Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Design of Robust Internal Models for a Class of Linear Hybrid Systems ⋆

Nicholas Cox * **Lorenzo Marconi** ** **Andrew R. Teel** *

* *ECE Dept., University of California, Santa Barbara, USA (e-mail: njcox@umail.ucsb.edu, teel@ece.ucsb.edu).*
** *C.A.SY. – DEI, University of Bologna, Bologna, Italy (e-mail: lorenzo.marconi@unibo.it).*

Abstract: This paper focuses on the design of internal models for a class of hybrid linear SISO systems and exosystems. The internal model design takes into consideration observability properties of the exosystem that have significance in the hybrid setting. In particular, visibility and invisibility of the so-called hybrid steady-state generator are discussed, as well as uniform observability properties.

Keywords: Hybrid Systems, Output Regulation, Internal Models, Robustness, Output Tracking, Disturbance Rejection, State Observers, Visibility

1. INTRODUCTION

There has been significant work in the field of output regulation for hybrid systems recently. Marconi and Teel (2013) lay out the relevant framework and develop hybrid regulation equations and a hybrid internal model property. The proposed framework considers the class of linear systems and exosystems that are subject to jumps according to a known clock that satisfies a dwell-time constraint.

The authors have published further related works in the same vein. For results on designing stabilizers in conjunction with hybrid internal models, see Cox et al. (2013). Furthermore, the case where the regulator may not be able to rely on a known jump clock, but still relies on periodic jumps, is covered by Cox et al. (2011). More recently, the authors have shown that hybrid internal models can be used to achieve robust global exponential tracking of spline trajectories, see Cox et al. (2012). The previous paper on spline tracking uses some restrictive assumptions on the zero-dynamics of the plant, but here we addresses those shortcomings. In fact, the design proposed here is motivated by the methods used to track spline trajectories, but works much more generally.

Recent work has also been done regarding stabilization and regulation goals achieved for MIMO systems, see Carnevale et al. (2012a) and Carnevale et al. (2012b), amongst others. They also study the application of hybrid output regulation to the problem of tracking a spinning and bouncing disk, see Carnevale et al. (2013).

In the following, the goal is to achieve hybrid output regulation for a class of hybrid linear SISO systems and exosystems. We use the framework of Marconi and Teel (2013), with a focus on linear systems described in normal form. As shown by Cox et al. (2012), the general internal model design method given by Marconi and Teel (2013) may not always be sufficient, and in some cases a more guided approach is necessary. The internal model developed here builds on a "visibility property" of the so-called hybrid steady state generator system, namely the hybrid system that generates the ideal control input able to keep the regulation error identically zero. In this way we give a consistent design method for hybrid internal models applicable in general. The internal model designed is similar to a state observer, but with the alternate goal of reproducing the output of the hybrid steady state generator, as opposed to the entire state. It will be shown that the design procedure presented in the paper is relevant in achieving robust output regulation goals.

Notation The unit disk is denoted by $\mathcal{D}_1$. The eigenvalues of a matrix M are denoted by eig(M). The real space is denoted by $\mathbb{R}$. The Kronecker product of two matrices A_1 and A_2 is denoted by $A_1 \otimes A_2$, while the Kronecker sum is $A_1 \oplus A_2$.

2. FRAMEWORK

Consider the system that flows according to

$$\begin{aligned} \dot{\tau} &= 1, \\ \dot{w} &= S(\tau)w, \\ \dot{z} &= A_{11}z + A_{12}y + P_1 w, \\ \dot{y} &= A_{21}z + A_{22}y + bu + P_2 w, \end{aligned} \tag{1}$$

for $(\tau, w, z, y) \in [0, \tau_{\max}] \times W \times \mathbb{R}^n \times \mathbb{R}$, and jumps according to

$$\begin{aligned} \tau^+ &= 0, \\ w^+ &= Jw, \\ z^+ &= M_{11}z + M_{12}y + N_1 w, \\ y^+ &= M_{21}z + M_{22}y + N_2 w, \end{aligned} \tag{2}$$

for $(\tau, w, z, y) \in \{\tau_{\max}\} \times W \times \mathbb{R}^n \times \mathbb{R}$.

The system, (z, y), and exosystem, w, jump periodically according to the clock variable, τ. The exosystem state w lives in the compact set $W \subset \mathbb{R}^s$ with the set $[0, \tau_{\max}] \times W$ that is assumed to be (forward) *invariant* for the (τ, w).

⋆ Research supported by AFOSR FA9550-12-1-0127, NSF ECCS-1232035 and by the European Project SHERPA (ICT 600958).

The goal of output regulation is to regulate the variable

$$e = y - Qw,$$

i.e. we want to find a regulator that processes only the error, e, and steers it, asymptotically, to zero.

We limit the analysis to minimum-phase systems. Namely, systems that fulfill the following assumptions.

Assumption 1. (Minimum-Phase). The matrices A_{11}, M_{11} are such that $\mathrm{eig}(M_{11}\exp(A_{11}\tau_{\max})) \in \mathcal{D}_1$.

Furthermore, in order to have z-dynamics with a well-defined hybrid steady state, we assume the following non-resonance condition between the zeros of the system and the "poles" of the hybrid exosystem.

Assumption 2. (Non-Resonance Condition). The following holds: $\mathrm{eig}(M_{11}\exp(A_{11}\tau_{\max})) \cap \mathrm{eig}(J\exp(S\tau_{\max})) = \emptyset$.

