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Invariant Manifold Approach for Time-Varying Extremum-Seeking Control Problem ⋆

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Abstract: In this paper, the minimization of an unknown but measurable cost function using extremum-seeking control is considered. The main contribution of the paper is to formulate the extremum-seeking problem as a time-varying estimation problem. The concepts of invariant manifold and adaptive parameter update law are used for adaptive estimation of the time-varying gradient of the unknown cost function. The proposed approach is shown to avoid the need for averaging analysis which minimizes the sensitivity of the closed-loop performance to the choice of dither signal.

Keywords: Extremum-seeking control; Parameter estimation; Invariant manifold.

1. INTRODUCTION

Optimization has a significant rule in control applications due to the increasing needs of products with specific qualities. The optimization techniques can be used to achieve optimal plant operating condition and to reduce the final cost. In many optimization problems, off-line methods are applied to reach the optimal conditions of the plant. However, most of the systems in the real world applications are subject to different initial conditions, dynamics uncertainties and external disturbances. Under these conditions, the off-line approaches cannot guarantee the acceptable performance of the system and real time optimization is crucial for superior performance (Guay and Adetola [2013]). Extremum seeking control (ESC) is an adaptive control method that is design to solve real-time optimization problems.

The ESC is a classical control approach that is used to steer a control system to the optimum of a measured objective function of interest (Tan et al. [2010]). The first precise stability proof of feedback ESC, based on the averaging and singular perturbations techniques has been provided in Krstic and Wang [2000]. Over the last few years, many researchers have considered various approaches to overcome the limitations of ESC. In Krstic [2000], some of these conditions were removed by using dynamic compensators, while the measurement noise rejection was also achieved. The non-local and semi-global stability analysis of ESC is established in Tan et al. [2006] and Tan et al. [2009]. The main contribution of these works was to find explicit expressions for the domain of attraction in the closed-loop system.

As highlighted in Krstic and Wang [2000], the stability analysis relies on a time-scale separation between the fast transients of the system dynamics and the slow quasi steady-state condition. This analysis demonstrates that the amplitude and frequency of the dither signal, must be chosen very carefully to guarantee convergence to a neighbourhood of the unknown optimum. More precise statements concerning the dependence of the stability properties on the choice of different dither signals is provided by Tan et al. [2008].

In Gelbert et al. [2012], extended Kalman filters are used instead of the low and high pass filters to estimate the gradient of the input to reference map. The main advantages of this nonlinear filter are the faster response and the extension of the algorithm to more than one input. On the other hand, the gain of the gradient update cannot be adjusted freely since the convergence of the ESC depends largely on the magnitude of the unknown Hessian of the steady-state measured output. A Newton-based approach is reported in Ghaffari et al. [2012], which provides an estimate for the inverse of the Hessian matrix of the unknown cost function. This technique can effectively alleviate the convergence problems associated with the gradient-based approach. The recent work of Nesic et al. [2013] is another example of non-gradient approach to the ESC problem.

In Guay and Zhang [2003], an adaptive ESC algorithm is considered for a system with a known objective function that depends on the system states and uncertain plant parameters. The proposed adaptive extremum seeking control has lots of applications in (bio)chemical processes (see Dochain et al. [2011], and the references therein).

In this paper, we provide an alternative extremum-seeking technique which is based on the estimation of the gradient as a time-varying parameter. The estimation scheme is based on the geometric concepts of invariant manifolds. For this purpose, a number of high gain estimators and filters are provided and an almost invariant manifold is generated from the filters. This allows to exploit an implicit function relating the known variables and the unknown variables. Then a parameter update law is presented using the almost invariant manifold and an adaptive estimator.

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The main contribution of this work is to remove the need for averaging of the quasi steady-state system, a crucial element of conventional ESC approach. It also avoids the need to use the frequency of the dither signal as a singular perturbation parameter. The proposed ESC algorithm can improve the transient performance of the closed-loop system.

The paper is organized as follows. The problem description is given in Section 2. In Section 3, the proposed ESC controller is presented for the case of process described by a static map. The extension of the algorithm to an unknown dynamical system is presented in Section 4. This is followed by a simulation example in Section 5 and a brief conclusion in Section 6.

2. PROBLEM DESCRIPTION

Consider a nonlinear system

$$\dot{x} = f(x, u) \tag{1}$$

$$y = h(x) \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, u is the control input taking values in $\mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathbb{R}$ is the unknown and measurable cost function to be minimized. The functions $f(x, u)$ and $h(x)$ are assumed to be C^∞ in all of their arguments.

