

Pricing for Coordination in Open-Loop Differential Games

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Abstract: We provide a method for designing a quadratic pricing scheme to induce a desired local Nash equilibrium in an open-loop differential game with nonlinear dynamics. In addition, we present conditions when the induced equilibrium is a global equilibrium. The results are applied to the problem of inducing network managers to invest in security in a multi-network with an epidemic model of the spread of malware.

Keywords: Game Theory, pricing design, nonlinear systems

1. INTRODUCTION

When resources are scarce, competition develops between self-interested agents. Game theory is an established technique for modeling this interaction, and it has emerged as an engineering tool for analysis and synthesis of systems comprised of dynamically-coupled decision-making agents possessing competing interests (Coogan et al. (2013), Li and Marden (2011), Ratliff et al. (2012)). In such scenarios, the strategies chosen by the selfish agents result in a solution that is often inefficient from a societal point of view. This motivates the design of coordinating mechanisms that induce agents to play a Nash equilibrium with desirable properties, namely an equilibrium that is socially optimal.

Engineering problems in which there are decision making agents, either with competing interests or different information sets, as well as a social planner who is tasked with coordinating the agents are appearing more frequently in the literature as technology is integrated into infrastructure (Oldewurtel et al. (2010)). It is important to accurately model these systems and develop control strategies accounting for the interests of all the participating agents while meeting the organizational objective which may represent social welfare or the common good.

The interaction of selfish agents in multi-agent system may be cast as a differential game by modeling dynamically-coupled agents as strategic players. The coordination problem may then be cast as an optimization problem whereby a social planner determines a *coordination mechanism* ensuring the agents play the desired equilibrium. The coordination mechanisms modify the agents' nominal utility functions thereby allowing for a social planner to shape the strategies of the agents in order to meet a desired global objective.

In Ratliff et al. (2012); Calderone et al. (2013); Coogan et al. (2013) the problem of finding prices to induce a socially optimal Nash equilibrium in a linear quadratic game is solved. In this paper, we show that the result can be generalized to open-loop differential games with

non-linear dynamics and possibly non-convex costs that are separable in the state and the control. In particular, we formulate the problem of designing prices to induce a desired (socially optimal) equilibrium for games with non-linear dynamics and non-convex costs as a feasibility problem. We show that if this feasibility problem has a solution, then the desired equilibrium is a local Nash equilibrium of the game resulting from imposition of prices on the players. Further, if the desired equilibrium is the unique equilibrium of the pricing induced game, the designed prices cause the agents to play the socially optimal solution. If in addition the dynamics are convex, then we provide an extended feasibility problem to design prices that force the socially optimal solution to be the unique Nash equilibrium of the pricing induced game. Finally, we apply the theory to the problem of security in multi-networks which is a rising problem in the study of cyber-physical systems (Cardenas et al., 2008; Bloem et al., 2009; Alpcan and Başar, 2010). In this example, we add an objective function and additional constraints to the pricing design feasibility problem to ensure a budget balanced solution.

The rest of the paper is organized as follows. In Section 2, we formulate the game. In Section 3, we define the pricing optimization problem and state our main results. In Section 4, we consider the application of designing prices to induce investment in security in a multi-network system with an epidemic model for the spread of malware. The pricing scheme guarantees that the network dynamics remains stable meaning the spread of malware does not destabilize the networks. In Section 5, we summarize the contributions and discuss future directions.

2. AGENT GAME

Consider a dynamic game with n agents where the system dynamics are given by the general nonlinear ordinary differential equation

$$\dot{x} = f(t, x, u), \quad x_0(t) = x_0 \quad (1)$$

where

$$x(t) = [x_1(t)^T \dots x_n(t)^T]^T \quad (2)$$

and

$$u(t) = [u_1(t)^T \dots u_n(t)^T]^T. \quad (3)$$

Each $x_i(t) \in \mathbb{R}^{n_i}$ where n_i is the dimension. The i -th agent has control over control input $u_i(t) \in \mathbb{R}^{m_i}$ where m_i is the dimension of input $u_i(t)$ and has *nominal* cost given by

$$J_i(x(t_0), u) = \frac{1}{2} \int_{t_0}^{t_f} q_i(t, x) + r_i(t, u) dt + q_i(t_f, x) \quad (4)$$

We suppress the dependence of the cost J_i on the initial condition $x(t_0)$ when it is clear from context. By an abuse of notation, we let u_i denote the strategy over the horizon $[t_0, t_f]$ and we drop the dependence on t in the control action $u_i(t)$ where it is clear from context. Each player is interested in minimizing their cost $J_i(u_i, u_{-i})$ with respect to their choice variable u_i and where $-i = \{1, \dots, i-1, i+1, \dots, n\}$. We restrict each agent's choice u_i to be an open-loop control and we denote the space of open-loop control strategies for agent i by Γ_i .