With assumption 2 in hand we let $\Pi_z(\tau) : [0, \tau_{\max}] \to \mathbb{R}^{n\times s}$ be the continuously differentiable function that is the unique solution of

$$\begin{aligned} \frac{d\Pi_z(\tau)}{d\tau} &= A_{11}\Pi_z(\tau) - \Pi_z(\tau)S(\tau) + A_{12}Q + P_1, \\ 0 &= M_{12}Q + M_{11}\Pi_z(\tau_{\max}) - \Pi_z(0)J + N_1\,. \end{aligned}$$

Furthermore we consider the change of variables

$$z \;\mapsto\; \tilde{z} = z - \Pi_z(\tau)w, \qquad y \;\mapsto\; e = y - Qw,$$

which transforms the system into a hybrid system flowing according to

$$\begin{aligned} \dot{\tau} &= 1, \;\; \dot{w} = S(\tau)w, \\ \dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12}e, \\ \dot{e} &= A_{21}\tilde{z} + A_{22}e + b(u - R(\tau)w), \end{aligned} \tag{3}$$

whenever $(\tau, w, \tilde{z}, e) \in [0, \tau_{\max}] \times \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}$, where

$$R(\tau) = \frac{1}{b}\left(QS - A_{22}Q - A_{21}\Pi_z(\tau) - P_2 \right), \tag{4}$$

and jumping according to

$$\begin{aligned} \tau^+ &= 0, \;\; w^+ = Jw, \\ \tilde{z}^+ &= M_{11}\tilde{z} + M_{12}\tilde{z}, \\ e^+ &= M_{21}\tilde{z} + M_{22}e + (M_{21}\Pi_z(0) + M_{22}Q - QJ + N_2)w, \end{aligned} \tag{5}$$

whenever $(\tau, w, \tilde{z}, e) \in \{\tau_{\max}\} \times \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}$.

The goal of the regulator is to make the set $\{(\tau, w, \tilde{z}, e) \in [0, \tau_{\max}] \times W \times \mathbb{R}^n \times \mathbb{R} \;:\; \tilde{z} = 0,\, e = 0\}$ globally exponentially stable for the error system (3)-(5), by compensating for the term $R(\tau)w$. This necessitates the following assumption on the jump dynamics because there is no feedback during jumps to compensate for any disturbance that may show up.

Assumption 3. The matrix equation $M_{21}\Pi_z(0) + M_{22}Q - QJ + N_2 = 0$ is satisfied.

A crucial role in the design of internal model-based regulators is played by the so-called "hybrid steady state generator system" defined as the following hybrid system

$$\begin{aligned} &\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{w} &= S(\tau)w \end{aligned}\right\} \quad (\tau, w) \in [0, \tau_{\max}] \times W \\ &\left.\begin{aligned} \tau^+ &= 0 \\ w^+ &= Jw \end{aligned}\right\} \quad (\tau, w) \in \{\tau_{\max}\} \times W \\ &y_w = R(\tau)w \end{aligned} \tag{6}$$

with output $y_w \in \mathbb{R}$. This system, in fact, generates all the ideal steady state control inputs required of the regulator in order to keep the regulation error, e, identically zero.

Due to the fact that the initial condition $(\tau(0), w(0))$ of the exosystem is arbitrary on $[0, \tau_{\max}] \times W$, it is apparent that the "visible" dynamics of system (6) must be embedded into any regulator that solves the problem of output regulation [1]. This observation is at the foundation of the celebrated internal model principle (see Francis and Wonham (1976) for continuous-time linear systems and Marconi and Teel (2013) for hybrid linear systems).

For reasons that are motivated by the problem of designing robust internal models, and to make the notion of visibility rigorous, it is useful to introduce the class of systems that are "state-output" equivalent to (6) as formally defined in Definition 1. In the definition we refer to an "equivalent" system defined by

$$\begin{aligned} &\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{\mathbf{w}} &= \mathbf{S}(\tau)\mathbf{w} \end{aligned}\right\} \quad (\tau, \mathbf{w}) \in [0, \tau_{\max}] \times \mathbf{W} \\ &\left.\begin{aligned} \tau^+ &= 0 \\ \mathbf{w}^+ &= \mathbf{J}\mathbf{w} \end{aligned}\right\} \quad (\tau, \mathbf{w}) \in \{\tau_{\max}\} \times \mathbf{W} \\ &\mathbf{y_w} = \mathbf{R}(\tau)\mathbf{w} \end{aligned} \tag{7}$$

where $\mathbf{w} \in \mathbb{R}^{\mathbf{s}}$, $\mathbf{s} \in \mathbb{N}$, and $\mathbf{W}$ is a compact subset of $\mathbb{R}^{\mathbf{s}}$ with $[0, \tau_{\max}] \times \mathbf{W}$ invariant for (7). We note that (6) and (7) have the same hybrid time domain (see Goebel et al. (2009)) dependent on the initial condition $\tau(0)$.