The objective of ESC is to bring the closed-loop control system to the unknown equilibrium x^* and u^* that minimizes the cost function y. The equilibrium (or steady-state) map is the n dimensional vector-valued function $\pi(u)$ which is such that $f(\pi(u), u) = 0$. The steady-state cost function is given by $y = h(\pi(u)) = \ell(u)$. Thus, at steady-state, the problem is reduced to finding the minimizer u^* of the $y = \ell(u)$. The following assumptions are required.

Assumption 1. The equilibrium cost is such that

$$\frac{\partial \ell(u^*)}{\partial u} = 0, \quad \frac{\partial^2 \ell(u)}{\partial u \partial u^T} > \alpha_1 I \quad \forall u \in \mathcal{U}$$

$$(u - u^*)^T \frac{\partial \ell(u)}{\partial u} \geq \alpha_2 \left\| u - u^* \right\|^2, \quad \forall u \in \mathcal{U}$$

where matrix I is an identity matrix with suitable dimension, and α_1, α_2 are strictly positive constants.

Assumption 2. The steady-state map is such that

$$\|y\| \leq Y, \; \left\| \tfrac{\partial \ell(u)}{\partial u} \right\| \leq L_1, \; \left\| \tfrac{\partial^2 \ell(u)}{\partial u \partial u^T} \right\| \leq L_2$$

$\forall u \in \mathcal{U}$ with positive constants Y, L_1 and L_2.

3. STATIC MAP

Since the minimization of y is performed in real-time, the input u is taken as a time-varying signal. That is,

$$y(t) = \ell(u(t)) \tag{3}$$

Assumption 3. The input signal $u(t)$ is such that $u(t) \in \mathcal{U}$ and $\|\dot{u}(t)\| \leq \alpha_3$, $\forall t \geq t_0 \geq 0$.

The differentiation of (3) with respect to time results in $\dot{y} = (\frac{\partial \ell(u)}{\partial u})^T \dot{u}$. Following the time-varying approach in Guay et al. [2013], we define the unknown gradient and the cost dynamic as

$$\theta(t) = \frac{\partial \ell(u)}{\partial u}, \quad \dot{y} = \theta(t)^T \dot{u}(t) \tag{4}$$

If one has access to the gradient $\theta(t)$, then it follows that the gradient descent $\dot{u} = -k_1 \theta(t)$ with $k_1 > 0$, will converge the optimum u^*.

Lemma 1. Consider the cost function subject to the all assumptions. The gradient descent update $\dot{u} = -k_1 \theta(t)$ is such that the objective function decreases monotonically and reaches to its minimum at u^*.

Proof. By convexity of the cost, it is shown in the following that the cost will decrease until a value of u is reached such that the gradient of the cost be zero. Let $\tilde{u} = u - u^*$ and consider the following Lyapunov function candidate for the input dynamics:

$$V_{\tilde{u}} = \tfrac{1}{2} \tilde{u}^T \tilde{u}$$

By considering Assumption 1, and differentiation with respect to t, the following inequality is obtained

$$\dot{V}_{\tilde{u}} = -k_1 \tilde{u}^T \tfrac{\partial \ell(u)}{\partial u} \leq -k_1 \alpha_2 \left\| \tilde{u}(t) \right\|^2$$

As a result, the system converges to the unknown minimizer u^*.

The design of the extremum seeking scheme is based on the unknown dynamics (4). The first step consists in the estimation of the time-varying parameter $\theta(t)$. In the second step, we define a suitable adaptive controller that accomplishes the extremum seeking task.

3.1 Parameter Estimation

In this section, an alternative technique based on the idea of invariant manifolds is proposed for adaptive estimation of the unknown time-varying parameters. The geometric concept of invariance has been widely used in nonlinear control theory (see Tian and Yu [2000], Astolfi and Ortega [2003]). This algorithm utilizes a number of high gain estimators and filters. An almost invariant manifold is generated from the filters, which allows to exploit implicit functions relating implicitly the known variables and the unknown variables. A parameter update law is assigned using the almost invariant manifold.

Invariant Manifold Design: The basic idea is to find a mapping from known variables to the unknown variables, which has an almost invariance property for sufficiently large value of an assignable design gain. This implicit mapping is used to facilitate the estimation the unknown parameters.

The estimator model for (4) is defined as

$$\dot{\hat{y}} = -k^2(\hat{y} - y), \quad k > 0 \tag{5}$$

and the filter is described along with the structure of the system by

$$\dot{\phi} = -k^2(\phi - \dot{u}) \tag{6}$$

Assumption 4. The vector-valued function $\phi(t)$ is bounded with a known bound as $\|\phi\| \leq \lambda$, $\forall t \geq t_0 \geq 0$.

