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Squaring-Up Method in the Presence of Transmission Zeros ⋆

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Abstract: Non-square MIMO systems are becoming increasingly common, as the addition of multiple sensors is becoming prevalent. However, square systems are needed sometimes as an leverage when it comes to design and analysis, as they possess desirable properties such as invertibility and strict positive realness. This paper presents a method to square-up a class of MIMO systems with stable transmission zeros while keeping the squared system minimum phase. The proposed method is used to carry out adaptive control of this class of systems and shown to lead to satisfactory performance in a numerical study of a 747 aircraft.

Keywords: Squaring-Up; Transmission Zero; Minimum Phase; Positive Real.

1. INTRODUCTION

Square systems play a key role in control theory development because of some unique properties they may possess such as left/right invertibility in Chen et al. [2004]. Additionally, in order for a system to be strictly positive real (SPR) it must necessarily be square (Weiss et al. [1994]). The SPR property is essential for prescribing the direction of parameter adaptation and guarantees stability through KYP lemma (Narendra and Annaswamy [2004]). Therefore, in adaptation design of multivariable parametric uncertainties (Narendra and Annaswamy [2004], Tao [2003]), square minimum-phase systems are commonly assumed. To extend these results to non-square systems, a squaring-up (or down) method is usually needed, which effectively produces a minimum-phase square system through addition (or deletion) of suitable inputs or outputs.

The squaring-down method is first attempted in 1970s by Kouvaritakis and MacFarlane [1976] and Sebakhy et al. [1985] and its zero placement was observed to be equivalent to pole-placement using output feedback in a transformed space. Since pole-placement using output feedback can be achieved only under some specific conditions, the squaring-down method can be restrictive. Literature on squaring-up methods were rather sparse until the work by Misra [1992, 1993]. It has been shown the zero-placement in the square-up method is equivalent to pole-placement using state feedback in a transformed coordinate and therefore is much more feasible. On the other hand, squaring-up methods involve the addition of pseudo inputs or outputs and therefore can only be used as a preliminary step in the overall control design.

Recently, the squaring-up method has gained increasing interest in adaptive control design (Lavretsky and Wise [2013], Qu et al. [2013]). One key finding is that the pseudo-inputs (or outputs) can be used for feedback gain design which yields good properties that usually only exists in a square system. The first procedure in these papers is to perform squaring-up, then design a feedback compensator so that an underlying sub-system becomes SPR. The design has been proved plausible (Huang et al. [1999]) but only in the "lifted" design space, which then is fulfilled by squaring-up. For the design to be implementable on the real control/measurement, the compensator has to be "projected" back into the original design space through suitable partition. Properties such as SPRness are preserved, which enables the design of adaptive output feedback control for general non-square MIMO systems (Qu et al. [2013]).

Squaring-up is therefore a critical component of multivariable adaptive control. In this paper, we present a modification to the square-up procedure proposed in Misra [1992]. This modification allows square-up to be possible even when the non-square plant has transmission zeros. With some preliminaries in Section 2, we describe in Section 3 the difficulty with the method in Misra [1992] and our proposed modification. Adaptive control of a nonsquare plant is shown to be feasible using the proposed method, in section 4. Due to the importance of this method in multivariable adaptive control, this simple extension to the square-up method is expected to be highly useful.

2. PRELIMINARIES

Given a system realization $\Sigma_p = \{A, B, C\}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, the system Σ_p is square if $m = p$, tall if $m < p$, and wide if $m > p$. Squaring-up is the process by which a non-square system is made square through the addition of more inputs or outputs until $m = p$. Squaring-down is a similar process where a square system is reached through the removal of inputs or outputs. Squaring methods usually introduce zeros into the squared system as demonstrated by Kouvaritakis and MacFarlane

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[1976]. To facilitate control designs, one underlying task of squaring-up is to place these zeros in the finite left half of complex plane, i.e. $\mathbb{C}^-$. In this context, definition of zeros are relevant and will be presented below. First, we need to define the normal rank of a matrix function.

Definition 1. The normal rank of a matrix function $X(s)$ is defined as the rank of $X(s)$ for almost all the values of $s \in \mathbb{C}$, or $\max_{s\in\mathbb{C}}(rank[X(s)])$.

