Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Robust High Order Sliding Mode Optimization for Linear Time Variant Systems ⋆

Manuel Jimenez-Lizarraga * **Efrain Ibarra** ??

* *Faculty of Physical and Mathematical Sciences, Autonomous University of Nuevo Leon, San Nicolás de los Garza, N.L., Mexico (e-mail: manalejimenez@ yahoo.com).*

Abstract: This paper studies the problem of higher order singular linear quadratic optimization for Linear Time Variant Systems affected by some sort of uncertainties. It is shown the natural connection between the order of singularity of the time varying quadratic cost and the order of Sliding Mode. An integral high order sliding mode is proposed to reach the corresponding higher order singular time varying optimal manifold in prescribed time. The transformation to the phase-variable form for the Linear Time Variant Systems becomes the key step solve the problem and the proposed solution provides insensitvity of trajectory w.r.t. matched bounded perturbation.

Keywords: Sliding Mode Control, Variable Structure Systems, Singular Optimal Control

1. INTRODUCTION

Singular Optimal Control (SOC) is well a known field of research (see (1), (8), (3)) motivated by problems that emerge in a variety of practical situations ranging from economy to aeronautics ((5), (17) and (10)). In this last field particularly the control engineers face the problem of time changing parameters such as variable mass of rockets, probes and tanks. Therefore these problems are better modeled by models with time variable coefficients. The main results in SOC were conceived during the 60th and 70th. At that time, one of the common approach to solve SOC problems, involves a transformation of the states and control variables from which a reduced dimension system is obtained and the original singular problem is transformed to a regular (non-singular) control problem in the new variables, as a result it is possible to find, the so-called, higher order singular manifold (SOM). If that does not hold, then a further transformation to a state space of smaller dimension is needed (see (11), (8) and (1)). Other approaches include the use of impulse control along with first order sliding mode controls to reach, the higher SOM, this is the surface in the state space in which the cost function achieve its minimum. It is suggested also the concept of cheap control converting the singular optimization stabilization (SOS) problem into singularly perturbed LQ optimal problems ensuring the fast convergence of the solutions to the SOM (see for example (3), (12), (18)). Most of this approaches studied time invariant systems.

In this paper we study the problem of singular control problem of higher order, this is, problems with arbitrary reduced dimension of the associated system for a linear plant with time-varing parameters. In the book of ((19) chapter 6), this problem was first studied and a time-varying sliding surface based on a time varying SOM (TVSOM) was designed, then an order one sliding mode is applied to solve the problem. However some importante aspects was not mentioned by that approach, such as: problems for singular system with smaller dimension or higher order problems, the starting time at the TVSOM was not specified and the robustnes of the solution w.r.t to bounded perturbation is not mention.

Contribution: In this paper we propose an Integral Higher Order Sliding Mode (IHOSM) approach ((14)) to solve arbitrary order singular LQ problems for uncertain time-varying systems. The main contributions are the follows:

- a notion of order of singularity of time-varying LQ problem is introduced;
- the natural connection between order singularity and order of sliding mode is shown;
- based on the order of singularity the integral quasi-continuous HOSM algorithm is designed allowing to:
  - reach the TVSOM in a desired time instant;
  - maintain the system solution on the TVSOM;
  - ensure the insensitivity of the system trajectory with respect to the bounded matched uncertainties.

2. GENERAL CASE

Consider the following perturbed linear time varying system

$$\begin{gathered}\dot{x} = A(t)\,x + B(t)\,(u + \zeta)\,, \\ x(t_0) = x_0,\end{gathered} \tag{1}$$

⋆ The authors wish to thank for the grants CONACYT 169734 as well as PAICyT CE851-11.

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ is the scalar control input, and $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times 1}$,are time varying matrices and the perturbation satisfies

$$|\zeta(t,x)| \le L,\ L > 0. \quad (2)$$

is bounded and Lebesgue-measurable on t . Moreover, here and always bellow we will suppose that the solution of the system (1)-(2) is unique in the sense of Filippov ((6)) for all $t \ge t_0$. Assume the system is uniformly controllable (see (2)). The optimization problem for dynamics (1) could not be not well posed because the system (1) is uncertain.