Proposition 2. Consider the estimator (5) and filter (6), the following implicit manifold

$$\lim_{k \to \infty} [k^2(\hat{y} - y) + \phi^T \theta(t)] = 0 \tag{7}$$

is such that the manifold is invariant and internally exponentially stable. Furthermore, the desired manifold is bounded and almost invariant for bounded and sufficiently large values of k.

Proof. First, it is shown that the manifold (7) has an invariance feature. The off-the-manifold coordinate variable $z(t)$ is defined as

$$z(t) = k^2(\hat{y} - y) + \phi^T\theta(t). \quad (8)$$

The derivative of $z(t)$ along the trajectories of the system is given by

$$\dot{z}(t) = k^2(\dot{\hat{y}} - \dot{y}) + \dot{\phi}^T\theta(t) + \phi^T\dot{\theta}(t). \quad (9)$$

It follows from (4)-(6) that

$$\dot{z}(t) = -k^2 z(t) + \phi^T\dot{\theta}(t) = -k^2(z(t) - \frac{\phi^T\dot{\theta}(t)}{k^2}). \quad (10)$$

By the fact that $\dot{\theta} = \frac{\partial^2 \ell(u)}{\partial u \partial u^T}\dot{u}$, and in view of Assumptions 2 and 3, it follows that $\left\|\dot{\theta}\right\| \leq L_2\alpha_3 = L_\theta$, so $\frac{\phi^T\dot{\theta}(t)}{k^2} \to 0$ as $k \to \infty$. Suppose there exists a time t_c such that $z(t_c) = 0$, then

$$z(t_c) = 0 \Rightarrow \dot{z}(t) \equiv 0 \Rightarrow z(t) \equiv 0, \quad \forall t \geq t_c$$

which shows that $z(t) = 0$ is an invariant manifold for nonlinear dynamics (4)-(6) as $k \to \infty$, and $\dot{z}(t) = -k^2 z(t)$ is globally exponentially stable.

For the case where k is bounded, a quadratic Lyapunov function can be defined as

$$V_{z(t)} = \frac{1}{2}z^2(t) \quad (11)$$

The first time derivative of $V_{z(t)}$ along its trajectories, result in

$$\dot{V}_{z(t)} = -k^2 z^2(t) + z(t)\phi^T\dot{\theta}(t)$$

As a result of Assumption 4, and applying Young's Inequality, one obtains the following inequality:

$$\begin{aligned} \dot{V}_{z(t)} &\leq -k^2 z^2(t) + \frac{\lambda k}{2}z^2(t) + \frac{\lambda}{2k}\dot{\theta}^T(t)\dot{\theta}(t) \\ &\leq -\left(k^2 - \frac{\lambda k}{2}\right)V_{z(t)} + \frac{\lambda}{2k}L_\theta^2 \end{aligned} \quad (12)$$

where $k \gg \frac{\lambda}{2}$. Hence, the manifold normal coordinate variables $z(t)$ enter a small neighborhood of the origin. The size of this neighborhood depends on the choice of the high gain k. Therefore, an almost invariant manifold is obtained by considering sufficiently large value for k.

Remark 1. It will be shown that the parameter estimation can be achieved by considering the proposed almost invariant manifold. Thus, unlike the sliding mode-based techniques, satisfactory performance can be attained without requiring that the system states reach the reference manifold.

Adaptive Estimation: The invariant manifold (7) can be re-written as

$$k^2(\hat{y} - y) = -\phi^T\theta(t). \quad (13)$$

This equation provides an implicit relationship from known variables (ϕ, $\hat{y}$, y) to the unknown variable $\theta(t)$. This mapping provides direct information about the parameter estimation error without requiring a priori knowledge of the time-varying parameters. This is achieved by defining the auxiliary variables p and q with the dynamics

$$\begin{aligned} \dot{p} &= -kp - \phi\phi^T\hat{\theta}(t) \\ \dot{q} &= -kq + \phi(k^2(\hat{y} - y)) \end{aligned} \quad (14)$$

An adaptive estimator is considered as

$$\dot{\Sigma}(t) = k[bI - \phi\phi^T(\Sigma(t))], \quad \Sigma(t_0) = bI \quad (15)$$

Based on (14) and (15), the proposed parameter update law is given by

$$\dot{\hat{\theta}}(t) = k^2\Sigma(t)[\dot{\delta} + k\delta] \quad (16)$$

where $b > 1$ and $\delta = p - q$.

Assumption 5. There exists constants $\alpha > 0$ and $T > 0$ such that

$$\int_t^{t+T} \phi(\tau)\phi^T(\tau)d\tau \geq \alpha I, \quad \forall t > t_0. \quad (17)$$

The condition of the Assumption 5 is equivalent to the standard persistent excitation (PE) condition required for parameter convergence.