In this paper, our focus is on the few s such that the rank of $X(s)$ becomes smaller than its normal rank, i.e. "looses" its normal rank. With Definition 1, we define transmission zeros as follows.

Definition 2. (MacFarlane [1975]). For a system Σ_p and its transfer function $G(s) = C(sI - A)^{-1}B$, the transmission zeros are defined as the values of $s \in \mathbb{C}$ such that $G(s)$ loses normal rank.

One pathological case of Definition 2 is that the transmission zeros coincide with the poles. To avoid the ambiguity, an advanced definition of transmission zero should be introduced (MacFarlane and Karcanias [1976]), which nevertheless, does not alter the result in this paper and therefore will not be considered. In the definition, the normal rank is used to take into account of the case of degenerate systems, which is defined below.

Definition 3. If for a system Σ_p, the rank of $G(s)$ is strictly less than $min(m, p)$ for any $s \in \mathbb{C}$, then the system is degenerate.

Controllable and observable systems can be degenerate. The system is degenerate when there are repeated states, or outputs. While $G(s)$ possesses transmission zeros, similarly, a system's realization Σ_p also has zeros, which are called "invariant zeros" and has a geometric definition as following.

Definition 4. (Rosenbrock [1970]). For a system $\Sigma_p = \{A, B, C\}$, the invariant zeros are the values of $s \in \mathbb{C}$ such that its Rosenbrock matrix $R(s)$ loses normal rank, where $R(s)$ is defined as

$$R(s) = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}. \tag{1}$$

It is noted that for ease of exploration, we used a negative version of Rosenbrock matrix. The name "invariant" comes from the following well-known proposition in MacFarlane and Karcanias [1976].

Proposition 1. Invariant zeros (and transmission zeros) are invariant for coordinate transformation, state feedback, observer feedback, output feedback, and invertible pre- or post-compensation.

It is also shown in MacFarlane and Karcanias [1976] that the set of invariant zeros contains "decoupling zeros", which are defined below.

Definition 5. (MacFarlane and Karcanias [1976]). For a system Σ_p, the input decoupling zeros correspond to the set of all $s_d \in \mathbb{C}$ such that the following $n \times (n+m)$ matrix loses normal rank:

$$R_I(s_d) = [\, s_d I - A \quad -B \,] \tag{2}$$

and the output decoupling zeros correspond to the set of all $s_d \in \mathbb{C}$ such that the following $(n + p) \times n$ matrix loses normal rank:

$$R_O(s_d) = \begin{bmatrix} s_d I - A \\ C \end{bmatrix}. \tag{3}$$

The decoupling zeros are a subset of system poles. They are actually the uncontrollable modes or the unobservable modes of the system. The following proposition relates invariant zeros, transmission zeros and decoupling zeros as shown by MacFarlane and Karcanias [1976].

Proposition 2. For a general non-square system, the set of {Invariant zeros} = the set of {transmission zeros + some of decoupling zeros}.

When calculating transmission zeros or performing zero placement, one should be very careful about the types of zeros in the results. Rather than using Definition 2, the squaring-up method proposed in this paper will use the system's Rosenbrock matrix to calculate and relocate transmission zeros. We guarantee the method only manipulates transmission zeros, through the proposition below.

Proposition 3. For a non-degenerate system realization $\Sigma_p = \{A, B, C\}$ that is controllable and observable, the transmission zeros are the values of $s \in \mathbb{C}$ such that $rank[R(s)] < min(n + m, n + p)$.

It is a direct result of Definitions 3, 4 and 5, and Proposition 2. The reader is referred to (MacFarlane and Karcanias [1976]) for the detail proof. Finally, we introduce the general definition of minimum phase systems.

Definition 6. The system is minimum phase if all its transmission zeros lie in $\mathbb{C}^-$.