Given the cost $J_i(u_i, u_{-i})$, the Hamiltonian for the i -th player is given by

$$\mathcal{H}_i(t, x, p_i, u_i, u_{-i}) = q_i(t, x) + r_i(t, u) + p_i(t)^T f(t, x, u) \quad (5)$$

and the optimized Hamiltonian for the i -th player is given by

$$H_i(t, x, p_i) = \min_{u_i \in \Gamma_i} \mathcal{H}_i(t, x, p_i, u_i, u_{-i}). \quad (6)$$

Note that the co-state for each player is a vector $p_i(t) \in \mathbf{R}^n$ for each $t \in [t_0, t_f]$. Let $\{u_i^*\}_{i=1}^n$ be an open-loop Nash equilibrium (either local or global). Then, in equilibrium, the optimality conditions for agent i 's optimization problem are

$$\dot{x}(t) = \frac{\partial \mathcal{H}_i}{\partial p_i}(t, x^*, p_i, u^*) \quad (7)$$

$$\dot{p}_i(t)^T = -\frac{\partial \mathcal{H}_i}{\partial x}(t, x^*, p_i, u^*) \quad (8)$$

$$0 = \frac{\partial \mathcal{H}_i}{\partial u_i}(t, x^*, p_i, u^*) \quad (9)$$

for all $t \in [t_0, t_f]$ where

$$x(t_0) = x_0 \quad \text{and} \quad p_i(t_f) = \frac{\partial q_i}{\partial x}(x^*(t_f)). \quad (10)$$

and

$$\frac{\partial \mathcal{H}_i}{\partial p_i}(t, x^*, p_i, u^*) = f(t, x^*, u^*), \quad (11)$$

$$\frac{\partial \mathcal{H}_i}{\partial x}(t, x^*, p_i, u^*) = \frac{\partial q_i}{\partial x}(t, x^*) + p_i^T \frac{\partial f}{\partial x}(t, x^*, u^*), \quad (12)$$

and

$$\frac{\partial \mathcal{H}_i}{\partial u_i}(t, x^*, p_i, u^*) = \frac{\partial r_i}{\partial u_i}(t, u^*) + p_i^T \frac{\partial f}{\partial u_i}(t, x^*, u^*). \quad (13)$$

Equation (7) is the state equation, Equation (8) is the co-state equation, and Equation (9) is the input stationarity condition.

Definition 1. A *Nash equilibrium* in the open-loop differential game defined by the costs (4) and dynamics (1) is a set of strategies $\{u_i^*[t_0, t_f]\}_{i=1}^n$ such that for each $i \in \{1, \dots, n\}$

$$J_i(u_i^*, u_{-i}^*) \leq J_i(u_i, u_{-i}^*) \quad \forall u_i \in \Gamma_i. \quad (14)$$

The above definition can be interpreted as saying a set of strategies $\{u_i^*\}_{i=1}^n$ is a Nash equilibrium if no player

can decrease their cost by unilaterally deviating from their strategy in $\{u_i^*[t_0, t_f]\}_{i=1}^n$.

The problem of finding an open-loop Nash equilibrium for the game defined by (1) and (4) is to find a set of control strategies $\{u_i^*[t_0, t_f]\}_{i=1}^n$ such that the inequality (14) is satisfied for each $i \in \{1, \dots, n\}$. In general, finding open-loop Nash equilibria of dynamic games is a difficult problem. Some recent work has explored the characterization and computation of local equilibria in continuous games including open-loop differential games Ratliff et al. (2013). In Section 4 we will use the techniques introduced in Ratliff et al. (2013) to compute the Nash equilibrium under the pricing scheme in order to validate that the pricing scheme results in the desired behavior modification.

