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Optimal Control Problems in Binocular Vision ⋆

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Abstract: Human eye movement can be looked at, as a rotational dynamics on the space **SO(3)** with constraints that have to do with the axis of rotation. A typical eye movement can be decomposed into two components, that go by the name 'version' and 'vergence'. The version component produces identical eye movement in both the eyes, and is used to follow a target located far away by taking the general direction of the target. In order to focus on a closer target, the eyes rotate in opposite directions, using the vergence component. A typical eye movement can be regarded as a concatenation of version followed by vergence. These two eye movements are modeled in this paper assuming that the eyes are perfect spheres with their mass distributed uniformly and that the eyes rotate about their own centers. An optimal control problem is considered where the goal is to rotate an eye pair from an initial 'parallel gaze direction' to a final 'gaze focusing on a target'. The eye pairs are to be actuated optimally using an external torque vector of minimum energy.

Keywords: Eye Movement, Binocular Vision, Listing's Plane, Mid-Sagittal Plane, Euler Lagrange's Equation, Optimal Control.

1. INTRODUCTION

Neurologists, physiologists and engineers have been interested in modeling and control of a single human eye (monocular control) since 1845 with notable studies conducted by Listing (1845), Donders (1848. Press, 1996) and von Helmholtz (1866). Specifically, it has been observed that the oculomotor system chooses just one angle of ocular torsion for any one gaze direction (see Donders (1848. Press, 1996)). Several studies have focused on three dimensional eye movements Crawford et al. (2003), Haslwanter (1995), including rigorous treatment of the topic from the point of view of modern control theory and geometric mechanics Murray (1997). Assuming the human eye to be a rigid sphere, the oculomotor system can be viewed as a mechanical control system and one can apply geometric theory with Lagrangian and Hamiltonian viewpoints Bullo and Lewis (2004), Murray et al. (1994). This paper extends our earlier studies Polpitiya et al. (2007), Ghosh and Wijayasinghe (2012), Wijayasinghe et al. (2014) from monocular control to controlling a pair of eyes (see binocular control paper by Wijayasinghe and Ghosh (2013)).

Any eye orientation can be reached, starting from one specific orientation called the primary orientation, by rotation about a single axis. Listing's law states that, starting from a frontal gaze, any other gaze direction is obtained by a rotation matrix whose axis of rotation is constrained to lie on a plane, called the Listing's plane. Consequently, the set of all orientations the eye can assume is a submanifold Boothby (1986) of **SO(3)** called **LIST**. Listing had shown and subsequently verified by others Tweed and Vilis (1987), Tweed and Vilis (1990), that while gazing targets located at optical infinity and keeping the head fixed, eye orientations are restricted to this submanifold **LIST** Polpitiya et al. (2007). In binocular vision, Listing's law is not valid for fixation of nearby targets, which is the main point of this paper.

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It has been observed Rijn and Berg (1993) (see also Nakayama (1983)), that when a pair of human eyes fixate on a nearby point target, the axes of rotations of the two eyes are not located on the Listing's plane. The eye rotations are not independently controlled (as proposed by von Helmholtz (1866)), but can be viewed as a concatenation of *version* followed by *vergence* (in the spirit of what was originally proposed by Hering (1868)). The versional component of the eye movement is identical for both the eyes and satisfies Listing's law. This is equivalent to saying that the versional part of the eye rotation belongs to **LIST**. On the other hand, the vergence component of the eye movement rotates the two eyes in opposite directions, in order to fixate nearby point targets. Following Rijn and Berg (1993), we would assume that for the vergence part, the rotation vector is restricted to the mid-sagittal plane with respect to the fixed head coordinate system [1]. Starting from the primary orientation, the set of all orientations that are achievable using rotations with axes in the mid-sagittal plane is a submanifold **MS** of **SO(3)**. Identical to what was done earlier for **LIST**, we introduce a Riemannian metric for **MS** and write down the associated Euler Lagrange equation that describes the *vergence eye dynamics* [2]. The paper combines version and vergence dynamics as a proposal for binocular control.

2. QUATERNIONIC REPRESENTATIONS

Representation of 'eye orientation' using quaternion has already been described in Polpitiya et al. (2007). For the sake of clarity, we revisit some of the main ideas in this section. A quaternion is a four tuple of real numbers denoted by Q. We write each element $a \in Q$ as

$$a = a_0\, 1 + a_1\, i + a_2\, j + a_3\, k.$$

[1] The point midway between the center of the two eyes is called the ego-center. The mid-sagittal plane is the plane perpendicular to the segment joining the center of the two eyes and passing through the ego-center.

[2] The versional part of the eye dynamics was studied earlier in Polpitiya et al. (2007).