3. SQUARING-UP METHOD

3.1 Problem Definition

Without loss of generality, we assume the given system is wide. A tall system can be squared-up using its duality. Given a wide system $\Sigma_p = \{A, B, C\}$, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $n > m > p$, the goal is to find an augmentation $C_a \in \mathbb{R}^{(m-p)\times n}$ such that the system $\Sigma_a = \{A, B, \bar{C}\}$ is square and minimum phase, where $\bar{C}^T = [C^T, C_a^T]$. We assume that Σ_p satisfies following assumptions:

Assumption 1. (A, B) is controllable, and (A, C) is observable;

Assumption 2. $rank(B) = m$;

Assumption 3. $rank(CB) = p$.

It is noted that the observability of (A, C) is not necessary for the existing squaring procedure to work (Misra [1992]). It is assumed here for the ease of exposition in the sense that no output-decoupling zeros will be involved in our exploration. Assumptions 1 through 3 guarantee that Σ_p is controllable, observable and non-degenerate. From Proposition 3, the only subset of $s \in \mathbb{C}$ that makes $R(s)$ lose normal rank is the set of transmission zeros.

Before proceeding to the squaring-up method, we examine closely system's Rosenbrock matrix and interpret the goal geometrically (Misra [1992]). The Rosenbrock matrix $R(s)$ of the given system Σ_p can be written in an orthogonal state coordinate, $\tilde{R}(s)$ as in

$$R(s) \xrightarrow{T} \tilde{R}(s) = \left[\begin{array}{cc|c} sI_m - A_{11} & -A_{12} & -B_1 \\ -A_{21} & sI_{n-m} - A_{22} & 0 \\ \hline C_{11} & C_{12} & 0 \end{array}\right] \tag{4}$$

where T is an invertible coordinate transformation matrix defined as

$$T = [B^T, (null(B^T))^T]. \tag{5}$$

It is easy to show that $rank[\tilde{R}(s)] = rank[R(s)]$ for all $s \in \mathbb{C}$ and as a result, the transmission zeros of $R(s)$ coincide with the transmission zeros of $\tilde{R}(s)$. The goal then is to design $C_{21} \in \mathbb{R}^{(m-p)\times m}$ and $C_{22} \in \mathbb{R}^{(m-p)\times(n-m)}$ such that the squared-up system $\tilde{R}_a(s)$ as

$$R_a(s) \xrightarrow{T} \tilde{R}_a(s) = \left[\begin{array}{cc|c} sI_m - A_{11} & -A_{12} & -B_1 \\ -A_{21} & sI_{n-m} - A_{22} & 0 \\ \hline C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \end{array}\right] \tag{6}$$

has all minimum phase transmission zeros. In other words, the goal is to design $C_a = [C_{21}, C_{22}]$ such that $\tilde{R}_a(s)$ only loses rank at the set of pre-specified $s \in \mathbb{C}^-$.

3.2 The Squaring-Up Method of Misra [1992]

For the given system Σ_p, a squaring-up method has been proposed in Misra [1992] to find C_a that places the transmission zeros of the squared-up system using state-feedback pole-placement in the control canonical coordinate, whose steps are briefly summarized below.

Step 1. First choose C_{21} such that C_1 is an invertible matrix, where $C_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$. Without loss of generality, we can choose

$$C_{21} = null(C_{11}^T) \tag{7}$$

where *null* forms a basis for the null space.

Step 2. Find C_{22} such that the eigenvalues of $(\tilde{A}_{22} - B_{ps2}C_{22})$ are at desired locations v_{s0}. For example the following MATLAB command is used to find C_{22}:

$$C_{22} = \texttt{place}(\tilde{A}_{22}, B_{ps2}, v_{s0}) \tag{8}$$

where $\tilde{A}_{22} \triangleq A_{22} - A_{21}C_1^{-1}\tilde{C}_2$, $\tilde{C}_2 \triangleq \begin{bmatrix} C_{12} \\ O_{(m-p)\times(n-m)} \end{bmatrix}$, and $B_{ps2} \in \mathbb{R}^{(n-m)\times(m-p)}$ is defined as:

$$B_{ps} = [B_{ps1}, B_{ps2}] \triangleq A_{21}C_1^{-1}. \tag{9}$$

It has been proved that the placed pole v_{s0} is exactly the transmission zeros of the squared-up system Σ_a (Misra [1992]). The proof uses the fact that $\tilde{R}(s)$ loses rank only if $(sI_{n-m} - A_{22} + A_{21}C_1^{-1}C_2)$ loses rank, where $C_2 = \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix}$. As a result, one can make the squared-up system minimum phase using a v_{s0} with all elements in $\mathbb{C}^-$.