3. PRICING DESIGN

The pricing design problem is defined to be the optimization problem solved by the social planner in which she designs pricing mechanisms to induce agents to use the desired equilibrium $\{u_i^*(t) = K_i(t)\}_{i=1}^n$. We will restrict ourselves to modifying each player's cost by adding a pricing mechanism, $P_i(t, u)$, that is composed of a quadratic term and a linear term in the control inputs. Namely, we define

$$P_i(t, u) = \int_{t_0}^{t_f} u^T(t) R_i(t) u(t) + c_i u(t) dt \quad (15)$$

where $R_i(t) = R_i(t)^T \succeq 0$. Thus each player's new cost with pricing is given by

$$\begin{aligned} \tilde{J}_i(u) = \int_{t_0}^{t_f} & q_i(t, x) + r_i(u, t) + u^T R_i(t) u + c_i u dt \\ & + q_i(t_f, x(t_f)). \end{aligned} \quad (16)$$

For convenience, we will partition $R_i(t)$ and $c_i(t)$ as follows:

$$R_i(t) := [R_i^1(t) \dots R_i^n(t)] \quad \text{and} \quad c_i(t) := [c_i^1(t) \dots c_i^n(t)]. \quad (17)$$

The Hamiltonian for the i -th agent generated under the pricing scheme is given by

$$\begin{aligned} \tilde{\mathcal{H}}_i(t, x, p_i, u) = & q_i(t, x) + r_i(t, u) + u(t)^T R_i(t) u(t) \\ & + c_i(t) u + p_i^T f(t, x, u) \end{aligned} \quad (18)$$

and the optimized Hamiltonian under pricing is given by

$$\tilde{H}_i(t, x, p_i) = \min_{u_i \in \Gamma_i} \tilde{\mathcal{H}}_i(t, x, p_i, u_{-i}). \quad (19)$$

The optimality conditions under pricing are

$$\dot{x}(t) = \frac{\partial \tilde{\mathcal{H}}_i}{\partial p_i}(t, x^*, p_i, u^*) = f(t, x^*, u^*) \quad (20)$$

$$\begin{aligned} \dot{p}_i^T = & -\frac{\partial \tilde{\mathcal{H}}_i}{\partial x}(t, x^*, p_i, u^*) \\ = & -\frac{\partial q_i}{\partial x}(t, x^*) - p_i^T \frac{\partial f}{\partial x}(t, x^*, u^*) \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial \tilde{\mathcal{H}}_i}{\partial u_i}(t, x^*, p_i, u^*) = & (R_i^i(t))^T u^* + c_i^i(t) + p_i^T \frac{\partial f}{\partial u_i}(t, x^*, u^*) \\ & + \frac{\partial r_i}{\partial u_i}(t, u^*) \\ = & 0. \end{aligned} \quad (22)$$

Note that since the costs are separable in the states and controls and since we do not allow the prices to depend

on the state, both the state equation and the co-state equation are independent of the pricing mechanism. Thus, given a set of desired controls, we can solve for the corresponding state trajectory, $x(t)$ and co-state trajectory $p_i(t)$ on the time interval $[t_0, t_f]$. Then, use the state and co-state to choose prices that satisfy the input stationarity condition.

Define the desired equilibrium

$$u^* = K(t) = [K_1(t) \ \cdots \ K_n(t)]^T. \quad (23)$$

Further, let $x^*(t)$ and $p_i^*(t)$ denote the state and co-state at time t under the desired equilibrium control $K(t)$ respectively. At the desired equilibrium, the input stationarity condition is

$$\begin{aligned} \frac{\partial r_i}{\partial u_i}(t, K(t)) + R_i^i(t)^T K(t) + c_i^i(t) \\ + p_i^*(t)^T \frac{\partial f}{\partial u_i}(t, x^*(t), K(t)) = 0 \end{aligned} \quad (24)$$

Rearranging (24), we define $\alpha_i(t)$ as follows:

$$\begin{aligned} \alpha_i(t) &= R_i^i(t)^T K(t) + c_i^i(t) \\ &= -p_i^*(t)^T \frac{\partial f}{\partial u_i}(t, x^*(t), K(t)) - \frac{\partial r_i}{\partial u_i}(t, K(t)) \end{aligned} \quad (25)$$

for each $i \in \{1, \dots, n\}$. Note that $\alpha_i(t)$ is completely known. Thus, we want to find $R_i(t)$ and $c_i(t)$ to satisfy (25).