3.2 Controller Design

The proposed extremum seeking control is given by

$$\dot{u} = -k_1\hat{\theta}(t) + d(t) \quad (18)$$

where $d(t)$ is a bounded dither signal with $\|d(t)\| \leq D$. The dither signal $d(t)$ can be chosen arbitrary, and it only need to satisfy the PE condition of the Assumption 5. A one proper choice of $d(t)$ signals are sinusoidal waves, because of their orthogonality feature (Tan et al. [2008]). Although, in our algorithm convergence to the optimal value is not sensitive to the choice of amplitude and frequency of the sinusoidal signal. Note that, since $\hat{\theta}(t)$ and $d(t)$ are assumed to be bounded then the controller (18) is such that $\|\dot{u}\| \leq k_1 L_1 + D \leq \alpha_3$.

Theorem 3. Let Assumptions 1 to 5 hold. Then the parameter update law (16) and the control law (18) are such that the closed-loop extremum seeking control system converges to a neighborhood of the minimizer u^* of the static nonlinear optimization problem (4). The size of this neighborhood is adjustable by the gains k and k_1.

Proof. By construction of parameter estimation error as $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$, the parameter update law (16) can be approximated implicitly in the form

$$\dot{\hat{\theta}}(t) = k^2\Sigma(t)\phi\phi^T\tilde{\theta}(t). \quad (19)$$

The quadratic Lyapunov function is defined as

$$V_{(\tilde{\theta},\tilde{u})} = \frac{1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + V_{\tilde{u}} \quad (20)$$

By differentiating of (20) along (19) and (18), we have

$$\begin{aligned} \dot{V}_{(\tilde{\theta},\tilde{u})} &= -k^2\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}(t) + \tilde{\theta}^T(t)\dot{\theta}(t) \\ &\quad -k_1\tilde{u}^T\theta(t) + k_1\tilde{u}^T\tilde{\theta}(t) + \tilde{u}^T d(t) \end{aligned}$$

By Assumption 1, one can write the following inequality

$$\begin{aligned} \dot{V}_{(\tilde{\theta},\tilde{u})} &\leq -k^2\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}(t) + \tilde{\theta}^T(t)\dot{\theta}(t) \\ &\quad -k_1\alpha_2\tilde{u}^T\tilde{u} + k_1\tilde{u}^T\tilde{\theta}(t) + \tilde{u}^T d(t) \end{aligned}$$

Applying Young's inequality to all indefinite terms of the last inequality, there exists a positive constant k_2 such that

$$\begin{aligned} \dot{V}_{(\tilde{\theta},\tilde{u})} &\leq -k^2\tilde{\theta}^T(t)(\Sigma(t)\phi\phi^T)\tilde{\theta}(t) + \frac{k}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) \\ &+\frac{1}{2k}\dot{\theta}^T(t)\dot{\theta}(t) - k_1\alpha_2\tilde{u}^T\tilde{u} + \frac{kk_1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) \\ &+\frac{k_1}{2k}\tilde{u}^T\tilde{u} + \frac{k_2}{2}\tilde{u}^T\tilde{u} + \frac{1}{2k_2}d^T(t)d(t) \end{aligned}$$

With collecting the similar terms, one can rewrite the above inequality as follows

$$\dot{V}_{(\tilde{\theta},\tilde{u})} \leq -\left(k^2 \left\|\Sigma(t)\phi\phi^T\right\| - \frac{kk_1+k}{2}\right)\tilde{\theta}^T(t)\tilde{\theta}(t) - \left(k_1\alpha_2 - \frac{k_1+kk_2}{2k}\right)\tilde{u}^T\tilde{u} + \frac{1}{2k}\dot{\theta}^T(t)\dot{\theta}(t) + \frac{1}{2k_2}d^T(t)d(t)$$

Next we claim the boundedness of the matrix $\Sigma(t)$ as follows. By integration, one gets

$$\begin{aligned}&\Sigma(t) = \exp\left[\int_{t_0}^{t} -k\phi(\tau)\phi^T(\tau)d\tau\right]\Sigma(t_0)\\&+kb\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k\phi(\xi)\phi^T(\xi)d\xi\right]d\tau >\\&kb\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k\phi(\xi)\phi^T(\xi)d\xi\right]d\tau \geq\\&\left(kb\int_{t_0}^{t} e^{-k\beta(t-\tau)}d\tau\right)I = \frac{b(1-e^{-k\beta(t-t_0)})}{\beta}I\end{aligned} \tag{21}$$

where the last inequality is achieved from the boundedness of matrix $\phi\phi^T$. As a result of Assumption 5, we obtain

$$\begin{aligned}&\Sigma(t) \leq \Sigma(t_0) + kb\int_{t_0}^{t}\exp\left[\int_{\tau}^{t} -k\phi(\xi)\phi^T(\xi)d\xi\right]d\tau \leq\\&\left(b + kb\int_{t_0}^{t} e^{-k\alpha(t-\tau)}d\tau\right)I \leq \frac{b\alpha + b(1-e^{-k\alpha(t-t_0)})}{\alpha}I\end{aligned} \tag{22}$$