Definition 1. System (6) is *state-output equivalent* to system (7) if for any $\tau(0) \in [0, \tau_{\max}]$ and $w(0) \in W$ there exists a $\mathbf{w}(0) \in \mathbf{W}$ such that, having denoted by $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ the corresponding hybrid time domain,

$$y_w(t, j) = \mathbf{y_w}(t, j) \qquad \forall\; (t, j) \in E\,.$$

By following the prescriptions of Marconi and Teel (2013), Section IV.A, we focus on a hybrid internal model-based regulator of the form

$$\begin{aligned} &\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{\eta} &= F_{\mathrm{im}}(\tau)\eta + G_{\mathrm{im}}(\tau)u \end{aligned}\right\} \; (\tau, \eta) \in [0, \tau_{\max}] \times \mathbb{R}^\nu, \\ &\left.\begin{aligned} \tau^+ &= 0 \\ \eta^+ &= \Sigma_{\mathrm{im}}\eta \end{aligned}\right\} \; (\tau, \eta) \in \{\tau_{\max}\} \times \mathbb{R}^\nu, \\ &u = \Gamma_{\mathrm{im}}(\tau)\eta + v, \end{aligned} \tag{8}$$

where $\nu \in \mathbb{N}$, $F_{\mathrm{im}} : [0, \tau_{\max}] \to \mathbb{R}^{\nu\times\nu}$, $G_{\mathrm{im}} : [0, \tau_{\max}] \to \mathbb{R}^{\nu\times 1}$ and $\Gamma_{\mathrm{im}} : [0, \tau_{\max}] \to \mathbb{R}^{1\times\nu}$ are *continuously differentiable* functions, Σ_{im} is a matrix, and v is a residual control input, all to be designed.

The following result, which can be proven by slightly adapting the arguments of Marconi and Teel (2013), Section IV, provides the main guidelines for the design of (8).

Proposition 1. Let Assumptions 1, 2 and 3 be fulfilled. Let (7) be a system that is state-output equivalent to the hybrid steady-state generator system (6). Assume that the controller (8) is designed so that for some continuously differentiable function $\Pi_\eta(\tau)$ the set

$$\mathcal{S} = \{(\tau, \mathbf{w}, \eta) \in [0, \tau_{\max}] \times \mathbf{W} \times \mathbb{R}^\nu : \eta = \Pi_\eta(\tau)\mathbf{w}\} \tag{9}$$

[1] The concept of visibility here is used loosely and will be better specified later by following D'Alessandro et al. (1973). Intuitively, visible dynamics are state trajectories of (6) that show up on the output y_w and, as such, must be reproduced by the regulator.

is globally exponentially stable for the hybrid system

$$\left.\begin{aligned} \dot{\tau} &= 1\,, \quad \dot{\mathbf{w}} = \mathbf{S}(\tau)\mathbf{w} \\ \dot{\eta} &= F_{\rm im}(\tau)\eta + G_{\rm im}(\tau)\mathbf{R}(\tau)\mathbf{w} \end{aligned}\right\} \begin{aligned} &(\tau, \mathbf{w}, \eta) \\ &\in [0, \tau_{\max}] \times \mathbf{W} \times \mathbb{R}^{\nu}, \end{aligned}$$

$$\left.\begin{aligned} \tau^+ &= 0\,, \quad \mathbf{w}^+ = \mathbf{J}\mathbf{w} \\ \eta^+ &= \Sigma_{\rm im}\eta \end{aligned}\right\} \begin{aligned} &(\tau, \mathbf{w}, \eta) \\ &\in \{\tau_{\max}\} \times \mathbf{W} \times \mathbb{R}^{\nu}, \end{aligned} \tag{10}$$

with

$$\Gamma_{\rm im}(\tau)\Pi_{\eta}(\tau) = \mathbf{R}(\tau) \qquad \forall\, \tau \in [0, \tau_{\max}]. \tag{11}$$

Then, there exists a $\kappa^\star > 0$ such that for all $\kappa \geq \kappa^\star$ the regulator (8) with $v = -\kappa e$ solves the problem of hybrid output regulation.

An internal model of the form (8) making the set (9) globally exponentially stable for (10) and fulfilling (11) always exists provided that the dimension ν is taken sufficiently large and the triplet $(F_{\rm im}, G_{\rm im}, \Sigma_{\rm im})$ satisfies some technical requirements (see Proposition 3 of Marconi and Teel (2013)). However, there are significant cases where the design tools proposed by Marconi and Teel (2013) do not directly apply as shown. For instance, see Cox et al. (2012), where the problem of robustly tracking a spline generated reference signal is dealt with. The main objective of this paper is to enrich these tools by presenting a general methodology for the design of the triplet $(F_{\rm im}, G_{\rm im}, \Sigma_{\rm im})$ so that the requirements of Proposition 1 are fulfilled.

As is clear from the statement of the proposition, the problem at hand is related to the problem of designing an *asymptotic output reproducer* for a hybrid system that is state-output equivalent to the hybrid steady state generator (6), namely to design a hybrid system that, forced by $\mathbf{y_w}$, is able to asymptotically reproduce all the possible output behaviors $\mathbf{y_w}(t)$ of (7), and thus all the possible output behaviors $y_w(t)$ of (6).

Of course the properties required of system (8) in Proposition 1 could be fulfilled by directly using, as system (7), the hybrid state-steady generator (6). The reason why it is worth introducing a different, although equivalent, system (6) in Proposition 1 is related to the design of *robust* internal models.