Thus designing prices to make $K(t)$ a local Nash equilibrium amounts to solving the feasibility problem defined below in Equation (26).

$$R(t)K(t) + c(t) = \alpha(t) \ \forall t \quad (26)$$

where

$$R(t) = [R_1^1(t) \ \cdots \ R_n^n(t)]^T \quad (27)$$

$$c(t) = [c_1^1(t) \ \cdots \ c_n^n(t)]^T \quad (28)$$

$$\alpha(t) = [\alpha_1(t) \ \cdots \ \alpha_n(t)]^T. \quad (29)$$

We will use the notation $R[t_0, t_f]$ and $c[t_0, t_f]$ to denote $R(t)$ and $c(t)$ for each $t \in [t_0, t_f]$. We can summarize the above results in the following theorem.

Theorem 1. Consider the game defined by nominal agent costs (4) and dynamics (1). Let $\{u^*[t_0, t_f]\}_{i=1}^n$ be the desired Nash equilibrium. If there exists a solution

$$(R[t_0, t_f], c[t_0, t_f]) \quad (30)$$

to the feasibility problem defined in Equation (26), then the desired solution $\{u^*[t_0, t_f]\}_{i=1}^n$ is a local Nash equilibrium to the pricing induced game defined by costs (16) and dynamics (1).

In the case that the desired solution is the unique Nash equilibrium to the induced game, we get the following result.

Corollary 1. Consider the game defined by nominal agent costs (4) and dynamics (1). Let $\{u^*[t_0, t_f]\}_{i=1}^n$ be the desired Nash equilibrium. If there exists a solution

$$(R[t_0, t_f], c[t_0, t_f]) \quad (31)$$

to the feasibility problem defined in Equation (26) and the desired Nash equilibrium is unique in the resulting game defined by costs (16) and dynamics (1), then the prices $(R[t_0, t_f], c[t_0, t_f])$ induce the agents to play $\{u^*[t_0, t_f]\}_{i=1}^n$.

3.1 Dynamics Convex in the State and Control

In the case that the desired equilibrium is unique, the pricing mechanisms are guaranteed to enforce the desired equilibrium strategies.

Let us recall the notion of strict diagonal convexity introduced by Rosen in Rosen (1965) and then extended to the infinite-dimensional case in Haurie and Moresino (2001).

Definition 2. A function $\sum_i L_i(x, u, t, p_i)$ is **diagonally strictly convex** in (x, u) if for all $\bar{u}, \hat{u}, \bar{x}$, and \hat{x} we have

$$\begin{aligned} \sum_i (\bar{u}_i - \hat{u}_i)^T \left(\frac{\partial L_i}{\partial u_i}(x, \bar{u}, t, p_i) - \frac{\partial L_i}{\partial u_i}(x, \hat{u}, t, p_i) \right) \\ - (\bar{x}_i - \hat{x}_i)^T \left(\frac{\partial L_i}{\partial x_i}(\bar{x}, u, t, p_i) - \frac{\partial L_i}{\partial x_i}(\hat{x}, u, t, p_i) \right) > 0 \end{aligned} \quad (32)$$

The following lemma and theorem provide conditions under which a Nash equilibrium of an open-loop differential game is unique.

Lemma 1. Assume that $L_i(x, u)$ is convex in (x, u) and assume that the total running cost

$$L(x, u) = \sum_i L_i(x, u) \quad (33)$$

is diagonally strictly convex in (x, u) . Further, assume that the dynamics $f(t, x, u)$ are convex in x and u . Then, the combined Hamiltonian

$$H(t, x, p) = \sum_i H_i(t, x, p_i) \quad (34)$$

is diagonally strictly convex in x and concave in p .

Theorem 2. If the combined Hamiltonian $H(t, x, p)$ is diagonally strictly convex in x and concave in p , then the open-loop Nash equilibrium is unique.

Lemma 1 is a modified version of Lemma 2.1 in Haurie and Moresino (2001). Theorem 2 is stated in Haurie and Moresino (2001) and its proof can be found in Carlson and Haurie (1996) using our version of the lemma. If the costs under pricing satisfy the assumptions of Lemma 1, the desired equilibrium is the unique Nash equilibrium of the open-loop differential game induced through pricing.

Define the total running cost

$$\sum_i L_i(x, u, t, p_i) \quad (35)$$

where

$$\begin{aligned} L_i(x, u, t, p_i) &= q_i(x, t) + r_i(u, t) + u^T R_i(t)u \\ &\quad + c_i(t)u(t). \end{aligned} \quad (36)$$

Assumption 1. $\sum_i q_i(x, t) + r_i(u, t)$ is diagonally strictly convex in (x, u) .