The space of unit quaternion is identified with the unit sphere in $\mathbb{R}^4$ and denoted by S^3. Each $q \in \mathrm{S}^3$ can be written as

$$q = \cos\frac{\phi}{2}\, 1 + \sin\frac{\phi}{2}\, n_1\, i + \sin\frac{\phi}{2}\, n_2\, j + \sin\frac{\phi}{2}\, n_3\, k, \quad (1)$$

where $\phi \in [0, 2\pi]$, and $n = (n_1, n_2, n_3)^T$ is a unit vector in $\mathbb{R}^3$. If q is an unit quaternion represented as in (1), one can show Altmann (2005), Kuipers (2002) the following, using simple properties of quaternion multiplication (denoted by $\bullet$):

"The vector component of $[q \bullet (v_1\, i + v_2\, j + v_3\, k) \bullet q^{-1}]$ is rotation of the vector (v_1, v_2, v_3) around the axis n by a counterclockwise angle ϕ."

If $\mathbf{S}^3$ is the space of unit quaternions, we define a map between $\mathbf{S}^3$ and **SO(3)** described as follows

$$rot\ :\ S^3 \to SO(3) \quad (2)$$

where

$$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \longmapsto \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2+q_2^2-q_1^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2+q_3^2-q_1^2-q_2^2 \end{pmatrix}. \quad (3)$$

Recall that $SO(3)$ is the space of all 3×3 matrices W such that $WW^T = I$, the identity matrix and $det\ W = 1$. It can be verified that for any nonzero vector v in $\mathbb{R}^3$ we have

$$rot(q)\, v = vec[q \bullet v \bullet q^{-1}].$$

Note that the map 'rot' in (2) is surjective but not $1-1$. This is because both q and $-q$ in S^3 has the same image. We now write down a parametrization of the unit vector 'n' in (1) using polar coordinates as

$$n = \begin{pmatrix} \cos\theta\ \cos\alpha \\ \sin\theta\ \cos\alpha \\ \sin\alpha \end{pmatrix}. \quad (4)$$

Combining (1) and (4), we have the following parametrization of unit quaternions

$$q = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\theta\ \cos\alpha \\ \sin\frac{\phi}{2}\sin\theta\ \cos\alpha \\ \sin\frac{\phi}{2}\sin\alpha \end{pmatrix}, \quad (5)$$

called the axis-angle parametrization. Using the coordinates (θ, ϕ, α) we construct the following sequence of maps

$$[0, \pi] \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{\rho} S^3 \xrightarrow{rot} SO(3) \xrightarrow{proj} S^2, \quad (6)$$

where

$$\rho((\theta, \phi, \alpha)) = q\ \text{(in (5))}, \qquad rot(q) = W$$

and

$$proj(W) = \begin{pmatrix} \sin\theta\ \sin\phi\ \cos\alpha + \cos\theta\ \sin^2\frac{\phi}{2}\ \sin 2\alpha \\ -\cos\theta\ \sin\phi\ \cos\alpha + \sin\theta\ \sin^2\frac{\phi}{2}\ \sin 2\alpha \\ \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\ \cos 2\alpha \end{pmatrix}. \quad (7)$$

Note that the matrix W in $SO(3)$ can be easily written from (3) and the details are omitted. The points in S^2 described by (7) provide a description of the gaze directions as a function of the coordinate angles θ, ϕ, α with respect to an initial gaze direction of $(0,0,1)^T$, i.e. obtained by rotating the vector $(0,0,1)^T$ using the rotation matrix W. The sub-manifolds **LIST** and **MS** can be easily parameterized by restricting $\alpha = 0$ and $\theta = \frac{\pi}{2}$ respectively in (5). This is done in the next section.

3. THE SUBMANIFOLD LIST

The law of rotation for the version eye movement, the Listing's law, asserts that the axis of rotation 'n' in (4) is restricted to the plane

$$\alpha = 0, \quad (8)$$

and one obtains the axis of rotation as

$$n = (\cos\theta,\ \sin\theta,\ 0)^T.$$

The corresponding unit quaternion vector is given by

$$q_L = (\cos\frac{\phi}{2},\ \sin\frac{\phi}{2}\cos\theta,\ \sin\frac{\phi}{2}\sin\theta,\ 0)^T.$$

The rotation matrix W is given by

$$W = \begin{pmatrix} \cos^2\frac{\phi}{2} + \cos 2\theta\ \sin^2\frac{\phi}{2} & \sin 2\theta\ \sin^2\frac{\phi}{2} & \sin\theta\ \sin\phi \\ \sin 2\theta\ \sin^2\frac{\phi}{2} & \cos^2\frac{\phi}{2} - \cos 2\theta\ \sin^2\frac{\phi}{2} & -\cos\theta\ \sin\phi \\ -\sin\theta\ \sin\phi & \cos\theta\ \sin\phi & \cos\phi \end{pmatrix}. \quad (9)$$

Under the Listing's constraint, we define **LIST** to be the associated submanifold of $\mathbf{S}^3$ and $\mathbf{SO}_L(3)$ to be the associated submanifold of **SO(3)**. They are both two dimensional submanifolds parameterizing the versional eye rotation in $\mathbf{S}^3$ and $\mathbf{SO}(3)$ respectively. The gaze direction $(0,\ 0,\ 1)^T$ is transformed to the direction

$$(\sin\theta\ \sin\phi,\ -\cos\theta\ \sin\phi,\ \cos\phi)^T$$

by the rotation matrix (9). Thus we have the following sequence of maps

$$[0, \pi] \times [0, 2\pi] \xrightarrow{\rho} \mathbf{LIST} \xrightarrow{rot} \mathbf{SO}_L(3) \xrightarrow{proj} \mathbf{S}^2, \quad (10)$$

obtained by restricting (6) under the constraint $\alpha = 0$.