In fact, the quadruplet $(F_{\rm im}(\tau), G_{\rm im}(\tau), \Gamma_{\rm im}(\tau), \Sigma_{\rm im})$ fulfilling the properties of Proposition 1 generally depends on the triplet $(\mathbf{S}, \mathbf{J}, \mathbf{R})$. In this respect, it turns out that the function $R(\tau)$ defined in (4) is, in general, affected by possible parametric uncertainties in the regulated plant (1)-(2), which makes the direct use of (6) to design the regulator (8) according to Proposition 1 ineffective if a robust regulator is sought. On the other hand, the presence of possible uncertainties in $R(\tau)$ can be overcome by defining an equivalent system in an appropriate way. This is certainly the case if $R(\tau)$ is linearly parametrized in the uncertainties, namely if there exists a $p \in \mathbb{N}$ and *known* continuously differentiable functions $R_i(\tau)$, $i = 1, \ldots, p$, such that

$$R(\tau) = \sum_{i=1}^{p} R_i(\tau)\mu_i \tag{12}$$

where μ_i are the uncertain parameters ranging in a *known* compact set $[\underline{\mu}_1, \bar{\mu}_1] \times \ldots \times [\underline{\mu}_p, \bar{\mu}_p]$. In fact, the next result follows.

Proposition 2. Let $\mathbf{S}(\tau) = I_p \otimes S(\tau)$, $\mathbf{J} = I_p \otimes J$, $\mathbf{R}(\tau) = (R_1(\tau), \ \ldots, \ R_p(\tau))$ and $\mathbf{W} = W_1 \times \ldots \times W_p$ where

$$W_i = \{w_i \in \mathbb{R}^s \ : \ w_i = \mu w, \, w \in W\,, \ \mu \in [\underline{\mu}_i, \, \bar{\mu}_i]\}$$

for $i = 1, \ldots, p$. Then system (7) is state-output equivalent to (6).

It is worth noting that system (7) as defined in the previous proposition is not affected by the actual values of the μ_i's. Thus, a quadruplet $(F_{\rm im}(\tau), G_{\rm im}(\tau), \Gamma_{\rm im}(\tau), \Sigma_{\rm im})$ fulfilling the properties of Proposition 1 immediately yields a robust regulator.

3. DESIGN OF THE REGULATOR

We approach the problem of designing a quadruplet $(F_{\rm im}(\tau), G_{\rm im}(\tau), \Gamma_{\rm im}(\tau), \Sigma_{\rm im})$ fulfilling the properties of Proposition 1 by designing an observer for the dynamics of (7) that are "visible" on the output $\mathbf{y_w}$. Toward this end, in the next subsection we present a decomposition of system (7) that isolates visible and invisible dynamics. Our goal is to identify a hybrid system that is state-output equivalent to (7) and for which an asymptotic observer can be designed. The design of the (hybrid) asymptotic observer is dealt with in Section 3.2. This, in turn, will lead to immediately obtaining an "output reproducer" of system (7). For notational convenience, in the following part we drop the bold notation for system (7), by using $S(\tau)$, J, $R(\tau)$, W and y_w instead of $\mathbf{S}(\tau)$, $\mathbf{J}$, $\mathbf{R}(\tau)$, $\mathbf{W}$ and $\mathbf{y_w}$.

3.1 Isolating invisible dynamics

Towards the final goal of isolating visible and invisible dynamics of the *hybrid* system (7), we start by focusing on the flow dynamics by identifying dynamics that do not affect the output during flow. Consider the continuous-time time-varying linear system of the form

$$\begin{aligned} \dot{w} &= S(\tau)w, \qquad w \in \mathbb{R}^s \\ y_w &= R(\tau)w, \end{aligned} \tag{13}$$

defined on the interval $\tau \in [0, \tau_{\max}]$ and let $\phi(\tau, \tau_0)$ be the state transition matrix associated with $\dot{w} = S(\tau)w$. As the system is time-varying, a Kalman-like decomposition related to observability can be rigorously obtained by the arguments of D'Alessandro et al. (1973). The definition of an *invisible state* is crucial to that paper and is recalled here.

Definition 2. We say that a state $w \in \mathbb{R}^s$ is *invisible* at time $\tau \in [0, \tau_{max}]$ in the interval $[0, \tau_{\max}]$ if it is unobservable and unreconstructable at time τ in the specified time interval, namely if

$$R(t)\phi(t, \tau)w = 0 \qquad \text{for all } t \in [0, \tau_{\max}]\,.$$

Furthermore, we define the invisible space as in the following.

Definition 3. We let $\mathcal{L}(\tau)$ be the space of states that are invisible at time τ in the interval $[0, \tau_{\max}]$.

Let Q be the gramian associated to the system in the interval defined as

$$Q(\tau_{\max}) = \int_0^{\tau_{\max}} \phi(t, \tau)^T \, R^T(t)R(t)\phi(t, \tau)dt\,.$$

The following result plays a crucial role in finding the change of variables that isolates the visible and invisible dynamics of (13).

Theorem 1. The following holds:

- $\mathcal{L}(\tau) = \mathrm{Ker}(Q(\tau))$;
- $\dim\mathcal{L}(\tau_1) = \dim\mathcal{L}(\tau_2) := s_{no}$ for all $\tau_1, \tau_2 \in [0, \tau_{\max}]$.

The proof is omitted, but follows the arguments in D'Alessandro et al. (1973) specialized to the present context.