3.3 Restrictions in the Existing Method

The existing squaring-up method works provided $(\tilde{A}_{22}, B_{ps2})$ is controllable. However, there is no guarantee that $(\tilde{A}_{22}, B_{ps2})$ is controllable for every given Σ_p satisfying assumptions 1 to 3. In what follows we precisely delineate cases where $(\tilde{A}_{22}, B_{ps2})$ can be uncontrollable.

$(\tilde{A}_{22}, B_{ps2})$ is not controllable if and only if there exists a scalar s_0 and a non-zero vector w_0 such that

$$w_0^T[s_0 I - \tilde{A}_{22}, B_{ps2}] = 0. \tag{10}$$

That is s_0 is an uncontrollable eigenvalue of $(\tilde{A}_{22}, B_{ps2})$ with w_0 being the uncontrollable mode. Then

$$w_0^T s_0 - w_0^T \tilde{A}_{22} = 0 \tag{11}$$

$$w_0^T B_{ps2} = 0. \tag{12}$$

Substituting the definition of $\tilde{A}_{22}$ transforms Eq.(11) into

$$w_0^T s_0 - w_0^T A_{22} + w_0^T A_{21} C_1^{-1} \tilde{C}_2 = 0. \tag{13}$$

Also, it is noted that

$$w_0^T A_{21} C_1^{-1} = w_0^T [B_{ps1}, B_{ps2}] = w_0^T [B_{ps1}, 0] \tag{14}$$

which follows from Eq.(9) and Eq.(12). Without loss of generality, C_1^{-1} can be written as

$$C_1^{-1} = [C_{11}^\dagger, C_{21}^T] \tag{15}$$

where $C_{11}^\dagger$ stands for the right pseudo-inverse of C_{11}. It is noted the representation Eq.(15) is not unique. One can easily verify Eq.(15) using the facts that

$$\begin{cases} C_{11}C_{21}^T = O_{p\times(m-p)} \\ C_{11}C_{11}^\dagger = I_p \\ C_{21}C_{11}^\dagger = O_{(m-p)\times p} \\ C_{21}C_{21}^T = I_{m-p} \\ C_{11}^\dagger C_{11} + C_{21}^T C_{21} = I_m \end{cases} \tag{16}$$

Using Eq.(15), B_{ps1} and B_{ps2} can be rewritten as

$$B_{ps1} = A_{21}C_{11}^\dagger \quad \text{and} \quad B_{ps2} = A_{21}C_{21}^T \tag{17}$$

Eq. (14) and Eq. (17) can be used to transform Eq. (13) into

$$w_0^T s_0 - w_0^T A_{22} + w_0^T A_{21} C_{11}^\dagger C_{12} = 0. \tag{18}$$

Now we will examine the implication of Eq. (18) on the original system (4). For this purpose, we return to $\tilde{R}(s)$, and note that

$$rank[\tilde{R}(s)] = rank\left(\tilde{R}(s)\left[\begin{array}{cc|c} I_m & -C_{11}^\dagger C_{12} & 0 \\ 0 & I_{n-m} & 0 \\ \hline 0 & 0 & I_m \end{array}\right]\right)$$
$$= rank\left[\begin{array}{cc|c} sI_m - A_{11} & -sC_{11}^\dagger C_{12} - A_{12} + A_{11}C_{11}^\dagger C_{12} & -B_1 \\ -A_{21} & sI_{n-m} - A_{22} + A_{21}C_{11}^\dagger C_{12} & 0 \\ \hline C_{11} & 0 & 0 \end{array}\right] \tag{19}$$

Eq.(19) shows that the rank of $\tilde{R}(s)$ fully depends on C_{11}, B_1 and $sI_{n-m} - A_{22} + A_{21}C_{11}^\dagger C_{12}$ and is given by

$$rank[\tilde{R}(s)] = rank(C_{11}) + rank(B_1) + rank(sI_{n-m} - A_{22} + A_{21}C_{11}^\dagger C_{12}). \tag{20}$$