4. THE SUBMANIFOLD MS

The law of rotation for the vergence eye movement (proposed in Rijn and Berg (1993)), asserts that the axis of rotation 'n' in (4) is restricted to the plane [3]

$$\theta = \frac{\pi}{2}, \quad (11)$$

and obtain the axis of rotation as

$$n = (0,\ \cos\alpha,\ \sin\alpha)^T.$$

The corresponding unit quaternion vector is given by

$$q_M = (\cos\frac{\phi}{2},\ 0,\ \sin\frac{\phi}{2}\cos\alpha,\ \sin\frac{\phi}{2}\sin\alpha)^T. \quad (12)$$

The rotation matrix W is computed to be

$$W = \begin{pmatrix} \cos\phi & -\sin\phi\ \sin\alpha & \sin\phi\ \cos\alpha \\ \sin\phi\ \sin\alpha & \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\ \cos(2\alpha) & \sin^2\frac{\phi}{2}\ \sin(2\alpha) \\ -\sin\phi\ \cos\alpha & \sin^2\frac{\phi}{2}\ \sin(2\alpha) & \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\ \cos(2\alpha) \end{pmatrix}. \quad (13)$$

Under this new constraint (11), we define **MS** to be the associated submanifold of $\mathbf{S}^3$ and $\mathbf{SO}_M(3)$ to be the associated submanifold of **SO(3)**. They are both two dimensional submanifolds parameterizing vergence eye rotation in $\mathbf{S}^3$ and $\mathbf{SO}(3)$ respectively. The gaze direction $(0,\ 0,\ 1)^T$ is transformed to the direction

$$(\sin\phi\cos\alpha,\ \sin^2\frac{\phi}{2}\sin(2\alpha),\ \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\cos(2\alpha))^T$$

[3] We shall call the constraint (11), the extended Listing's constraint. The plane (11) is precisely the mid-sagittal plane, justifying the acronym **MS** for the submanifold.

by the rotation matrix (13). Thus we have the following sequence of maps

$$[0,2\pi] \times [-\frac{\pi}{2},\frac{\pi}{2}] \xrightarrow{\rho} \mathbf{MS} \xrightarrow{rot} \mathbf{SO}_M(3) \xrightarrow{proj} \mathbf{S}^2, \quad (14)$$

obtained by restricting (6) under the constraint $\theta = \frac{\pi}{2}$.

The following two results are easy consequences of the Listing's Theorem already sketched in Ghosh and Wijayasinghe (2012).

Theorem (Listing): Under the Listing's constraint (8), the map

$$\mathbf{SO}_L(3) - \left\{ \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \xrightarrow{\mathbf{proj}} \mathbf{S}^2 - \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

described by (10) is one to one and onto. □

Theorem (Extended Listing): Under the Extended Listing's constraint (11), the map

$$\mathbf{SO}_M(3) - \left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \xrightarrow{\mathbf{proj}} \mathbf{S}^2 - \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (15)$$

described by (14) is one to one and onto. □

Proof of Extended Listing's Theorem: There are two parts of the proof. The first part is to show that the mapping **'proj'** in (15) is $1-1$. In the second part we show that the mapping **'proj'** in (15) is onto.

Part I (Proof of $1-1$): Consider $\chi = proj \circ rot$ as a mapping from S^3 to S^2 in (6). It is known from an unpublished result of Helmke, that if p and q are points in S^3 and S^2 respectively such that

$$\chi(p) = q.$$

then the set of all $p_1 \in S^3$ that satisfies

$$\chi(p_1) = q$$

has the property that

$$p_1 = p \bullet (a,0,0,b) \quad (16)$$

where $a^2+b^2 = 1$ and the symbol $\bullet$, as before, denotes multiplication as a quaternion. Writing p as in (12), it follows that if $p \in MS$, then every p_1 satisfying (16) belongs to MS if $\sin\frac{\phi}{2}\cos\alpha = 0$ or $b = 0$. If $b = 0$, it would follow that $p_1 = p$ and χ is $1-1$. If $\phi = 0$, it would follow that $p = (1,0,0,0)^T$ and $q = (0,0,1)^T$ which is excluded from the range of **'proj'** in (15). Finally if $\alpha = \pm\frac{\pi}{2}$, it follows that $p = (\cos\frac{\phi}{2}, 0, 0, \pm\sin\frac{\phi}{2})^T$ and $q = (0,0,1)^T$ which is excluded from the range of **'proj'** in (15).

Part II (Proof of onto): For details we would like to refer to Wijayasinghe and Ghosh (2013), where this proof has already been sketched. □

It is easily inferred from the above two theorems that for both, versional and vergence eye movements, the gaze direction uniquely specifies the orientation of the eye, except perhaps when the gaze is backwards for the versional eye movement and axis of rotation is 'pure torsional' for the vergence eye movement [4].