Provided Assumption 1, by Lemma 1 and Theorem 2, we only need to ensure that the pricing mechanism is diagonally strictly convex in u , i.e. we need to enforce

$$(\hat{u}(t) - \bar{u}(t))^T R(t)(\hat{u}(t) - \bar{u}(t)) > 0 \quad (37)$$

for all \hat{u} and \bar{u} and for each $t \in [t_0, t_f]$. Thus in order to ensure diagonal strict convexity in u , we simply need to ensure that the symmetric component of $R(t)$, i.e. $R(t) + R(t)^T$, is positive definite for all t .

$$\begin{cases} R(t)K(t) + c(t) = \alpha(t) \quad \forall t \\ R(t) + R(t)^T \succ 0 \quad \forall t \end{cases} \quad (38)$$

We have the following result.

Theorem 3. Consider the game defined by nominal agent costs (4) and dynamics (1). Suppose that $f(t, x, u)$ is convex in x and u . Let $\{u^*[t_0, t_f]\}_{i=1}^n$ be the desired Nash equilibrium. Then, if there exists a solution

$$(R[t_0, t_f], c[t_0, t_f]) \quad (39)$$

to the feasibility problem defined in Equation (38), then the prices $(R[t_0, t_f], c[t_0, t_f])$ induce the agents to play $\{u^*[t_0, t_f]\}_{i=1}^n$ to the game defined by (16) and dynamics (1). Further, the desired Nash equilibrium is the unique equilibrium in the pricing induced game.

4. PRICING IN MULTI-NETWORKS

We use the epidemic model for the spread of malware in a multi-network introduced in Bloem et al. (2009). Self-spreading attacks on computer networks are expensive owing to the damage they cause and the security investment required to defend against them. The social planner's goal is to design pricing mechanisms that coordinate the networks so that the overall multi-network is stabilized.

Suppose that we have n networks with N_i nodes in the i -th network and let $x_i(t) \in \mathbb{R}$ denote the number of infected hosts in a network i where hosts can be fractionally infected. Let $u_i(t)$ be the malware removal rate for network i , α be the cross-network pairwise rate of infection, and β be the pairwise rate of infection within networks. In general, computers within a network are more likely to communicate with one another than across networks; hence, we assume $\beta > \alpha$. The spread of malware is then captured in the following epidemic model:

$$\dot{x}_i(t) = \beta(N_i - x_i(t))x_i(t) + \sum_{j=1, j \neq i}^n \alpha(N_j - x_j(t))x_j(t) - u_i(t) \quad (40)$$

Each network independently tries to choose u_i so that the i th network is stabilized. For each network in the multi-network, we consider a cost that is quadratic in the state, i.e. the number of infected hosts, and quadratic in the control, i.e. the patching rate. The nominal cost for network i is

$$J_i(u) = \int_0^{t_f} x^T Q_i x + u^T M_i u dt. \quad (41)$$

where Q_i and M_i are the cost of an infected network host and the cost of the implemented patching response respectively. The social planner designs pricing mechanisms to coordinate the networks by inducing them to choose a desired control action which stabilizes the entire multi-network.

We consider a group of six networks with $N_1 = 3500$, $N_2 = 500$, $N_3 = 2000$, $N_4 = 1000$, $N_5 = 500$, and $N_6 = 1000$. For each network, we take Q_i to be a diagonal matrix with random positive entries where the (i, i) th element of Q_i is larger than the others. The M_i matrices are chosen to be 0 except for the (i, i) th element which is 1. The ratio between $Q_i(i, i)$ and $M_i(i, i)$ is 10 to 1. We scale the time horizon to be over the interval $[0, 1]$ and take the initial number of infected nodes in each network

to be half of the total number of nodes. As in Bloem et al. (2009), we take $\beta = 5.6 \times 10^{-5}$. We set $\alpha = \frac{2}{3}\beta$.

Using the discretization scheme for optimal control problems described in Chapter 4 of Polak (1997), we compute a centralized solution using the sum of all the agents costs and standard nonlinear programming techniques. In general, this only gives us a local optimum to the centralized problem, but we will see that it does improve the performance of the system as compared to the Nash equilibrium. From the centralized solution, we determine a desired set of controls, $K(t) = [K_1(t) \cdots K_n(t)]^T$.