Eq.(18) says there exists a s_0 such that $s_0 I_{n-m} - A_{22} + A_{21}C_{11}^\dagger C_{12}$ loses rank. Eq.(20) implies that such s_0 will cause the system $\tilde{R}(s)$ to loose rank. From Proposition 3, it is clear that s_0 is a transmission zero of $\tilde{R}(s)$. From Proposition 1, it follows that $R(s)$ has a transmission zero at s_0. From Eq.(10), we see that this s_0 corresponds to the uncontrollable mode of $(\tilde{A}_{22}, B_{ps2})$. In summary, the square-up method of Misra [1992] fails when the original system has a transmission zero.

This finding is not entirely surprising. Suppose that the transfer function matrix representation $G(s)$, of Σ_p,

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{p1}(s) & \cdots & G_{pm}(s) \end{bmatrix} \tag{21}$$

has a normal rank r, and $r \leq p < m$. If $G(s)$ has a transmission zero at s_o, then by Definition 1 it means $rank[G(s_o)] < r$. We append $(m-p)$ outputs leading to a C_a that is independent of C. This in turn implies that we append $(m-p)$ rows to $G(s)$ and produce a square matrix $G_a(s)$. It is easy to show that $rank[G_a(s)] = (r+m-p)$ and that $rank[G_a(s_o)] < (r+m-p)$, which by Definition 1 means that there is a transmission zero s_o in the squared-up system. This argument holds for any arbitrary C_a. It therefore can be concluded that squaring-up, by appending inputs/outputs, preserves pre-existing transmission zeros in the given non-square system. This finding motivates following modification.

3.4 A Modified Squaring-Up Method

We show in this section that by introducing an additional assumption on the pre-existing transmission zeros, a modified method can be employed to square-up the system.

Assumption 4. The system Σ_p has only minimum-phase transmission zeros.

Under assumption 4, it follows that all uncontrollable modes s_0 in $(\tilde{A}_{22}, B_{ps2})$, if there are any, lie in $\mathbb{C}^-$. This in turn implies that the pair $(\tilde{A}_{22}, B_{ps2})$ is stabilizable. We will show below that in this case, squaring-up is possible.

Since $(\tilde{A}_{22}, B_{ps2})$ is stabilizable, an LQR approach can be used to determine the corresponding "gain" C_{22}. For example, the following MATLAB command can be used to calculate C_{22}.

$$C_{22} = \texttt{lqr}(\tilde{A}_{22}, B_{ps2}, Q_{22}, R_{22}) \tag{22}$$

where Q_{22} and R_{22} are LQR weights and are positive definite. An additional advantage of Eq.(22) over Eq.(8) is the numerical stability in the magnitude of C_{22}. For instance, we can choose a large R_{22} to penalize the use of C_{22} leading to a small C_a.

We summarize the overall results in Theorem 1, which is an important preliminary step in multivariable adaptive control design.

Theorem 1. Given a system satisfying assumptions 1 to 4, there exists a $E \in \mathbb{R}^{m\times m}$ and a $C_a \in \mathbb{R}^{(m-p)\times n}$ such that 1) the system $\Sigma_{su} = \{A,\ BE,\ \bar{C}\}$ is minimum phase; 2) $\bar{C}BE = (\bar{C}BE)^T > 0$, where $\bar{C} = [C,\ C_a]$.

Proof. The existence of C_a is proved by virtue of Eq.(22). The existence of E is specified by the construction

$$E = (\bar{C}B)^T \tag{23}$$

and Proposition 1.

We summarize our modified square-up procedure below:

M.Step 1. Check if Σ_p satisfies assumption 1 through 4;

M.Step 2. Transform Σ_p into the control canonical form as in Eq.(4) using Eq.(5);

M.Step 3. Find C_{21} using Eq.(7);

M.Step 4. Calculate the stabilizable pair $(\tilde{A}_{22}, B_{ps2})$;

M.Step 5. Find C_{22} using Eq.(22);

M.Step 6. Augment C with C_a and transform the system back to its original coordinate.