To end this section, we make the following remark:

Remark: Under vergence eye movement, the left and the right eyes rotate in opposite directions. This can be implemented by considering the unit quaternion (12) for the left eye and defining

$$q_M = (\cos\frac{\phi}{2},\ 0,\ -\sin\frac{\phi}{2}\cos\alpha,\ -\sin\frac{\phi}{2}\sin\alpha)^T, \quad (17)$$

for the right eye. This would be equivalent to reversing the sign of the axis of rotation going from left to the right eye. □

[4] Human eye does not rotate with pure torsion.

5. EYE MOVEMENTS UNDER BINOCULAR CONTROL

In this section, we describe the binocular control of human eyes, by combining the version system on **LIST** and vergence system on **MS**. Figs. 1a, 1b, and 1c illustrate how the eye pair moves to see a bird in the sky starting from an initial parallel gaze to a final focused gaze on the bird. Between Figs. 1a, 1b, the version control system rotates the eye-pair maintaining parallel gaze, in the general direction of the bird. We assume that this rotation satisfies the Listing's constraint (8). Between Figs. 1b, 1c, the vergence control system focuses the eye-pair on the bird. The focusing movement assumes that the rotation satisfies the Extended Listing's constraint (11). Although in Figs. 1a, 1b, and 1c, the vergence has been shown following the version, in practice, the vergence and version control can act simultaneously and independently. Typically, however, the versional movements are rapid compared to the vergence eye movements.

The combined effect of the version and vergence movements can be described by the product

$$q_c = q_{vs} \bullet q_{vg}, \quad (18)$$

where q_{vs} is the quaternion from the version part of the dynamics, q_{vg} is the quaternion from the vergence part of the dynamics and $\bullet$ denotes quaternion multiplication. Note that the quaternions q_{vs} and q_{vg} are both defined with respect to the inertial frame (i.e. frame fixed to the earth). If vergence movements are to be viewed with respect to coordinate frame rotating with the versional movement, the quaternion q_{vg} is to be replaced by $\bar{q}_{vg} = q_{vs} \bullet q_{vg} \bullet q_{vs}^{-1}$. The combined effect of version followed by vergence rotation is given by $\bar{q}_{vg} \bullet q_{vs} = q_{vs} \bullet q_{vg}$, which is precisely described in (18). The quaternions q_{vs} and q_{vg} are assumed to satisfy Listing's and Extended Listing's constraints, respectively.

6. VERSION AND VERGENCE AS DYNAMICAL SYSTEMS

6.1 Dynamical systems in alpha-parametrization

Following Polpitiya et al. (2007), one can compute a Riemannian Metric, see Weyl (1964; Dover Publications (Paperback), on **LIST** given by

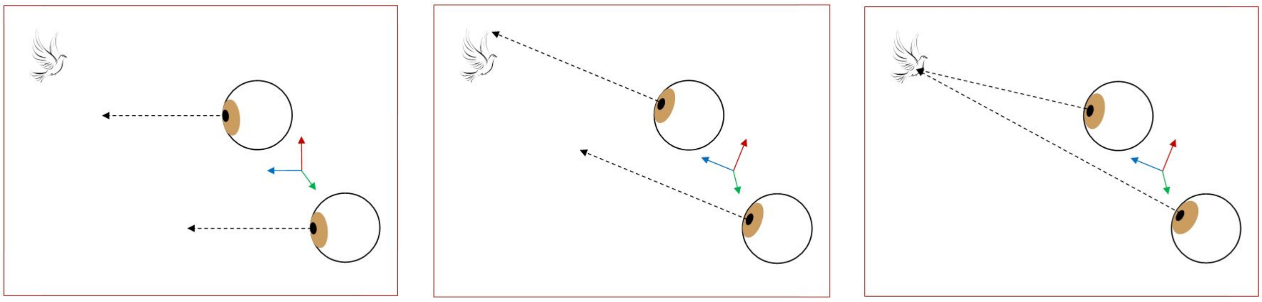
$$g = \sin^2(\phi_L/2)\, d\theta_L^2 + \frac{1}{4} d\phi_L^2. \quad (19)$$

The associated geodesic equation on **LIST** reduces to the following pair of equations, already described in Polpitiya et al. (2007), given by

$$\begin{aligned} \ddot{\theta}_L + \dot{\theta}_L\dot{\phi}_L \cot(\phi_L/2) &= 0, \\ \ddot{\phi}_L - (\dot{\theta}_L)^2 \sin(\phi_L) &= 0. \end{aligned} \quad (20)$$

The above calculation can be easily repeated, and one can compute a Riemannian Metric on **MS** given by

$$g = \sin^2(\phi_M/2)\, d\alpha_M^2 + \frac{1}{4} d\phi_M^2. \quad (21)$$



(a) Initially, the two eyes are gazing in parallel along the primary direction.

(b) The version eye movement rotates the eyes satisfying Listing's Law. The two eyes stay parallel.

(c) The vergence eye movement rotates the eyes in opposite direction satisfying extended Listing's Law. The two eyes focus on a target.

Fig. 1. Movement of two eyes focusing on a target.