That is why together with system (1) consider the nominal singular finite time LQ problem with time coefficients dynamics:

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0, \quad (3)$$

We consider a quadratic in the states and a *"control-free"* cost objective where the weight matrix is also time variable, i.e.,

$$J = \frac{1}{2}\int_{t_1}^{t_f} x^{\intercal}(t)\, Q(t)\, x(t)\, dt, \quad (4)$$

where $Q(t) = Q^{\intercal}(t) \ge 0$, is a semipositive definite matrix for $\forall t \ge t_0$ and $t_1 > t_0$, t_0 is the initial time instant. The cost function does not depend on control that is why this problem leads to the solution of the singular optimal control problem.

The solution of nominal finite time singular problem (3)-(4) lies on the time variable singular optimal manifold.

3. HIGHER ORDER SINGULAR OPTIMIZATION FOR NOMINAL LTVS

Transformation to Phase-Variable canonical form

Let us consider that the system (3) as a uniformly controllable (9). The following phase-variable transformation is based on the procedure given in ((16)), consider the nonlinear transformation

$$y(t) = T(t)\, x(t), \quad (5)$$

applying into (3) yields to

$$\dot{y} = \left(T(t)A(t) + \dot{T}\right)T(t)^{-1}y + T(t)B(t)u \quad (6)$$

where $T(t)$ is matrix such that $A_c = \left(T(t)A(t) + \dot{T}(t)\right)T(t)^{-1}$ and $B_c = T(t)B(t)$ given by:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_1(t) & \beta_2(t) & \beta_3(t) & \cdots & \beta_{n-1}(t) & \beta_n(t) \end{pmatrix},\ B_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (7)$$

which transform the system (3) into the canonical form:

$$\dot{y} = A_c(t)\, y + B_c u \quad (8)$$

in its turn the cost function (4) it is transformed into

$$J = \frac{1}{2}\int_{t_1}^{t_f} y^T \tilde{Q}(t)\, y(t)\, dt \quad (9)$$

where $\tilde{Q}(t) = \left(T^{-1}\right)^T Q(t)\, T^{-1} Q(t)$ and the cost is transform to following block structure:

$$\tilde{Q}(t) = \begin{pmatrix} \tilde{Q}_{11}(t) & \tilde{Q}_{12}(t) & 0 & \cdots & 0 \\ \tilde{Q}_{21}(t) & \tilde{Q}_{22}(t) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad\quad 0}_{n-k\ columns} & & \underbrace{0 \ \cdots \ 0}_{k-columns} \end{pmatrix} \quad \begin{array}{c}\tilde{Q}_{22}(t) > 0,\ \tilde{Q}_{21}(t) \ge 0 \\ \forall t \ge t_0\end{array} \quad (10)$$

the involved block matrices have dimension $\tilde{Q}_{22}(t) \in \mathbb{R}$, $\tilde{Q}_{11}(t) \in \mathbb{R}^{(n-k-1)\times(n-k-1)}$, $\tilde{Q}_{12}(t) = \tilde{Q}_{21}^{\intercal}(t) \in \mathbb{R}^{1\times(n-k-1)}$. The previous example in which the control variable can appear in higher derivative of the TVSOM motivate the following definition. For a given singular control problem, the **Order of Singularity** can be seen as $i = k+1$, and k is the number of zero columns in (10). For the porpouse of clarity, we define the vectors $\tilde{y}_1 = (y_1, y_2, ..., y_{n-k-1})^{\top}$ that represent the sliding mode reduced dynamics, and $y_2 = y_{n-k}$ a variable virtual control.

Procedure to find the transformation $T(t)$ We assume the entries of the matrices $A(t)$ and $B(t)$ to be infinitely differentiable functions. In this section we omit the time-dependent notation some places where it is clear from the previous context.