In order to design prices, the social planners solves a modified version of the feasibility problem outlined in (38) at each time step. Since the nominal costs are quadratic in the control, we replace each M_i with R_i so that the new cost for each player becomes

$$\tilde{J}_i(u) = \int_0^{t_f} x^T Q_i x + u^T R_i(t)u + c_i(t)u dt. \quad (42)$$

In this case, Equation (25) becomes

$$\begin{aligned} \bar{\alpha}_i(t) &= R_i^i(t)^T K(t) + c_i^i(t) \\ &= -p_i^*(t)^T \frac{\partial f}{\partial u_i}(t, x^*(t), K(t)) \end{aligned}$$

In addition, we add an objective and several more constraints to make the problem budget balanced. Our final optimization problem is given by

$$\min_{\{R_i(t), c_i(t)\}_{i=1}^n} \sum_{i=1}^n \|R_i(t) - M_i\| + \|c_i(t)\| \quad (43)$$

$$\text{subject to: } R(t)K(t) + c(t) = \bar{\alpha}(t) \quad (44)$$

$$R(t) + R(t)^T \succ 0 \quad (45)$$

$$\sum_{i=1}^n R_i(t) - \sum_{i=1}^n M_i \succ 0 \quad (46)$$

$$R_i(t) = R_i(t)^T \succ 0, \quad \forall i \quad (47)$$

at each time step. Recall that $R(t) = [R_1^1(t) \cdots R_n^n(t)]^T$. Equation (46) forces the sum of the quadratic components of the prices to be greater than the sum of the nominal quadratic components. Given this constraint, (43) seeks to make the costs with prices as close as possible to the nominal costs.

Using the same discretization scheme as in the centralized problem, we numerically approximate local Nash equilibria under both the nominal costs and the costs with pricing using the steepest descent algorithm presented in Ratliff et al. (2013). Since the dynamics are not convex in the state, we cannot guarantee global uniqueness of the induced equilibrium; however by checking the 2nd-order sufficient conditions presented in Ratliff et al. (2013), we can show that the Nash under pricing is an isolated local Nash. Thus control signals initialized close to the equilibrium will converge to it under the steepest descent algorithm.

By solving the pricing problem, we make the centralized solution a local Nash equilibrium of the game with prices. Moreover, we find that a wide variety of initializations for the controls actually converge to the desired equilibrium. Computation of basins of attraction of equilibria for this problem and other general nonlinear problems is left as future work.

It should be noted that the 2nd-order sufficient conditions for isolated local Nash equilibria are only applicable to finite dimensional problems. Thus we can not guarantee that we induce an isolated equilibria of the actual infinite dimensional optimization problem but only of the finite-dimensional discretized problem.

Figure 1 shows the control inputs and Figure 2 shows the state trajectories for the nominal Nash equilibrium, the centralized optimal solution, and the Nash equilibrium under pricing.

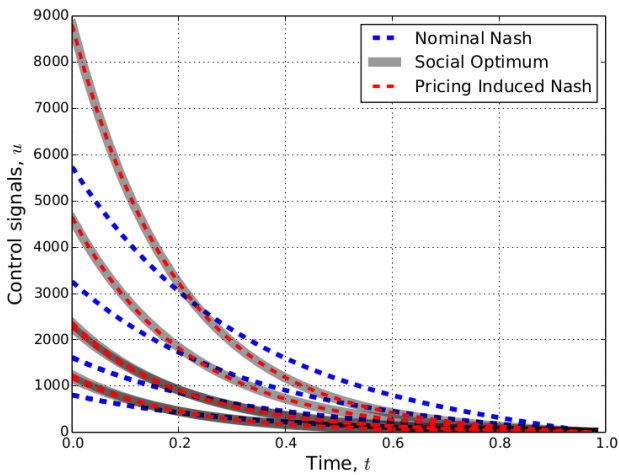


Fig. 1. Control signals at the nominal Nash equilibrium, the centralized optimum, and pricing induced Nash equilibrium.