The associated geodesic equation on **MS** reduces to the pair of equations given by

$$\begin{aligned} \ddot{\alpha}_M + \dot{\alpha}_M \dot{\phi}_M \cot(\phi_M/2) = 0, \\ \ddot{\phi}_M - (\dot{\alpha}_M)^2 \sin(\phi_M) = 0. \end{aligned} \tag{22}$$

Note that in the equations (19) - (22) the subscripts L and M describe variables in **LIST** and **MS** respectively. Combining the version and the vergence systems, we define a control system given by

$$\begin{aligned} \ddot{\theta}_L &= -\dot{\theta}_L \dot{\phi}_L \cot(\phi_L/2) + \csc^2(\phi_L/2)\tau_{\theta_L}, \\ \ddot{\phi}_L &= \left(\dot{\theta}_L\right)^2 \sin(\phi_L) + 4\tau_{\phi_L}, \\ \ddot{\alpha}_M &= -\dot{\alpha}_M \dot{\phi}_M \cot(\phi_M/2) + \csc^2(\phi_M/2)\tau_{\alpha_M}, \\ \ddot{\phi}_M &= (\dot{\alpha}_M)^2 \sin(\phi_M) + 4\tau_{\phi_M}, \end{aligned} \tag{23}$$

where the τ-s in the right hand side are the generalized torques (see Polpitiya et al. (2007)). If we assume that (23) is the dynamics of the left eye, then the corresponding dynamics of the right eye is given by the set of equations

$$\begin{aligned} \ddot{\theta}_L &= -\dot{\theta}_L \dot{\phi}_L \cot(\phi_L/2) + \csc^2(\phi_L/2)\tau_{\theta_L}, \\ \ddot{\phi}_L &= \left(\dot{\theta}_L\right)^2 \sin(\phi_L) + 4\tau_{\phi_L}, \\ \ddot{\alpha}_M &= -\dot{\alpha}_M \dot{\phi}_M \cot(\phi_M/2) + \csc^2(\phi_M/2)\tau_{\alpha_M}, \\ \ddot{\phi}_M &= (\dot{\alpha}_M)^2 \sin(\phi_M) - 4\tau_{\phi_M}. \end{aligned} \tag{24}$$

Note that the dynamics (23) and (24) differ by one sign in the fourth equation and as a result the variables α_M and ϕ_M are different between the left and the right eyes (although the variables θ_L and ϕ_L are identical between the two eyes).

6.2 Dynamical system in q-parametrization

Using the axis angle parametrization that we have been using so far, the dynamical systems (23), (24) have singularities at $\phi_L = 0$ and $\phi_M = 0$, which correspond to the frontal gaze direction, $(0\ 0\ 1)^T$ for both, the vergence as well as the version. In this section, we write down the version and the vergence dynamics using q-parametrization introduced recently by Kahagalage et al. (2014). The details are described as follows.

Let $q = (q_0\ q_1\ q_2\ q_3)^T$ be a unit quaternion in S^3 and $\omega = (\omega_1\ \omega_2\ \omega_3)^T$ be the angular velocity vector of the eye ball. The rotational dynamics of the eye can be described by

$$\dot{\omega} = \tau(t), \tag{25}$$

where we assume that the eye is a perfect and homogeneous sphere, and where $\tau(t) = (\tau_1\ \tau_2\ \tau_3)^T$ is the external torque vector (which is not the same as the generalized torques in 23, 24). In the inertial coordinate system, the angular velocity vector is related to the unit quaternion (1), describing the orientation of the rigid body as follows:

$$\frac{d}{dt}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \bullet \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{26}$$

where $\bullet$ is the quaternion product. By considering the coordinates $\bar{q} \in \mathbb{R}^3$ in the space of unit quaternions, we define

$$\bar{q}_i = \frac{q_i}{q_0} \tag{27}$$

for $i = 1, 2, 3$, and combine (25) and (26) to obtain

$$\begin{pmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \\ \dot{\bar{q}}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\left(\omega_1 + \bar{q}_3\omega_2 - \bar{q}_2\omega_3 + \omega_1\bar{q}_1^2 + \omega_2\bar{q}_2\bar{q}_1 + \omega_3\bar{q}_1\bar{q}_3\right) \\ -\left(\omega_2 + \bar{q}_1\omega_3 - \bar{q}_3\omega_1 + \omega_1\bar{q}_2\bar{q}_1 + \omega_2\bar{q}_2^2 + \omega_3\bar{q}_2\bar{q}_3\right) \\ -\left(\omega_3 + \bar{q}_2\omega_1 - \bar{q}_1\omega_2 + \omega_1\bar{q}_3\bar{q}_1 + \omega_2\bar{q}_2\bar{q}_3 + \omega_3\bar{q}_3^2\right) \\ 2\tau_1 \\ 2\tau_2 \\ 2\tau_3 \end{pmatrix}. \tag{28}$$