It follows from (21) and (22) that

$$\begin{aligned}&\frac{b(1-e^{-k\beta(t-t_0)})}{\beta}\phi\phi^T \leq \Sigma(t)\phi\phi^T\\&\leq \frac{b\alpha + b(1-e^{-k\alpha(t-t_0)})}{\alpha}\phi\phi^T\end{aligned} \tag{23}$$

and

$$\begin{aligned}&b(1-e^{-k\beta(t-t_0)}) \leq \left\|\Sigma(t)\phi\phi^T\right\|\\&\leq \frac{b\alpha + b(1-e^{-k\alpha(t-t_0)})}{\alpha}\beta.\end{aligned} \tag{24}$$

Consider the function $\dot{V}_{(\tilde{\theta},\tilde{u})}$ and inequality (24), the gain should be chosen such that

$$\begin{aligned}&(1-e^{-k\beta(t-t_0)})k \geq \frac{k_1+1}{2}\\&k_1\alpha_2 - \frac{k_1+kk_2}{2k} > 0\end{aligned}$$

or equivalently

$$\begin{aligned}&k\beta(t-t_0) = k\beta\sum_{i=1}^{N}(t_i - t_{i-1}) \geq \ln(\frac{k_1+1}{2k})\\&k_1 > \frac{kk_2}{2k\alpha_2 - 1}\end{aligned}$$

Based on the last inequality, $k\beta NT' \geq \ln(k_1 + 1/2k)$, where T' is the sampling time and N is an integer. If the estimation gain is chosen large enough as $k \geq \frac{\ln(k_1+1/2k)}{\beta T'}$, then $(1-e^{-k\beta(t-t_0)})k \geq \frac{k_1+1}{2}$, $\forall t > t_0 \geq 0$. For the given k, there exist strictly positive constants k_a, k_b and $k' = \min\{k_a, k_b\}$ such that

$$\begin{aligned}\dot{V}_{(\tilde{\theta},\tilde{u})} &\leq -k_a\tilde{\theta}^T(t)\tilde{\theta}(t) - k_b\tilde{u}^T\tilde{u} + \frac{1}{2k}L_\theta^2 + \frac{1}{2k_2}D^2\\&\leq -k'V_{(\tilde{\theta},\tilde{u})} + \frac{1}{2k}L_\theta^2 + \frac{1}{2k_2}D^2\end{aligned} \tag{25}$$

It follows that $\tilde{\theta}$ and $\tilde{u}$ converge exponentially to a small neighborhood of the origin. The size of this neighborhood depends on the choice of gains k, k_1 and the magnitude of the dither signal. This completes the proof of Theorem 3.

Remark 2. By construction, one can decrease the contribution from $d(t)$ and $\dot{\theta}(t)$ by increasing the optimization gain k_1 and the estimation gain k, respectively. Since, the rate of change of the parameter is proportional to the optimization gain, $\left\|\dot{\theta}(t)\right\| \leq L_2\left(k_1\left\|\hat{\theta}(t)\right\| + D\right)$, the convergence to the small neighborhood is achieved by ensuring that $k \gg k_1$.

4. OPTIMIZATION IN DYNAMICAL SYSTEMS

In this section, we consider the initial extremum control system which consists in steering the unknown dynamical system (1) to the equilibrium that minimizes the measure cost function (2). The closed-loop extremum seeking control system is given by (Guay et al. [2013]):

$$\begin{aligned}\epsilon_1\dot{x} &= f(x,u)\\\dot{u} &= -k_1\hat{\theta}(t) + d(t)\\\dot{\hat{\theta}}(t) &= k^2\Sigma(t)[\dot{\delta} + k\delta]\\\dot{\hat{y}} &= -k^2(\hat{y} - y)\\\dot{\phi} &= -k^2(\phi - \dot{u})\\\dot{\Sigma}(t) &= k[bI - \phi\phi^T(\Sigma(t))]\end{aligned} \tag{26}$$

As in other works on extremum-seeking control, the closed-loop dynamics of the system are written in error form in a two time-scale system where t is the slow time-scale and the system's dynamics are assumed to evolve over a fast time-scale $\tau = \frac{t}{\epsilon_1}$. The parameter $\epsilon_1 > 0$ is a small strictly positive parameter to be assigned.