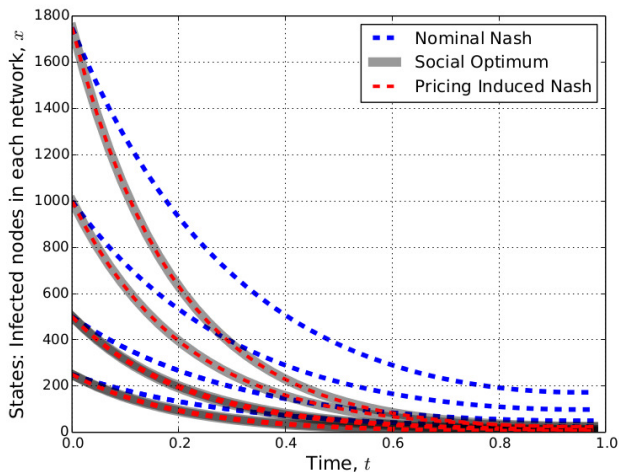


Fig. 2. Number of infected nodes in each network over time. Note that the nominal Nash strategies do not eliminate all infected nodes whereas the socially optimal strategies and pricing induced Nash strategies do.

Figure 3 compares the sum of the running costs. The centralized solution as well as the Nash under pricing reduces the total cost to the system by 8.2%. We also see that we are able to force the sum of all the running costs with prices to be equal to the running cost of the

centralized problem at each time step. Each individual player's running cost with prices is not guaranteed to be equal to their portion of the social cost, however. As an example in Figure 4, we plot the running costs of players 1 and 6.

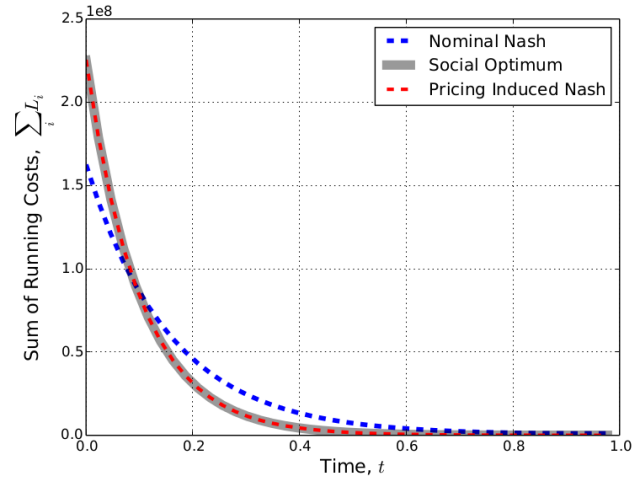


Fig. 3. Sum of individual running costs at the nominal Nash, social optimum, and pricing induced Nash. The fact that the social optimum cost and pricing induced Nash cost are equal means that we can achieve budget balance.

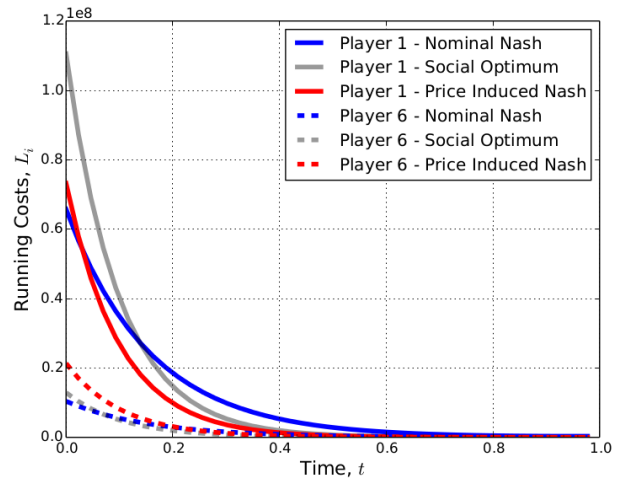


Fig. 4. Individual running costs for players 1 and 6. Though the sum of all the running costs is the same as the social optimum under pricing, each individual player's running cost is different from their portion of the social cost.

5. CONCLUSION

In summary, we have formulated a feasibility problem for finding quadratic and linear prices to induce a socially optimal local Nash equilibria in the context of open-loop differential games with non-linear dynamics and general nominal costs. In addition, we have shown that under special conditions on the dynamics and prices, the induced

equilibrium is the global equilibrium of the game with prices. We apply these techniques to the problem of incentivizing network managers to invest in security to prevent the spread of epidemics. In this particular example, we are able to design budget balanced prices that make the socially optimal controls an isolated local Nash equilibria.

We are currently investigating computation of basins of attraction for the equilibria of the induced game. We are also studying the stability of the equilibria as well as conditions to ensure uniqueness of equilibria in non-convex games.

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