The Listing's constraint in q-parametrization corresponds to $\bar{q}_3 = 0$, and the version-dynamics on LIST is given by

$$\begin{pmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_1\left(1 + \bar{q}_1^2 + \bar{q}_2^2\right) \\ \omega_2\left(1 + \bar{q}_1^2 + \bar{q}_2^2\right) \\ 2\tau_1 \\ 2\tau_2 \end{pmatrix}. \tag{29}$$

The extended Listing's constraint in q-parametrization corresponds to $\bar{q}_1 = 0$, and the vergence-dynamics on MS for left eye is given by

$$\begin{pmatrix} \dot{\bar{q}}_{2L} \\ \dot{\bar{q}}_{3L} \\ \dot{\omega}_{2L} \\ \dot{\omega}_{3L} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_{2L}\left(1 + \bar{q}_{2L}^2 + \bar{q}_{3L}^2\right) \\ \omega_{3L}\left(1 + \bar{q}_{2L}^2 + \bar{q}_{3L}^2\right) \\ 2\tau_{2L} \\ 2\tau_{3L} \end{pmatrix}. \tag{30}$$

The vergence dynamics for right eye can be obtained by reversing the axis of rotation of left eye since the eye pair rotates in opposite directions for the vergence system, and is given by

$$\begin{pmatrix} \dot{\bar{q}}_{2R} \\ \dot{\bar{q}}_{3R} \\ \dot{\omega}_{2R} \\ \dot{\omega}_{3R} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\omega_{2R}\left(1+\bar{q}_{2R}^2+\bar{q}_{3R}^2\right) \\ -\omega_{3R}\left(1+\bar{q}_{2R}^2+\bar{q}_{3R}^2\right) \\ 2\tau_{2R} \\ 2\tau_{3R} \end{pmatrix}. \quad (31)$$

Finally, we describe the the dynamics of the binocular vision by combining (29), (30), and (31).

7. OPTIMALLY CONTROLLING BINOCULAR EYE MOVEMENT

Hering (1868) has stated that the two human eyes share a common innervation and thus cannot be controlled independently [5]. Hering proposed that the primary control for the eyes are identical, resulting in identical movement pattern (version control), except perhaps when the eyes have to focus at a target (vergence control). In this section, we model the two eyes with the same version control system (given by (29)), and the version control to the two eyes are assumed to be identical. In order to focus at a target, the vergence control system is assumed to be (30) and (31) for the left and the right eyes respectively. Note that the two vergence dynamics differ only in sign. The vergence control input to the two dynamical systems are also assumed to be identical, and is imposed as a constraint in the cost function described below.

We would consider a dynamical system with 12 state variables given by (29), (30) and (31). Let us consider the quadratic cost function

$$\int_0^T \left[\frac{1}{2}\,\tau^T\tau + p(t)^T(F - \dot{x}) + \lambda^T(\tau_L - \tau_R) + \frac{1}{2}\varepsilon\dot{\lambda}^T\dot{\lambda}\right] dt \quad (32)$$

where the state vector x is described as

$$x = [\bar{q}_1\ \bar{q}_2\ \omega_1\ \omega_2\ \bar{q}_{2L}\ \bar{q}_{3L}\ \omega_{2L}\ \omega_{3L}\ \bar{q}_{2R}\ \bar{q}_{3R}\ \omega_{2R}\ \omega_{3R}]^T.$$

The first four elements of the state vector are the states of version system, the next four with subscript 'L' are the states of vergence system for the left eye and the last four with subscript 'R' are states of vergence system for right eye. The vector F is a 12-vector containing right hand sides of (29) – (31), and

$$p(t) = [p_1\ p_2\ p_3\ p_4\ p_{1L}\ p_{2L}\ p_{3L}\ p_{4L}\ p_{1R}\ p_{2R}\ p_{3R}\ p_{4R}]^T$$

is a 12-vector of Lagrange multipliers. The vector

$$\tau = [\tau_1\ \tau_2\ \tau_{2L}\ \tau_{3L}\ \tau_{2R}\ \tau_{3R}]^T$$

is the externally applied torque vector to the version and vergence systems of the two eyes. The first two elements of τ are for the version system, the next two with subscript 'L' are for the vergence in the left eye and the last two with subscript 'R' are for the vergence in the right eye. In order to satisfy Hering's law, we impose the constraint that τ_L and τ_R must be equal. This constraint is imposed in the cost function by multiplying with a Lagrange multiplier vector λ. The last term in the cost function is a penalty term which makes λ smooth.

In order to obtain a necessary condition for optimality, we define a Hamiltonian given by

$$H(x,p,\lambda) = \frac{1}{2}\,\tau^T\tau + p(t)^T F + \lambda^T(\tau_L - \tau_R) + \frac{1}{2}\varepsilon\dot{\lambda}^T\dot{\lambda}, \quad (33)$$

and write down Hamilton's equations given by

$$\dot{x} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial x}. \quad (34)$$

[5] It is conceivably harder to synchronize two eyes that are controlled independently.