Let us define the deviation variables $\tilde{x} = x - \pi(u)$ and $\tilde{u} = u - u^*$ where u^* is the local minimizer of the steady-state map $y = \ell(u)$. The auxiliary variable is defined as above. However, one must take into account the measurement of the cost function over the fast and the slow time-scale, as $h(\tilde{x} + \pi(\tilde{u} + u^*))$. Therefore, new dynamics are defined for the off-the-manifold coordinate variable $z(t)$, as

$$\begin{aligned}\dot{z} = k^2&\left(\dot{\hat{y}} - \left(\frac{1}{\epsilon_1}\frac{\partial h(x)}{\partial x}f(x,u) + \frac{\partial h(x)}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u}\right)\right)\\&+\dot{\phi}^T\theta(t) + \phi^T\dot{\theta}(t)\end{aligned} \tag{27}$$

Substituting for $\dot{\hat{y}}$ and $\dot{\phi}$, one obtains

$$\begin{aligned}\dot{z}(t) &= -k^2 z(t) + k^2\left(\frac{\partial h(\pi(u))}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u}\right)\\&-k^2\left(\frac{1}{\epsilon_1}\frac{\partial h(x)}{\partial x}f(x,u) + \frac{\partial h(x)}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u}\right) + \phi^T\dot{\theta}(t)\end{aligned} \tag{28}$$

As a result, the $z(t)$ dynamic is affected by the fast and slow dynamics. One can write the system (26) and implicit manifold (28) in deviation form as follows

$$\begin{aligned}\epsilon_1\dot{\tilde{x}} &= f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*) - \epsilon_1\frac{\partial \pi(u)}{\partial u}\dot{\tilde{u}}\\\epsilon_1\dot{z}(t) &= -k^2\frac{\partial h(x)}{\partial x}f(x,u) - k^2\epsilon_1 z(t) + \epsilon_1\phi^T\dot{\theta}(t)\\&\quad+k^2\epsilon_1\left(\frac{\partial h(\pi(u))}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u} - \frac{\partial h(x)}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u}\right)\\\dot{\tilde{u}} &= -k_1\hat{\theta}(t) + d(t)\\\dot{\tilde{\theta}}(t) &= -k^2\Sigma(t)[\dot{\delta} + k\delta] + \dot{\theta}(t)\end{aligned} \tag{29}$$

It is assumed that the gain k is such that

$$k^2 = k_s^2 + \frac{1}{\epsilon_1}k_f^2$$

This definition of k indicates that the predictor equation must be operated as a low-pass in the fast time-scale, by fixing k_f to be large. The parameter estimation routine operates following the slower time-scale with gain k_s. Following the standard singular perturbation technique, the reduced system ($\tilde{x} = \pi(u)$) is given by

$$\begin{aligned} \dot{z} &= -k^2 z(t) + \phi^T \dot{\theta}(t) \\ \dot{\tilde{u}} &= -k_1 \hat{\theta}(t) + d(t) \\ \dot{\tilde{\theta}}(t) &= -k^2 \Sigma(t)[\dot{\delta} + k\delta] + \dot{\theta}(t) \end{aligned} \tag{30}$$

The boundary layer system is given by

$$\begin{aligned} \frac{d\tilde{x}}{d\tau} &= f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*) \\ \frac{dz}{d\tau} &= -k_f^2 z - k^2 \frac{\partial h(\tilde{x} + \pi(\tilde{u} + u^*))}{\partial x} \\ & f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*) + k_f^2 (\frac{\partial h(\pi(\tilde{u} + u^*))}{\partial x} \\ & \frac{\partial \pi(\tilde{u} + u^*)}{\partial u} \dot{\tilde{u}} - \frac{\partial h(\tilde{x} + \pi(\tilde{u} + u^*))}{\partial x} \frac{\partial \pi(\tilde{u} + u^*)}{\partial u} \dot{\tilde{u}}) \end{aligned} \tag{31}$$

Assumption 6. The origin of the nonlinear system (1) is locally exponentially stable $\forall u \in \mathcal{U}$. Let $\chi_r = \{\tilde{x} \in \mathbb{R}^n | \, \|\tilde{x}\| \leq r\}$ for $r > 0$, a positive constant. Similarly, let $\mathcal{E}_r = \{z \in \mathbb{R} \, | \, |z| \leq r\}$.

Assumption 7. The vector field $f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*)$ is such that

$$\|f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*)\| \leq L_f \|\tilde{x}\|$$

$\forall \tilde{x} \in \chi$ and $\forall u \in \mathcal{U}$ where $L_f > 0$ is a positive constant.