4. ADAPTIVE CONTROL APPLICATION

Consider the following tall linear plant model

$$\begin{aligned} \dot{x} &= Ax + B\Lambda u_c + B_{\text{ref}} z_{\text{cmd}} \\ y &= Cx \\ z &= C_z x \end{aligned} \tag{24}$$

where z is the regulated output, and $\Lambda > 0$ is an unknown uncertainty in control effectiveness. An adaptive controller is proposed by augmenting a robust output feedback baseline control law with an adaptive component, where adaptation is driven by monitoring the response of the plant and comparing it with that of the nominal plant response as given by a reference model. The following observer-like reference model was designed by using the plant model in Eq. (24) with $\Lambda = I$, and adding output error feedback through L as given by

$$\begin{aligned} \dot{x}_m &= Ax_m + Bu_{\text{bl}} + B_{\text{ref}} z_{\text{cmd}} + L(y - y_m) \\ y_m &= Cx_m \\ z_m &= C_z x_m. \end{aligned} \tag{25}$$

u_{bl} denotes a baseline control component and is given by

$$u_{\text{bl}} = -K_{\text{lqr}}^\top x_m \tag{26}$$

where K_{lqr} is selected such that

$$A_m = A - BK_{\text{lqr}}^\top \tag{27}$$

is Hurwitz. The observer gain L, specially for adaptive control, can be calculated as

$$\begin{aligned} L &= \bar{B}R^{-1}S \\ R^{-1} &= (\bar{C}\bar{B})^{-1}((\bar{C}A\bar{B})^T + \bar{C}A\bar{B})(\bar{C}\bar{B})^{-1} + \epsilon I > 0 \\ \bar{B} &= [B,\ B_a] \\ B_a &= C_a^\top \\ S &= \left[S_{1,m\times p},\ S_{2,m\times(p-m)}\right] = (CB)^\top \\ \bar{C} &= SC \end{aligned} \tag{28}$$

It has to be emphasized that in order to find the squaring-up matrix B_a in this tall system case, we transposed the system to its duality, used M.Step 1-6 to find a C_a, and finally transposed the C_a to become B_a. Roughly speaking, we want a small L and therefore a small B_a is preferred.

The total control input is given by augmenting the baseline control law given in Eq. (26) with an adaptive component as

$$u_c = -K_{\text{lqr}}^\top x_m - \Theta^\top x_m \tag{29}$$

With the adaptive parameter Θ adjusted as

$$\dot{\Theta} = \Gamma x_m e_y^\top S_1^\top \text{sign}(\Lambda) \tag{30}$$

where $\Gamma > 0$ is a tuning parameter and $e_y = y - y_m$ is the measured output error. With the controller as in (25-30), z follows z_{cmd} satisfactorily as shown in Qu et al. [2013].

It should be noted that the critical components of the above adaptive control design are the feedback gain L and the mixing matrix S_1. Both have to be selected such that the transfer function matrix

$$\bar{C}(sI - A + LC)B \tag{31}$$

is strictly positive real. This in turn is made possible through the augmentation matrix C_a, a product of the square-up method proposed in this paper.

It should also be noted that the class of MIMO plants considered in this paper is the one that satisfies Assumptions 1 through 4. Of these, assumptions 1 and 2 are rather standard. Assumption 3 is a multi-variable counterpart of relative degree being unity. Assumption 4 is ubiquitous in adaptive control. Theorem 1 guarantees the existence of C_a for this entire class. The results of Qu et al. [2013] uses Theorem 1 as a leverage to guarantee that an adaptive controller as in (25-30) can be designed for all plants in this class. While in this paper, we have restricted our attention to uncertainties of the form Λ in the input matrix as in (24), further extensions for uncertainties in A are possible and are currently being investigated.