As is shown in ((16)) the necessary and sufficient condition for the unique existence of the transformation matrix $T(t)$ is the uniform controllability of the system (3) (see (16)). In what follows we outline the procedure to find $T(t)$. Let the components of the vector $y(t)$ in (8) be:

$$\begin{aligned} y_1(t) &= T_{11}x_1 + Tx_2 + \cdots + T_{1n}x_n = T_1 x \\ y_2(t) &= T_{21}x_1 + T_{22}x_2 + \cdots + T_{2n}x_n = T_2 x \\ &\vdots \\ y_n(t) &= T_{n1}x_1 + T_{n2}x_2 + \cdots + T_{nn}x_n = T_n x \end{aligned} \quad (11)$$

where $x(t)$ is the state vector of (3), and $T_{ij}(t)$, $(i,j = 1,2,\cdots,n)$ are the components of the time varying rows of $T(t)$, that is

$$T_i(t) = (T_{i1}, T_{i2}, \ldots, T_{in})$$

taking the time derivative of (5) we get

$$\dot{y}(t) = \left(\dot{T} + TA\right)x(t) + TB_c u(t) \quad (12)$$

the system (3) in canonical variables (6) makes the row of (12) equal to:

$$\begin{aligned} y_2 = \dot{y_1} &= \left(\dot{T_1} + T_1 A\right) x + T_1 B u\left(t\right) \\ y_3 = \dot{y_2} &= \left(\dot{T_2} + T_2 A\right) x + T_2 B u\left(t\right) \\ &\vdots \\ y_n = \dot{y_{n-1}} &= \left(\dot{T_{n-1}} + T_{n-1} A\right) x + T_{n-1} B u\left(t\right) \end{aligned} \tag{13}$$

making $T_1 B = T_2 B = \ldots = T_{n-1} B = 0$,the components of $T\left(t\right) x\left(t\right)$ becomes

$$\begin{aligned} T_2 x &= \left(\dot{T_1} + T_1 A\right) x \\ T_3 x &= \left(\dot{T_2} + T_2 A\right) x \\ &\vdots \\ T_n x &= \left(\dot{T_{n-1}} + T_{n-1} A\right) x \end{aligned} \tag{14}$$

Therefore in order to find the matrix $T\left(t\right)$ the row must satisfy:

$$T_k = \left(\dot{T_{k-1}} + T_{k-1} A\right) \text{ for } k = 2, \ldots, n \tag{15}$$

along with the equalities $T_1 B = T_2 B = \ldots = T_{n-1} B = 0$. Also because of (8) $TB = B_c$ we obtain

$$\begin{pmatrix} T_1 B \\ T_2 B \\ \vdots \\ T_{n-1} B \\ T_n B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{16}$$

Transformation of the functional

The cost function (9) involves cross terms in the state variables, in order to get a simpler problem, define a new auxiliary variable ν_i ((19)):

$$\nu_i\left(t\right) = y_2 + L_i\left(t\right) \tilde{y}_1; \quad L_i\left(t\right) := \left(\tilde{Q}_{22}\left(t\right)\right)^{-1} \tilde{Q}_{12}\left(t\right) \tag{17}$$

Using the definition of for the states and the control variables $\left(\mathbf{z}_1, \nu_i\right)$, the cost function (9) the is transform to:

$$J = \frac{1}{2} \int_{t_1}^{t_f} \left(\tilde{y}_1^{\intercal}(t) \hat{Q}_{11}\left(t\right) \tilde{y}_1(t) + \nu_i^{\intercal}(t) \tilde{Q}_{22}\left(t\right) \nu_i(t)\right) dt. \tag{18}$$

where the matrix

$$\hat{Q}_{11}\left(t\right) = \tilde{Q}_{11}\left(t\right) - \tilde{Q}_{12}\left(t\right) \left(\tilde{Q}_{22}\left(t\right)\right)^{-1} \tilde{Q}_{12}^{\intercal}\left(t\right).$$

Design of the TVSOM

For every high order singular optimization problem for LTVS we have the following result. For any order of singularity i, minimize the performance index (9), subject to the dynamics (8). Notice that in the system given by canonical variables (8) only the last coordinate is time dependent, nonetheless the singular optimal manifold (for every singular problem) remains time variable due to associated matrices of the cost function. To find out the optimality conditions for each index, we take again the last coordinate as the virtual control variable and minimize (18), subjected to the partial dynamics

$$\dot{\tilde{y}}_1 = \left(A_i + B_i \left(\tilde{Q}_{22}\left(t\right)\right)^{-1} \tilde{Q}_{12}^{\intercal}\left(t\right)\right) \tilde{y}_1 + B_i \nu_i\left(t\right) \tag{19}$$

where A_i and B_i are constant matrices with form:

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{(n-i)\times(n-i)}; \; B_i = \left.\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right\} \; n-i \; rows$$
$$with \;\; \tilde{y}_1 \in \mathbb{R}^{n-i}, \; \tilde{Q}_{22}\left(t\right) \in \mathbb{R}, \quad \tilde{Q}_{12}\left(t\right) \in \mathbb{R}^{n-i} \tag{20}$$

the dimension of the matrices are well defined. The next theorem is a higher order extension of the results concerning the

first-order SOS problem ((19)).