We are now in the position of finding the τ-dependent smooth change of variables $w = T(\tau)x$ such that, in the new coordinates, system (13) reads as

$$\begin{aligned} \dot{x}_{\mathrm{no}} &= 0 \\ \dot{x}_{\mathrm{o}} &= 0 \\ y_w &= R_{\mathrm{o}}(\tau)x_{\mathrm{o}}, \end{aligned} \tag{14}$$

with $x_{no} \in \mathbb{R}^{s_{no}}$, $x_o = \mathbb{R}^{s_o}$, where $s_o := s - s_{no}$, for some appropriately defined continuously differentiable function $R_{\mathrm{o}}(\cdot)$ such that the gramian

$$Q_o(\tau_{\max}) = \int_0^{\tau_{\max}} R_o^T(t)R_o(t)dt \tag{15}$$

associated to the x_o subsystem is not singular.

As a matter fact, pick any $\tau_0 \in [0, \tau_{\max}]$ and let $\{\mathrm{v}_i\}_{i=1}^{s_{no}}$ be a basis of $\mathcal{L}(\tau_0)$ (namely a basis of $\mathrm{Ker}(Q(\tau_0))$). Furthermore, let $\mathcal{C}(\tau_0)$ be the complement of $\mathcal{L}(\tau_0)$ relative to $\mathbb{R}^s$, namely the space such that $\mathcal{C}(\tau_0) \oplus \mathcal{L}(\tau_0) = \mathbb{R}^s$, and let $\{\mathrm{v}_i\}_{i=s_o+1}^{s}$ be a basis of $\mathcal{C}(\tau_0)$.

In the following we derive a basis for $\mathcal{L}(\tau)$ and $\mathcal{C}(\tau)$, with $\mathcal{C}(\tau)$ such that $\mathcal{L}(\tau) \oplus \mathcal{C}(\tau) = \mathbb{R}^s$ for all $\tau \in [0, \tau_{\max}]$. Those bases are obtained by flowing forward and backward in time the bases of $\mathcal{L}(\tau_0)$ and $\mathcal{C}(\tau_0)$. To this end, it turns out that (see D'Alessandro et al. (1973))

$$\mathcal{L}(\tau) = \mathrm{span}\,\{\phi(\tau, \tau_0)\,[\,\mathrm{v}_1 \;\cdots\; \mathrm{v}_{s_{no}}\,]\}\; \forall \tau \in [0, \tau_{\max}]. \tag{16}$$

To prove this, note that $\phi(\tau, \tau_0)\mathrm{v}_i \in \mathcal{L}(\tau)$ for all $i = 1, \dots, s_{no}$. As a matter of fact

$$\begin{aligned} Q(\tau)\phi(\tau, \tau_0)\mathrm{v}_i &= \phi(\tau, \tau_0)^{-T}\phi(\tau, \tau_0)^T Q(\tau)\phi(\tau, \tau_0)\mathrm{v}_i \\ &= \phi(\tau, \tau_0)^{-T} Q(\tau_0)\mathrm{v}_i = 0\,. \end{aligned}$$

Furthermore,

$$\mathrm{rank}\;\phi(\tau, \tau_0)\;[\,\mathrm{v}_1 \;\cdots\; \mathrm{v}_{s_{no}}\,] = s_{no}\,.$$

Since $\dim\mathcal{L}(\tau) = s_{no}$, the previous facts prove (16). Similarly,

$$\mathcal{C}(\tau) = \mathrm{span}\,\{\phi(\tau, \tau_0)\,[\,\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,]\} \tag{17}$$

for all $\tau \in [0, \tau_{\max}]$. As a matter of fact,

$$\mathrm{rank}\;\phi(\tau, \tau_0)\;[\,\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,] = s_o$$

and

$$\mathrm{rank}\;\phi(\tau, \tau_0)\;[\,\mathrm{v}_1 \;\cdots\; \mathrm{v}_{s_{no}} \;\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,] = s$$

by which, using (16), (17) follows.

By using the previous results we thus consider the (smooth) change of variable $w = T(\tau)x$ with

$$T(\tau) = \phi(\tau, \tau_0)V \qquad V := [\,\mathrm{v}_1 \;\cdots\; \mathrm{v}_{s_{no}} \;\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,]\,.$$

By construction it turns out that

$$S(\tau)w = \dot{w} = \dot{T}(\tau)x + T(\tau)\dot{x},$$

from which

$$\dot{x} = T(\tau)^{-1}(S(\tau)T(\tau) - \dot{T}(\tau))x\,.$$

Using,

$$\dot{T}(\tau) = \dot{\phi}(\tau, \tau_0)V = S(\tau)\phi(\tau, \tau_0)V = S(\tau)T(\tau),$$

the previous relations yield

$$\dot{x} = 0\,.$$

Furthermore, by construction and by the definition of an invisible state space,

$$R(\tau)\phi(\tau, \tau_0)V = [\,0 \;\; R_o(\tau)\,]\,,$$

where

$$R_o(\tau) = R(\tau)\phi(\tau, \tau_0)\,[\,\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,]\,.$$

Simple arguments can be finally used to show $Q_o(\tau)$ is not singular for all τ in the interval. Rewrite (15) as

$$\mathcal{Q}_o(\tau_{\max}) = [\,\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,]^T\, \mathcal{Q}(\tau_{\max})\,[\,\mathrm{v}_{s_{no}+1} \;\cdots\; \mathrm{v}_s\,]\,.$$

Then by construction of $\mathcal{C}(\tau_0)$, $\mathcal{Q}_o(\tau_{\max})$ is non-singular.

We note that the subspace $\{x \;:\; x_o = 0\}$ is invariant and composed of invisible states. On the other hand the subsystem

$$\begin{aligned} \dot{x}_o &= 0 \\ y_w &= R_o(\tau)x_o \end{aligned}$$

is "visible" in the interval, namely the subspace of invisible states $\mathcal{L}_o(\tau)$ associated to the previous system is such that $\mathcal{L}_o(\tau) \equiv \{0\}$.