4.1 Numerical Example

We present a numerical example of the proposed squaring up method as applied to the design of an adaptive output feedback controller for the lateral-directional dynamics of a Boeing 747 transport aircraft. This linear model is represented as

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u_c \\ y_p &= C_p x_p \\ z &= C_{pz} x_p \end{aligned} \tag{32}$$

where the state, control, output and regulated output are given by

$$\begin{aligned} x_p &= [\beta \;\, p \;\, r \;\, \phi]^\top \\ u_c &= [\delta_{\text{ail}} \;\, \delta_{\text{rud}}]^\top \\ y_p &= [p \;\, r \;\, \phi]^\top \\ z &= \phi \end{aligned} \tag{33}$$

where β represents the sideslip angle, p represents roll rate, r represents yaw rate and ϕ represents roll angle. The control inputs δ_{ail} and δ_{rud} represent the aileron and rudder deflection angles, respectively. All states are measurable except β. Integral augmentation is used on the regulated output $z = \phi$ to enforce reference tracking of roll angle commands ϕ_{cmd}, where the integral error state is defined by

$$\dot{x}_e = \phi_{\text{cmd}} - \phi \tag{34}$$

With integral error augmentation, the state and output vector become

$$\begin{aligned} x &= [\beta \;\, p \;\, r \;\, \phi \;\, x_e]^\top \\ y &= [p \;\, r \;\, \phi \;\, x_e]^\top \end{aligned} \tag{35}$$

The integral-error augmented 747 model with uncertainty due to actuator faults or degradation can be represented by the linear system in Eq. (24). The A,B and C matrices were calculated using the aerodynamic and mass data from Roskam [1995] for a steady, level trim flight condition at an altitude of 40,000 ft with and airspeed of 516 knots. These matrices were transposed to give a wide system with Rosenbrock matrix R(s) as

$$R(s) = \left[\begin{array}{ccccc|cccc} s+0.0605 & 0.0015 & -0.0011 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s+0.4603 & 0.0208 & -1 & 0 & -1 & 0 & 0 & 0 \\ 871 & -0.28 & s+0.141 & 0 & 0 & 0 & -1 & 0 & 0 \\ 32.3 & 0 & 0 & s & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & -1 \\ \hline 0 & -0.1860 & 0.0061 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.0380 & 0.1 & -0.4419 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{36}$$

The control problem is to design an adaptive controller such that command tracking in roll is achieved in the presence of any arbitrary positive definite uncertainty Λ. An adaptive controller is designed for the 747 using Eqs. (25-30). In order to choose L and S in Eqs. (28) and (30) we need to carry out M.Step 1-6 in section 3.4.

M.Step 1 follows by inspection. We note that the system in (36) has a transmission zero at -0.0511. M.Step 2 gives

$$\tilde{R}(s) = \left[\begin{array}{ccccc|cccc} s+0.4603 & 0.0208 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.28 & s+0.1410 & 0 & 0 & 871 & 0 & -1 & 0 & 0 \\ 0 & 0 & s & 1 & 32.3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & -1 \\ 0.0015 & -0.0011 & 0 & 0 & s+0.0605 & 0 & 0 & 0 & 0 \\ \hline -0.1860 & 0.0061 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & -0.4419 & 0 & 0 & 4.038 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{37}$$

which yields

$$A_{22} = -0.0605 \qquad A_{21} = [-0.0015,\ 0.0011,\ 0,\ 0] \tag{38}$$

and

$$\tilde{C}_2 = \begin{bmatrix} 0 \\ 4.038 \\ 0 \\ 0 \end{bmatrix}. \tag{39}$$

M.Step 3 yields

$$C_{21} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

M.Step 4 clearly shows

$$\tilde{A}_{22} = -0.0511 \quad \text{and} \quad B_{ps2} = [0 \;\, 0] \tag{41}$$

It is obvious that the pair $(\tilde{A}_{22}, B_{ps2})$ is uncontrollable and the uncontrollable mode $s_0 = -0.0511$ is exactly the transmission zero of the given system. Using M.Step 5 with LQR weights $Q_{22} = 1$ and $R_{22} = I$ yields

$$C_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{42}$$

and therefore

$$C_a = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{43}$$

Finally, M.Step 6 produces an augmented system in the original coordinate

$$R_a(s) = \left[\begin{array}{ccccc|cccc} s+0.0605 & 0.0015 & -0.0011 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s+0.4603 & 0.0208 & -1 & 0 & -1 & 0 & 0 & 0 \\ 871 & -0.28 & s+0.141 & 0 & 0 & 0 & -1 & 0 & 0 \\ 32.3 & 0 & 0 & s & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & -1 \\ \hline 0 & -0.1860 & 0.0061 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.0380 & 0.1 & -0.4419 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \tag{44}$$

The last two rows are the designed pseudo-output matrix. It can be easily verified that the augmented system $R_a(s)$ has only one transmission zero at -0.0511.