Theorem 1. The optimal value acting as virtual minimizing control in (9) is

$$y_2 = -\left(\tilde{Q}_{22}\left(t\right)\right)^{-1} \left(B_i P_i + \tilde{Q}_{12}^{\intercal}\right) \tilde{y}_1 \tag{21}$$

where P_i, is the solution of the differential matrix Riccati

$$\begin{aligned} -\dot{P}_i\left(t\right) = P_i\left(t\right) \tilde{A}_i\left(t\right) + \tilde{A}_i^{\intercal}\left(t\right) P_i\left(t\right) + \hat{Q}_{11}\left(t\right) \\ -P_i\left(t\right) B_i \left(\tilde{Q}_{22}\left(t\right)\right)^{-1} B_i^{\intercal} P_i\left(t\right) \end{aligned} \tag{22}$$

with boundary conditions $P_i\left(t_f\right) = 0$, where:

$$\tilde{A}_i\left(t\right) = A_i + B_i \left(\tilde{Q}_{22}\left(t\right)\right)^{-1} \tilde{Q}_{12}\left(t\right);$$
$$\hat{Q}_{11}\left(t\right) = \tilde{Q}_{11}\left(t\right) - \tilde{Q}_{12}\left(t\right) \left(\tilde{Q}_{22}\left(t\right)\right)^{-1} \tilde{Q}_{12}^{\intercal}\left(t\right)$$

follows from the standard optimal control theory.

Design of Transient Trajectory to Reach SOM

In the general case the cost function (9) is not optimized from the begginig of the process (8), the initial condition of does not belong to the SOM, therefore the initial time t_1 should be specified. Consider the SOM as

$$S_i\left(y, t\right) = y_2 + M_i\left(t\right) \tilde{y}_1 = 0 \tag{23}$$

Assume that art least $j, 0 \leq j \leq i-1$ we have $S_i^{(j)}\left(y_0, t_0\right) \neq 0$.We would like for the system trajectories reaches the ith order sliding mode set $S_i(y(t_1)) = \dot{S}_i(y(t_1)) = ... = S_i^{(i-1)}(y(t_1)) = 0$ at the reaching time t_1.

Let define the transient trajectory $\mu_i(t)$ defined for the interval time $t_0 \le t \le t_1$ as a polynomial of the form:

$$\mu_i(t) = (t - t_1)^i \times \left(c_0 + c_1(t - t_0) + ... + c_{i-1}(t - t_0)^{i-1}\right). \tag{24}$$

satisfying the initial conditions

$$\mu_i(t_0) = S_i(y_0), \dot{\mu}_i(t_0) = \dot{S}_i(y_0), ..., \mu_i^{(i-1)}(t_0) = S_i^{(i-1)}(y_0). \tag{25}$$

At the arrival time on SOM t_1 we have

$$\mu_i(t_1) = \dot{\mu}_i(t_1) = ... = \mu_i^{(i-1)}(t_1) = 0$$

and $\mu_i(t) = 0\ \forall t > t_1$. The parameters c_i could be uniquely defined from (25). Define the function $t_1 - t_0$ as a positive-defined function of the initial conditions as

$$t_1 - t_0 = T_i\left(S_i(y_0), \dot{S}_i(y_0), ..., S_i^{(i-1)}(y_0)\right); \tag{26}$$

For any $\lambda, p =$const> 0 the function T_i could be uniquely defined as (14) :

$$T_i = \lambda\left(|S_i(y_0)|^{p/i} + \left|\dot{S}_i(y_0)\right|^{p/(i-1)} + \ldots + \left|S_i^{(i-1)}(y_0)\right|^p\right)^{1/p} \tag{27}$$

The function $\mu_i(t)$ is uniquely determined by (25), (24) and (26).

