean field equation system is iterated 100 times. The state trajectories of a single realization can be seen for a population of all agents with their estimates. Only 10 minor agents are displayed for clarity. The effect of the major agent's state on minor agents' states is seen on the horizon. In the case when the minor agents are not allowed to directly observe the major agent, they apply Kalman filtering, and their estimates are also plotted in the graph. As expected, the estimates closely follow the true state values on the whole horizon.

5. CONCLUDING REMARKS

Building upon the MM-LQG-MFG theory for partially observed MM systems and the Nonlinear MM-MFG theory, one important next step is to generate a MM-MFG theory for partially observed nonlinear MM systems. In terms of significant applications, the application of these methods to power markets has been initiated in Caines and Kizilkale (2013); this formulation builds upon application of MM-MFG Theory to markets where minor agents (customers and suppliers) receive intermittent observations on active major agents (such as utilities and international energy prices) and on passive major agents (such as wind and ocean behaviour) (Kizilkale et al. (2012)).

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