The following simulation results in Figure 1 and Figure 2 show the efficacy of this adaptive output feedback control design based on the squaring up approach and selection of S and L as in Eq. (28).

Figure 1 shows the response of the aircraft with the robust baseline LQR-PI control law, with 30% control effectiveness on the ailerons and rudders when tracking a 30 degree roll angle doublet. Even with good stability margins, significant overshoot and oscillations are present due to the reduction in control effectiveness, and the aircraft response is highly unsatisfactory. Figure 2 shows the response of the aircraft with the same uncertainty and with the addition of the adaptive control component. Overshoot and settling time are significantly reduced, and the response with the reduced control effectiveness and the adaptive controller recovers that of the nominal baseline controller with no uncertainty. The sideslip angle does not exceed 1.1 degrees throughout the maneuver. It is noted that the improvement in the performance is achieved by using more control effort in the first fiew seconds after the command is issued. A higher control rate is observed in the adaptive system. The trade-off between performance and control effort is currently under investigation.

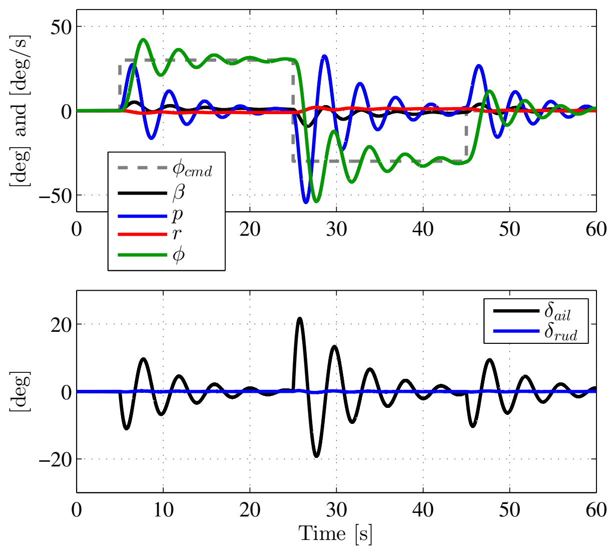

Fig. 1. Baseline control response, 30% control effectiveness

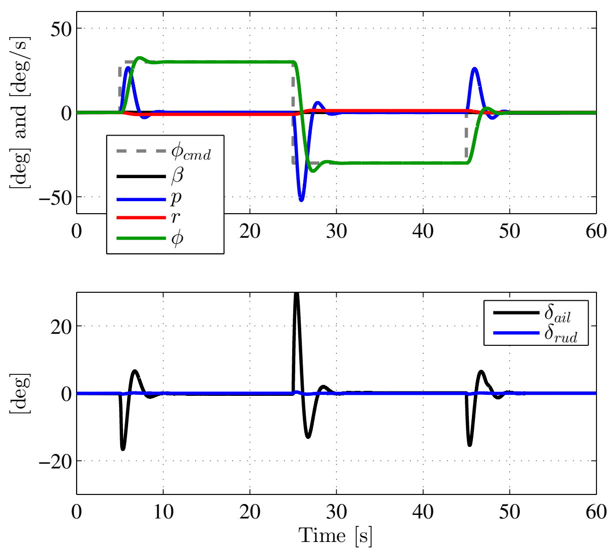

Fig. 2. Adaptive control response, 30% control effectiveness

5. CONCLUSIONS

This paper presents an extension to the square-up method proposed in Misra (1992) when the underlying system has finite and stable transmission zeros. The resulting augmentation matrix C_a is applied to adaptive control of a non-square plant to produce successful tracking under parametric uncertainties. Both the squaring-up procedure and the overall output-feedback based MIMO adaptive controller are numerically validated using a linear model of the lateral-directional dynamics of a Boeing 747 with unknown actuator faults.

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