We define an auxiliary surface $\sum_i(y) = S_i(y) - \mu_i(t)$, in such a way when the ith order quasicontinuous controller achieves to do that $\sum_i(y)$ is zero in some finite time t_r, we have that $S_i(y(t_r)) = \mu_i(t_r)$, in addition we remember $\mu_i(t \geqslant t_1) = 0$, and if in addition there is fulfilled that $t_1 \geqslant t_r$, then the time of convergence of S_i will be the same one in which the polynomial μ_i does zero to itself for the first time. that is to say t_1.

IHOSM Design

Now from ((15)). we can conclude following theorem:

Theorem 2. The controller

$$v_i = \Phi_i \Psi_{i-1,i}(\Sigma_i, \dot{\Sigma}_i, ..., \Sigma_i^{(i-1)});\ \varphi_{0,i} = \Sigma_i;\ N_{0,i} = |\Sigma_i|; \tag{28}$$

$$\Psi_{0,i} = \varphi_{0,i}/N_{0,i} = sign\Sigma_i;\ \varphi_{l,i} = \Sigma_i^{(l)} + \beta_i N_{l-1,i}^{(i-l)/(i-l+1)}\Psi_{l-1,i}$$

$$N_{l,i} = \left|\Sigma_i^{(l)}\right| + \beta_i N_{l-1,i}^{(i-l)/(i-l+1)};\ \ \Psi_{l-1,i} = \varphi_{l,i}/N_{l,i},$$

$$\Sigma_i(t, y) = \begin{cases} S_i(y) - \mu_i(t), & t_0 \le t \le t_1 \\ S_i(y), & t \ge t_1 \end{cases} \tag{29}$$

established the finite-time stable r-sliding mode $S_i(y_0) = \dot{S}_i(y(t)) = ... = S_i^{(i-1)}(y(t)) \equiv 0$ for $t \ge t_1$.

The equality $S_i(y(t)) = \mu_i(t)$, is kept during the transient process $t_1 \ge t \ge t_0$.

Remark 2.

The control quasi-continuous higher order controller for $i - 1, 2, 3, 4$ takes the form :

$$v_1 = -\Phi_1 sign\Sigma_1,$$

$$v_2 = -\Phi_2\left(\dot{\Sigma}_2 + |\Sigma_2|^{1/2} sign\Sigma_2\right) / \left(\left|\dot{\Sigma}_2\right| + |\Sigma_2|^{1/2}\right),$$

$$v_3 = -\Phi_3 \frac{\left[\ddot{\Sigma}_3 + 2\left(|\dot{\Sigma}_3| + |\Sigma_3|^{2/3}\right)^{-1/2}\left(\dot{\Sigma}_3 + |\Sigma_3|^{2/3} sign\Sigma_3\right)\right]}{\left[|\ddot{\Sigma}_3| + 2(|\dot{\Sigma}_3| + |\Sigma_3|^{2/3})^{1/2}\right]},$$

$$v_4 = -\Phi_4 \varphi_{3,4}/N_{3,4},$$

$$\varphi_{3,4} = \dddot{\Sigma}_4 + 3\left[\left|\ddot{\Sigma}_4\right| + \left(\left|\dot{\Sigma}_4\right| + 0.5|\Sigma_4|^{3/4}\right)^{2/3}\right]^{1/2} \times \left[\ddot{\Sigma}_4 + \left(\left|\dot{\Sigma}_4\right| + 0.5|\Sigma_4|^{3/4}\right)^{-1/3}\left(\dot{\Sigma}_4 + 0.5|\Sigma_4|^{3/4} sign\Sigma_4\right)\right] \tag{30}$$

and

$$N_{3,4} = \left|\dddot{\Sigma}_4\right| + 3\left[\ddot{\Sigma}_4 + \left(\left|\dot{\Sigma}_4\right| + 0.5|\Sigma_4|^{3/4}\right)^{-2/3}\right]^{1/2}, \tag{31}$$

4. DESCRIPTION OF THE ALGORITHM

To summarize the procedure design we have the following algorithm:

Step 1: Transformation of the system (1) into phase-variable form (8).
Step 2: Transformation of the functional (4) into transformed form (9) .
Step 3: Solving analytical or numerically the corresponding Riccati differential equation (22) in order to design the time-variable surface $S_i(y, t)$.
Step 4: Design the auxiliary Surface $\sum_i(y, t) = S_i(y) - \mu_i(t)$ finding the coefficients $c_0, c_1, \ldots, c_{i-1}$ of the polynomial $\mu_i(t)$, using the initial conditions (25).
Step 5: Design a convergence time t_1 with a given initial condition t_0 and a proposed value λ, considering (26) and (27).
Step 6: Design the corresponding IHOSM control.