Remark It is worth noting that the previous visibility property of the pair $(0, R_o(\cdot))$ does not imply, in general, that the pair is uniformly observable, namely that the observability matrix $\mathcal{O}_o(\tau)$ associated to the pair $(0, R_o(\tau))$ is not singular for all τ in the interval. The latter property is related to a uniform observability that is time-wise. $\triangleleft$

Now we consider the *hybrid* system (7), with the goal of identifying visible and invisible dynamics for this system. By applying the change of variable $T(\tau)$ discussed above, the jump relation of system (7) transforms according to

$$x^+ = [T(\tau_{\max})^{-1}]^+ w^+ = T(0)^{-1}Jw = T(0)^{-1}JT(\tau_{\max})x,$$

where[2] $x = x(\tau_{\max})$ and $w = w(\tau_{\max})$. By partitioning $T(0)^{-1}JT(\tau_{\max})$ consistently with x,

$$\begin{aligned} x_o^+ &= J_o\, x_o(\tau_{\max}) + J_{ono}\, x_{no}(\tau_{\max}) \\ x_{no}^+ &= J_{no}\, x_{no}(\tau_{\max}) + J_{noo}\, x_o(\tau_{\max}) \end{aligned} \tag{18}$$

where the matrices J_o, J_{ono}, J_{no}, J_{noo} do not have any special properties.

We note that, by construction, the hybrid system flowing according to (13) and jumping according to (18) is state-output equivalent to (7). Furthermore, we note that the x_{no} state component, that is invisible for the continuous-time system (13) during flows, might become visible for the hybrid system (7). As a matter of fact, the x_{no} component might show up during jumps by affecting x_o through the jump relation $x_o^+ = J_o x_o + J_{ono}x_{no}$, thus affecting the output $y_w(t)$ in the "subsequent" flow interval. This means that in the attempt to identify a system that is state-output equivalent to (7) and for which an asymptotic observer can be designed, it cannot be ignored.

This observation motivates the forthcoming developments in which the goal is to compute a system that is state-output equivalent to the hybrid system flowing according

[2] Here and in the following we compactly denote by $\xi(\tau_{\max})$ and $\xi(0)$ the value of a state variable ξ at the end and at the beginning of a generic time interval.

to (14) and jumping according to (18) by isolating the component of x_{no} that is also invisible during jumps. To this purpose, let $\Upsilon \in \mathbb{R}^{s_{no}} \times \mathbb{R}^{s_{no}}$ be the change of variable that puts the pair (J_{no}, J_{ono}) in observable canonical form. Namely,

$$\Upsilon J_{no} \Upsilon^{-1} = \begin{pmatrix} J'_{no} & 0 \\ \star & \star \end{pmatrix}, \qquad J_{ono}\Upsilon^{-1} = \begin{pmatrix} J'_{ono} & 0 \end{pmatrix},$$

where

$$(J'_{no}, J'_{ono}) \in \mathbb{R}^{s'_{no}\times s'_{no}} \times \mathbb{R}^{(s-s_{no})\times s'_{no}}, \quad s'_{no} \geq 0\,,$$

is an observable pair, with $\star$ denoting generic blocks of no interest in the subsequent developments. By changing coordinates as $x_{no} \mapsto x'_{no} = \Upsilon x_{no}$ and by partitioning $x'_{no} = \mathrm{col}(x'_{noo}, x'_{nono})$ with $x'_{noo} \in \mathbb{R}^{s'_{no}}$, $x'_{nono} \in \mathbb{R}^{s_{no}-s'_{no}}$, it turns out that the dynamics of x_o and x'_{no} are described by the flow dynamics

$$\begin{aligned} \dot{x}_{\mathrm{o}} &= 0 \\ \dot{x}'_{\mathrm{no}} &= 0\,, \end{aligned}$$

and by the jump relation

$$\begin{aligned} x_o^+ &= J_o x_o + J'_{ono} x'_{no} \\ x'^+_{noo} &= J'_{no} x'_{noo} + J'_{noo} x_o \\ x'^+_{nono} &= \star \end{aligned}$$

where $J'_{noo} \in \mathbb{R}^{s'_{no}\times s_o}$ is the matrix obtained by extracting the first s'_{no} rows from the matrix ΥJ_{noo}, and where $\star$ denotes a linear combination of $x'_{noo}(0)$, $x'_{nono}(0)$ and $x_o(\tau_{\max})$ of no interest in the following. By keeping in mind that the output y_w is only affected by the x_{o} component, it is immediately seen that x'_{nono} has no effect on the output, neither during flows nor during jumps. Hence, we conclude that system (13), (18) is state-output equivalent to the hybrid system

$$\begin{aligned} \dot{z}_o &= 0 \\ \dot{z}_{no} &= 0 \\ z_o^+ &= N_o z_o + N_{ono} z_{no} \\ z_{no}^+ &= N_{no} z_{no} + N_{noo} z_o \\ y_w &= R_o(\tau) z_o \end{aligned} \tag{19}$$

where $N_o = J_o$, $N_{ono} = J'_{ono}$, $N_{no} = J'_{no}$, $N_{noo} = J'_{noo}$.

All the state components of the previous system are visible, as is shown by the result in the next section, where an asymptotic hybrid observer for this system is presented.