The constraint equations are given by

$$\frac{\partial H}{\partial \lambda} = 0, \quad (35)$$

which reduces to

$$\ddot{\lambda} = \frac{1}{\varepsilon}(\tau_L - \tau_R).$$

The optimal values of the control are obtained by setting

$$\frac{\partial H}{\partial \tau} = 0.$$

The optimal controls are obtained as follows

$$\tau = [(-p_3,\ -p_4,\ -p_{3L} - \lambda_1,\ -p_{4L} - \lambda_2,\ -p_{3R} + \lambda_1,\ -p_{4R} + \lambda_2)]^T \quad (36)$$

In order to compute the optimal control, we would require to solve (34) and (35) . Since only the initial and final conditions on $x(t)$ are known and we do not have any boundary condition on $p(t)$ and $\lambda(t)$, the optimal control problem we need to solve is a 'two point boundary value problem'.

8. SIMULATION RESULT AND DISCUSSION

For the dynamical system described by (34), (35), the optimal control (36) is computed using COMSOL (see Zimmerman (2006)). The initial and the final condition for the left and the right eye for the versional system is given by $(1,0,0,0)^T$ and $(0.9492,-0.3126,-0.1563,0)^T$ respectively. The initial and the final condition for the left eye, vergence system is given by $(0.9732,0,0.2426,0.0001)^T$ and $(0.9982,0,0.0607,0)^T$ respectively. The initial and the final condition for the right eye, vergence system is given by $(0.9732,0,-0.2426,-0.0001)^T$ and $(0.9982,0,-0.0607,0)^T$ respectively. The time interval is chosen as $[0,1]$. The moment of inertia matrix of the eye is assumed to be the identity matrix.

Fig. 2 displays the behavior of the states and the external torques for the version system. Figs. 3 and 4 display the behavior of the states and the external torques in vergence system for left eye and right eye respectively. The boundary values for each of the angular velocities are set to zero. Note that the torque profiles in Figs. 3 and 4 are identical because they were constrained to be so. The angular velocities ω_L and ω_R are identical because they are initialized at the same values.

The version control system typically moves the eye faster than the vergence system. This feature has not been incorporated in our simulations so far. We alter the speed of the eye movement by adding an extra term to the cost function. This is derived as follows: let us define the gaze direction associated with the version state vector to be

$$\vartheta(t) = \left[2\bar{q}_2,\ -2\bar{q}_1,\ 1 - \bar{q}_1^2 - \bar{q}_2^2\right]^T.$$

We normalize this vector by writing $\bar{\vartheta} = \frac{\vartheta}{\|\vartheta\|}$. Let $\bar{\vartheta}^*$ be the final gaze direction of the version control system, corresponding to the target position. We define

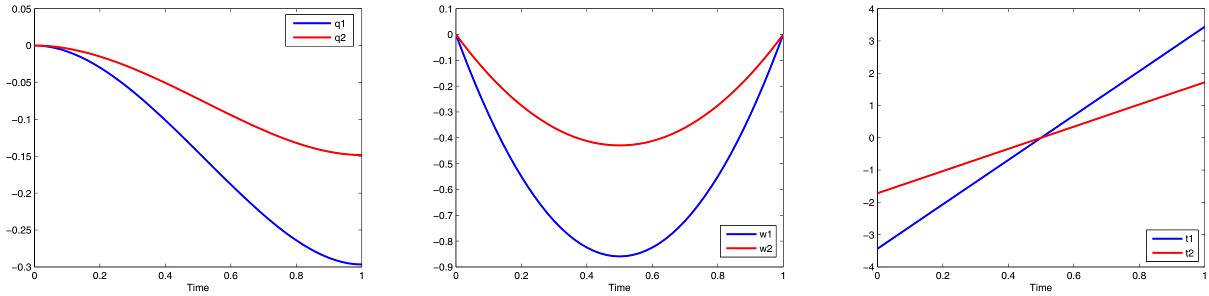
$$C(t) = 1 - \bar{\vartheta}^{*T}\bar{\vartheta}(t).$$

The cost function (32) is modified as

$$\int_0^T \left[\frac{1}{2}\,\tau^T\tau + \frac{\beta}{2}\,C^T C + p(t)^T(F - \dot{x}) + \lambda^T(\tau_L - \tau_R) + \frac{1}{2}\varepsilon\dot{\lambda}^T\dot{\lambda}\right] dt. \quad (37)$$

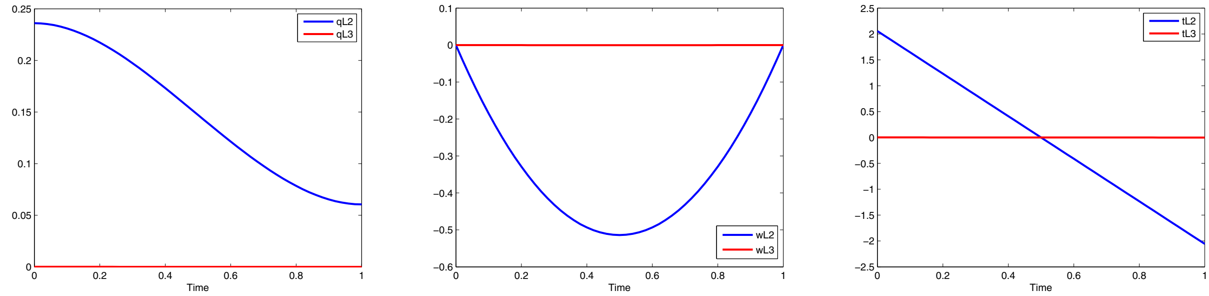
By choosing higher values of β in (37), the version system can be speeded up.