Assumption 8. The cost function $h(x)$ is such that

$$\left\| \frac{\partial h(\tilde{x} + \pi(\tilde{u} + u^*))}{\partial x} \right\| \leq L_h$$

$$\left\| \frac{\partial h(\tilde{x} + \pi(u))}{\partial x} - \frac{\partial h(\tilde{y} + \pi(u))}{\partial x} \right\| \leq L_h \|\tilde{x} - \tilde{y}\|$$

$\forall \tilde{x}, \, \tilde{y} \in \chi$ and $\forall u \in \mathcal{U}$ where $L_h > 0$ is a positive constant.

Finally, we make the following assumption concerning the steady-state map, $\pi(u)$.

Assumption 9. The steady-state map $\pi(u)$ is such that

$$\left\| \frac{\partial \pi(u)}{\partial u} \right\| \leq L_\pi$$

$\forall u \in \mathcal{U}$ where $L_\pi > 0$ is a positive constant.

By Assumptions 7, 8 and 9, it follows that there exists a k_f such that the origin of the boundary layer (31) is locally exponentially stable. It then follows that there exists a Lyapunov function and positive constants, β_i for $i = 1, ..., 6$ such that

$$\begin{aligned} \beta_1(\|\tilde{x}\|^2 + |z|^2) &\leq V(\tilde{x}, z) \leq \beta_2(\|\tilde{x}\|^2 + |z|^2) \\ \frac{dV}{d\tau} &= \frac{dV}{d\tilde{x}} \frac{d\tilde{x}}{d\tau} + \frac{dV}{dz} \frac{dz}{d\tau} \leq -\beta_3 \|\tilde{x}\|^2 - \beta_4 |z|^2 \\ & \left\| \left[\frac{dV}{d\tilde{x}}, \frac{dV}{dz} \right] \right\| \leq \beta_5 \|\tilde{x}\|^2 + \beta_6 |z|^2 \end{aligned}$$

$\forall u \in \mathcal{U}, \tilde{x} \in \chi_r$.

Theorem 4. Consider the nonlinear system (1) and the cost function (2). Let Assumptions 1 to 9 be fulfilled then the time-varying parameter estimation scheme and the extremum-seeking controller (18) is such that for every $\epsilon_1 \in (0, \epsilon^*)$, the closed-loop system converges to a neighborhood of the unknown local minimizer of the nonlinear optimization problem. The size of the neighborhood depends on the choice of the gains k, k_1 and the magnitude of the dither signal $d(t)$.

The proof is given in Appendix A.

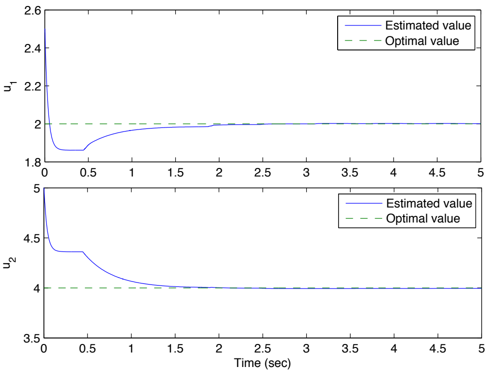

Fig. 1. Trajectories of the control inputs

5. SIMULATION EXAMPLE

The unknown static input-output map is given by Ghaffari et al. [2012] as

$$y = 100 + \frac{1}{2} \left(u - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)^T \begin{bmatrix} 100 & 30 \\ 30 & 20 \end{bmatrix} \left(u - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$$

The objective is to minimize the output y with respect to input u. We apply the proposed extremum-seeking control algorithm with $b = 1.1$, $k = 10\sqrt{10}$, $k_1 = 0.3$, and arbitrary dither signal

$$d(t) = 0.002 \begin{bmatrix} \sin(10t) \\ \sin(20t) \end{bmatrix}.$$

The initial conditions are chosen as $u(0) = [2.5, 5]^T$ and $\hat{\theta}(0) = [10, 10]^T$, where $\hat{\theta}(t)$ is estimation of the unknown gradient.

The simulation results are shown in Figs. 1-3. The extremum-seeking control input trajectories re shown in Fig. 1. The changes of the unknown cost function y is depicted in Fig. 2. The results demonstrate that the proposed extremum-seeking control algorithm provides a rapid progression to the unknown minimizer of the optimization problem. In addition, the control system provides satisfactory transient behaviour for both the inputs and the objective function.

The normal manifold coordinate variable $z(t)$ is shown in Fig. 3. As confirmed in this figure, the implicit manifold $z(t)$ converges to the small neighbourhood of zero. It is important to point out that the value of this variable is not known.