3.2 Design of the internal model

The goal of this section is to present a methodology for the design of the internal model having the output reproducer capabilities required in Proposition 1. The idea that is followed in the design is to construct a hybrid asymptotic observer for the dynamics of (19). The design of the observer for the z_{no} part (which is invisible during flows but which show up during jumps) follows the intuition that a discrete time observer could be designed using the "measure" $z_o^+ - N_o z_o$ to construct an innovation term. As z_o^+ is not measurable we "inject" it in the z_{no} jump dynamics through the change of variable

$$z_{no} \mapsto \xi_{no} = z_{no} + K_2 z_o$$

with K_2 to be fixed. By also letting $\xi_o = z_o$, in the new coordinates system (19) reads as

$$\begin{aligned} \dot{\xi}_o &= 0 \\ \dot{\xi}_{no} &= 0 \\ \xi_o^+ &= \bar{N}_o \xi_o + \bar{N}_{ono}\xi_{no} \\ \xi_{no}^+ &= \bar{N}_{noo}\xi_o + (N_{no} + K_2 N_{ono})\xi_{no} \\ y_w &= R_o(\tau)\xi_o \end{aligned} \tag{20}$$

where $\bar{N}_{ono} := N_{ono}$ and

$$\begin{aligned} \bar{N}_o &:= (N_o - N_{ono}K_2) \\ \bar{N}_{noo} &:= N_{noo} - N_{no}K_2 + K_2 N_o - K_2 N_{ono} K_2\,. \end{aligned}$$

Using the fact that the pair (N_{no}, N_{ono}) is observable we now choose K_2 such that

$$\mathrm{eig}(N_{no} + K_2 N_{ono}) \in \mathcal{D}_1\,.$$

We now develop two different observers for (20) according to the properties fulfilled by the pair $(0, R_o(\tau))$.

(A)The pair $(0, R_o(\tau))$ *is not uniformly observable.* As remarked in Section 3.1, there is no guarantee that the "visible" pair $(0, R_o(\tau))$ is uniformly observable. The only guarantee is that the gramian (15) is not singular. In such a case the asymptotic properties of the observer are mainly obtained through jumps, by integrating the output y_w in the interval $[0, \tau_{\max}]$ to compute a "finite-time" estimate of ξ_o (by inversion of the gramian). Specifically, the following proposition, for which the proof is omitted to save space, holds.

Proposition 3. As system (8), consider the system

$$\begin{aligned} \dot{\eta}_i &= R_o^T(\tau) u \\ \dot{\eta}_o &= 0 \\ \dot{\eta}_{no} &= 0 \\ \eta_i^+ &= 0 \\ \eta_o^+ &= \bar{N}_o\, Q_o(\tau_{\max})^{-1}\, \eta_i + \bar{N}_{ono}\eta_{no} \\ \eta_{no}^+ &= (N_{no} + K_2 N_{ono})\eta_{no} + \bar{N}_{noo}\, Q_o(\tau_{\max})^{-1}\, \eta_i \\ y_\eta &= R_o(\tau)\eta_o\,. \end{aligned} \tag{21}$$

Then, there exists a differentiable function $\Pi_\eta(\tau)$ satisfying the properties of Proposition 1, that is the set $\mathcal{S}$ in (9) is globally exponentially stable for (10) and (11).

We observe that the role of the variable η_i, which is reset at every jump, is to estimate the state component ξ_o of (20) at the beginning ($\xi_o(0)$) and at the end ($\xi_o(\tau_{\max})$) of the time interval through the relations

$$\xi_o(0) = \xi_o(\tau_{\max}) = Q_o(\tau_{\max})^{-1}\eta_i(\tau_{\max}) \tag{22}$$

and to use those estimates at the jump time to enforce a cascade structure of the error system.

Remark The flow dynamics of the η_i variable in (21) are those of a pure integrator, which could open the door to some criticism about the sensitivity of the proposed observer to noise superimposed on the input u. With regards to this, a variant of the observer structure in (21) could consider flow η_i dynamics of the form

$$\dot{\eta}_i = H\eta_i + R_o^T(\tau)u,$$

where H is a Hurwitz matrix introduced to filter possible noise superimposed on u. The previous construction continues to hold having care to substitute $Q_o(\tau_{\max})$ in the expressions of η_o+ and η_{no}^+, with the "filtered" gramian

$$Q_o^f(\tau_{\max}) = \int_0^{\tau_{\max}} \exp(H(\tau_{\max} - \tau))\, R_o^T(t) R_o(t)\, dt\,.$$

$\triangleleft$

(B) The pair $(0, R_o(\tau))$ is uniformly observable. An alternative observer design can be proposed if the visible pair $(0, R_o(\tau))$ is also uniformly observable in the interval. The uniform observability condition is formalized in the next assumption.

Assumption 4. The observability matrix

$$\mathcal{O}(\tau) = \begin{pmatrix} R_o(\tau) \\ \dot{R}_o(\tau) \\ \vdots \\ R_o^{(s_o-1)}(\tau) \end{pmatrix}$$

is non-singular for all $\tau \in [0, \tau_{\max}]$.