In Fig. 5a, the gaze directions of the left and the right eyes are shown. In this simulation, β is assumed to be zero, i.e. the



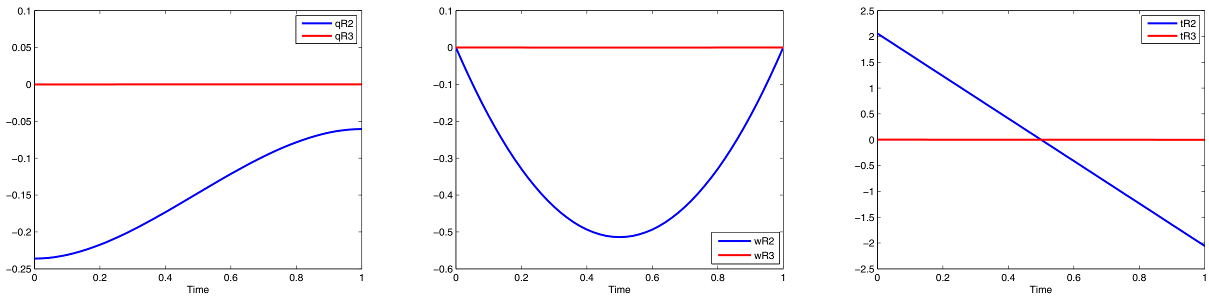
(a) States of $\bar{q}$.　(b) States of ω.　(c) Torques.

Fig. 2. States for $\bar{q}$ and ω, and τ in version system.



(a) States of $\bar{q}_L$.　(b) States of ω_L.　(c) Torques.

Fig. 3. States for $\bar{q}_L$ and ω_L, and τ_L for left eye in vergence system.



(a) States of $\bar{q}_R$.　(b) States of ω_R.　(c) Torques.

Fig. 4. States for $\bar{q}_R$ and ω_R, and τ_R for right eye in vergence system.

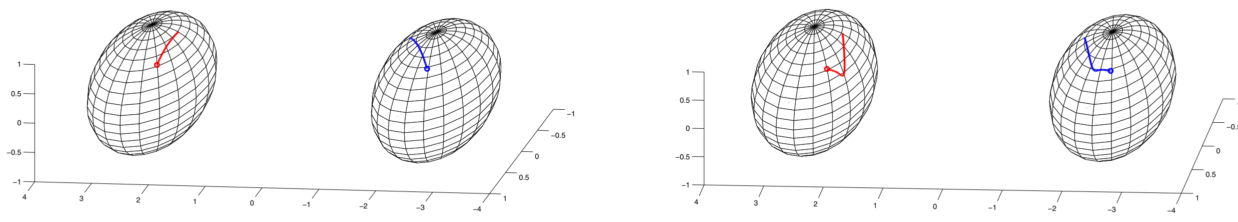
speed of version and vergence were chosen to be comparable. In Fig. 5b, a large value of β speeds up (by approximately 4 times) the version compared to vergence. The gaze trajectories, shown in the figure, sharply turns when the influence of version control gives way to the influence of vergence control. Note from the simulation that the effect of vergence control is to move the two eyes in opposite directions.

9. CONCLUSION

Binocular control problem has been modeled using two separate control systems, consistent with the framework originally proposed by Hering. Each of the two systems evolve in appropriate sub-manifolds of S^3 and we have described them as Lagrangian systems. We have also described these systems using a recently proposed q-parametrization. An optimal control problem has been formulated and simulation results from the associated two point boundary value problem has been displayed, using the q-parametrization.

The optimal control, we derive, follows Hering's principle of equal innervation to both the eyes, except that the effect of vergence control is such that the eyes move in opposite directions. The control, we derive, are not necessarily asymptotically stabilizing. In other words, deviated away from LIST and MS, the states are not necessarily able to recapture the associated sub-manifolds.

As a final remark, two different parametrizations have been introduced to describe binocular control. Axis Angle parameters, enable us to describe the control systems as a Lagrangian system. Using the q-parametrization, on the other hand, the dynamical systems are written using the Newton-Euler method (see Junkins and Turner (1986)). For a Lagrangian system, one can easily construct potential controllers (see Wijayasinghe et al.



(a) Equal speed for version and vergence. (b) Version is 4 times faster than vergence.

Fig. 5. The gaze trajectories of eye pair under the influence of version and vergence. North pole is the front gaze.

(2014)), and asymptotically stabilizing controllers (see Ghosh et al. (2014)), topics that have not been covered in this paper. Unfortunately, using the axis angle parameters, the dynamical systems (23), (24), have singularities along the frontal gaze direction. Using the q-parametrization, the systems (29) - (31) do not have singularities [6], but one cannot avail the potential controller synthesis methods available for Lagrangian systems.

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[6] Hence q-parametrization is particularly suitable for the optimal control problem, studied in this paper.

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