6. CONCLUSION

In this paper, an alternative time-varying ESC technique was proposed. The technique is based on the time-varying estimation of the unknown gradient which relies on definition of an invariant manifold principle. The ESC algorithm is shown to provide local exponential convergence of the closed-loop system to the unknown optimum. The

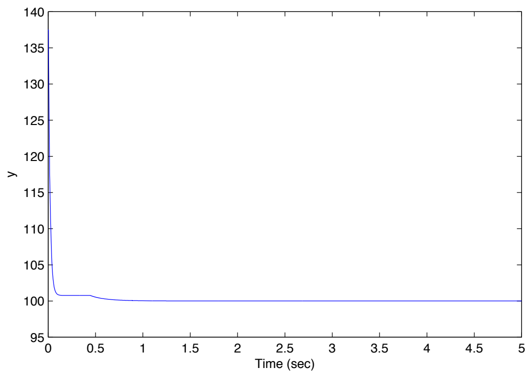

Fig. 2. The unknown cost function versus time

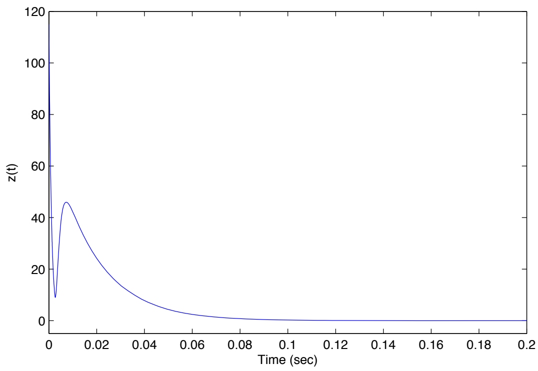

Fig. 3. Trajectory of the implicit manifold

technique simplifies the tuning of the designed gains by avoiding the limitations associated with choice of dither.

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Appendix A. PROOF OF THEOREM 4

Proof follows standard singular perturbation technique based on the Lyapunov function candidate

$$W = \mu V_{(\tilde{\theta},\tilde{u})} + \eta V_{z(t)} + (1-\mu-\eta)V(\tilde{x}, z) \tag{A.1}$$

where $\mu, \eta \in (0,1)$ and $\mu + \eta < 1$.
Differentiating with respect to t, one obtains

$$\dot{W} \leq -k'\mu V_{(\tilde{\theta},\tilde{u})} + \frac{\mu}{2k}L_\theta^2 + \frac{\mu}{2k_2}D^2 - \eta(k^2 - \frac{\lambda k}{2})V_{z(t)} + \frac{\eta\lambda}{2k}L_\theta^2 + \eta k_s^2 z(t)\left(\frac{\partial h(\pi(u))}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u} - \frac{\partial h(x)}{\partial x}\frac{\partial \pi(u)}{\partial u}\dot{u}\right) - \frac{(1-\mu-\eta)}{\epsilon_1}\beta_3\|\tilde{x}\|^2 - \frac{(1-\mu-\eta)}{\epsilon_1}\beta_4|z|^2$$

By Assumptions 7, 8 and 9, the inequality becomes

$$\begin{aligned}\dot{W} \leq & -[\|\tilde{x}\|, |z|]\, A \begin{bmatrix}\|\tilde{x}\| \\ |z|\end{bmatrix} - k'\mu V_{(\tilde{\theta},\tilde{u})} \\ & -\eta(k^2 - \frac{\lambda k}{2})V_{z(t)} + \frac{\mu+\eta\lambda}{2k}L_\theta^2 + \frac{\mu}{2k_2}D^2\end{aligned} \tag{A.2}$$

where

$$A = \begin{bmatrix} \frac{(1-\mu-\eta)}{\epsilon_1}\beta_3 & -\frac{1}{2}\eta k_s^2 L_\pi L_h(k_1L_1 + D) \\ -\frac{1}{2}\eta k_s^2 L_\pi L_h(k_1L_1 + D) & \frac{(1-\mu-\eta)}{\epsilon_1}\beta_4 \end{bmatrix}$$

The matrix A is positive definite if ϵ_1 is chosen such that

$$\epsilon_1 < \frac{2\sqrt{\beta_3\beta_4}(1-\mu-\eta)}{\eta k_s^2 L_\pi L_h(k_1L_1 + D)} = \epsilon^*$$

It then follows that $\forall \epsilon_1 \in (0, \epsilon^*)$,

$$\begin{aligned}\dot{W} \leq & -k'\mu V_{(\tilde{\theta},\tilde{u})} - \left(k_c + \eta(k^2 - \frac{\lambda k}{2})\right)V_{z(t)} \\ & -k_c\|\tilde{x}\|^2 + \frac{\mu+\eta\lambda}{2k}L_\theta^2 + \frac{\mu}{2k_2}D^2\end{aligned} \tag{A.3}$$

where k_c is a strictly positive constant. This completes the proof.