Under this assumption, the observer for the system (20) can be done by using high-gain tools to estimate, during flows, the observable component of the system as presented in the following. Let $P(\tau)$ be a matrix defined by its inverse

$$P^{-1}(\tau) = \left[\, q(\tau)\ \tilde{\mathcal{L}}q(\tau)\ \ldots\ \tilde{\mathcal{L}}^{s_o-1}q(\tau)\,\right]$$

where $q(\tau)$ is the last column of $\mathcal{O}(\tau)^{-1}$ and $\tilde{\mathcal{L}}(\cdot)$ is the differential operator

$$\tilde{\mathcal{L}}q(\tau) := -\dot{q}(\tau)\,.$$

It turns out (see Bestle and Zeitz (1983)) that $P^{-1}(\tau)$ is non-singular in the interval, and is continuously differentiable by construction. Furthermore, the change of variables

$$\xi_o \ \mapsto\ \chi_o = P(\tau)\xi_o, \qquad \xi_{no} \ \mapsto\ \chi_{no} = \xi_{no},$$

transforms system (20) into the following form.

$$\begin{aligned} \dot{\chi}_o &= A\chi_o + r(\tau)C\chi_o \\ \dot{\chi}_{no} &= 0 \\ \chi_o^+ &= M_o\chi_o + M_{ono}\chi_{no} \\ \chi_{no}^+ &= (N_{no} + K_2N_{ono})\chi_{no} + M_{noo}\chi_o \\ y_w &= C\chi_o \end{aligned} \tag{23}$$

where $r(\tau)$ is appropriately defined,

$$A = \begin{pmatrix} 0_{1\times s_o-1} & 0 \\ I_{s_o-1} & 0_{s_o-1\times 1} \end{pmatrix}, \quad C = \left(\, 0_{s_o-1}\ 1\,\right),$$

and

$$\begin{aligned} M_o &:= P(0)\bar{N}_oP^{-1}(\tau_{\max}) \\ M_{ono} &:= P(0)\bar{N}_{ono} \\ M_{noo} &:= \bar{N}_{noo}P^{-1}(\tau_{\max})\,. \end{aligned}$$

It should also be noted that the change of variables is a Lyapunov Transformation if the following assumption holds. This guarantees that the transformed system will have the same stability properties as the original system.

Assumption 5. The matrices $P(\tau)$ and $P^{-1}(\tau)$ are bounded for all $\tau \in [0, \tau_{\max}]$.

Proposition 4. Under assumptions 4 and 5, let system (8) be taken as

$$\begin{aligned} \dot{\eta}_o &= A\eta_o + r(\tau)y_w + K_1(C\eta_o - u) \\ \dot{\eta}_{no} &= 0 \\ \eta_o^+ &= M_o\eta_o + M_{ono}\eta_{no} \\ \eta_{no}^+ &= (N_{no} + K_2N_{ono})\eta_{no} + M_{noo}\eta_o \\ y_\eta &= C\eta_o \end{aligned} \tag{24}$$

in which

$$K_1 = \left(\, c_{n_o}\ell^{n_o}\ c_{n_o-1}\ell^{n_o-1}\ \ldots\ c_1\ell\,\right)^T$$

with the c_i's chosen as coefficients of a Hurwitz polynomial and where ℓ is a design parameter. Then there exists an $\ell^\star \geq 1$ such that for all $\ell \geq \ell^\star$ there exists a differentiable function $\Pi_\eta(\tau)$ satisfying the properties of Proposition 1, that is the set $\mathcal{S}$ in (9) is globally exponentially stable for (10) and (11).

4. CONCLUSIONS

We have given a general internal model design in the context of hybrid output regulation for linear systems. The key notion of visibility introduced for linear time varying systems has been used in order to obtain a systematic design procedure of the regulator. The proposed procedure is also able to systematically deal with uncertainties in the regulated plant as long as they affect the steady state control map in a linear way. Future works on the subject will attempt to extend the theory to nonlinear hybrid systems.

REFERENCES

Bestle, D. and Zeitz, M. (1983). Canonical form observer design for non-linear time-variable systems. *International Journal of Control*, 38(2), 419–431.

Carnevale, D., Galeani, S., and Menini, L. (2012a). Output regulation for a class of linear hybrid systems. part 1: trajectory generation. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, 6151–6156.

Carnevale, D., Galeani, S., and Menini, L. (2012b). Output regulation for a class of linear hybrid systems. part 2: stabilization. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, 6157–6162.

Carnevale, D., Galeani, S., and Menini, L. (2013). A case study for hybrid regulation: Output tracking for a spinning and bouncing disk. In *Control Automation (MED), 2013 21st Mediterranean Conference on*, 858–867.

Cox, N., Marconi, L., and Teel, A. (2011). Hybrid output regulation with unmeasured clock. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, 7410–7415.

Cox, N., Marconi, L., and Teel, A. (2012). Hybrid internal models for robust spline tracking. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, 4877–4882.

Cox, N., Marconi, L., and Teel, A. (2013). High-gain observers and linear output regulation for hybrid exosystems. *International Journal of Robust and Nonlinear Control*.

D'Alessandro, P., Isidori, A., and Ruberti, A. (1973). A new approach to the theory of canonical decomposition of linear dynamical systems. *SIAM Journal on Control*, 11(1), 148–158.

Francis, B. and Wonham, W. (1976). The internal model principle of control theory. *Automatica*, 12(5), 457 – 465.

Goebel, R., Sanfelice, R., and Teel, A. (2009). Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2), 28 –93.

Marconi, L. and Teel, A. (2013). Internal model principle for linear systems with periodic state jumps. *Automatic Control, IEEE Transactions on*, 58(11), 2788–2802.