5. NUMERICAL EXAMPLE

Consider the following LTVS

$$\dot{x} = A(t)x + B(t)(u + \xi), \quad x \in \mathbb{R}^3, u \in \mathbb{R}, \quad x(0) = x_0$$

and the cost function associated as:

$$J=\frac{1}{2}\int_{t_1}^{t_f} y^T\tilde{Q}(t)\,y(t)\,dt$$

where:

$$A=\begin{pmatrix}-1 & e^{-t} & t\\ 1 & -3 & 0\\ 0 & 1 & t^2\end{pmatrix};\quad B=\begin{pmatrix}e^{-t}\\ 0\\ 0\end{pmatrix}$$

$$x\in\mathbb{R}^3;\ \xi=2\cos(2t)+\cos(x_1+x_2+x_3)+1.5;$$

and

$$Q(t)=\begin{pmatrix}0 & 0 & 0\\ 0 & 4e^{2t} & 4e\left(t^2+1\right)\\ 0 & 4e^{2t}\left(t^2+1\right) & 4e^{2t}\left(t^2+1\right)^2+e^{2t}\end{pmatrix}$$

We can see that the perturbation is bounded $|\xi|\le 4.5$ and $Q(t)=Q^{\intercal}(t)\ge 0$. Applying the Step 1 we obtain that the transformation matrix $T(t)$ is

$$T(t)=\begin{pmatrix}0 & 0 & e^t\\ 0 & e^t & \left(t^2+1\right)e^t\\ e^t & \left(t^2-1\right)e^t & \left(t^4+2t^2+2t+1\right)\end{pmatrix}$$

and using the Step 2 the cost function finally becomes

$$J=\frac{1}{2}\int_{t_1}^{t_f}\left(y_1^2+4y_2^2\right)dt \tag{32}$$

subject to the dynamics

$$\dot{y}_1=A_2y_1+B_2y_2$$
$$A_2=0, B_2=1$$

the optimal virtual control y_2 is:

$$y_2=-\frac{1}{4}P_2(t)\,y_1 \tag{33}$$

where $P_2(t)\in\mathbb{R}$ is the solution of the Riccati differential equation:

$$-\dot{P}_2(t)=-P_2(t)^2/4+1 \tag{34}$$

with $P_2(t_f)=0$. If we take $t_f=10$, now following the Procedure of Step 3, the analytic solution is

$$P_2(t)=2\left(1-e^{(t-10)}\right)/\left(e^{(t-10)}+1\right)$$

and the time varying manifold is

$$S_2=y_2+M_2(t)\,y_1$$
$$M_2(t)=\frac{1}{4}P_2(t).$$

the auxiliary surface become:

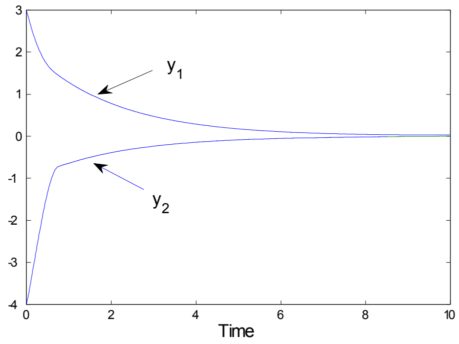

Fig. 1. States of the System.

$$\sum_2(y)=S_2(y)-\mu_2(t) \tag{35}$$

the order of quasicontinuous controller is two, therefore the polynomial (24) takes the form:

$$\mu_2(t)=(t-t_1)^2\left(c_0+c_1(t-t_0)\right)$$

where $t_0=0$, is the initial time of the states variables x, therefore we get

$$\mu_2(t)=(t-t_1)^2(c_0+c_1t). \tag{36}$$

In order to design the auxiliary surface $\sum_2(y)$ we follow the Step 4, then the coefficients of the polynomial $\mu_2(t)$ are given by:

$$c_0=S_2(0)/t_1^2$$

$$c_1=\dot{S}_2(0)/t_1^2+2c_0/t_1$$

now considering Step 5 with a initial condition $t_0=0$ and a proposed value λ, we have that the reaching time is given by:

$$t_1=T_2\left(S_2(0),\dot{S}_2(0)\right)$$

$$T_2=\lambda\left(|S_2(0)|^3+\left|\dot{S}_2(0)\right|^6\right)^{1/6}$$

for the initial condition $y_1(0)=3$, $y_2(0)=-4$, $y_3(0)=5$,with $P_2(10)=0$, and $k_{11}=50$; $k_{12}=50$, $S_2(0)=-2.5001$, $\dot{S}_2(0)=3.0003$ it is found that $c_0=-4.4125$, $c_1=-6.4286$ and taking $\lambda=0.25$; we get $t_1=T_2=0.7527$. Finally using Step 6 the second order quasicontinuous controller is expressed as:

$$u=-\Phi_1(y)\left(\frac{\dot{\Sigma}_2+|\Sigma_2|^{1/2}sign\Sigma_2}{{}^6\left|\dot{\Sigma}_2\right|+|\Sigma_2|^{1/2}}\right)$$

with the variable gain as:

$$\Phi_1(y)=50\sqrt{y_1^2+y_2^2+y_3^2}+50$$

in what follows we show the corresponding figures to the simulation:

7. CONCLUSION

This paper shows natural connection between order of singularity of the singular LQ and order of sliding modes. The

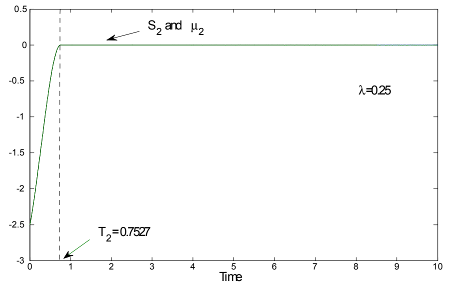

Fig. 2. Reaching Time $t_1 = 0.7527$.

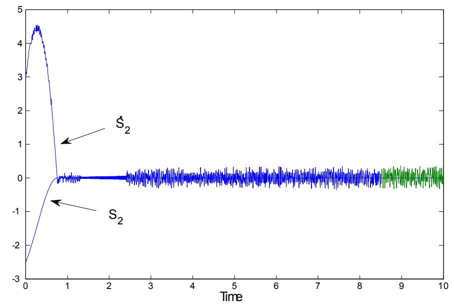

Fig. 3. Sliding Surfaces.

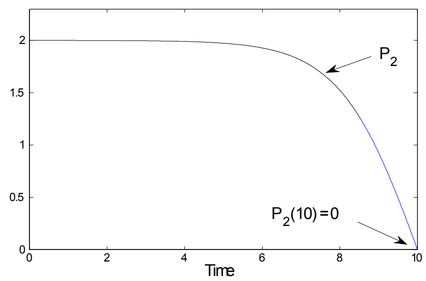

Fig. 4. Solution of Riccati equation P_2.

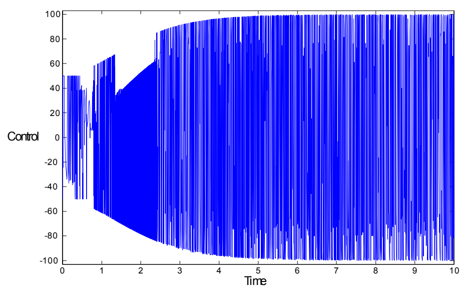

Fig. 5. Second Order Quasicontinuous Controller.

time variable SOM is considered as the sliding manifold for HOSM of corresponding order.

The HOSM algorithm is suggested ensuring:

- that the system trajectory will arrive on the TVSOM;
- maintain the system trajectory on the TVSOM;
- the insensitivity of the system trajectory with respect to the matched uncertainties.

REFERENCES

[1] Bell, J.B. and Jacobson, D.H., Singular Optimal Control Problems. Academic Press, London, 1975.

[2] Chen, C.T., Linear System Theory and Design. CBS College Pub, 1984.

[3] Dmitriev, M.G. and Kurina, G.A., Singular Perturbations in Control Problems. *Automation and Remote Control,* 67, 3-51.

[4] Edwards, C. and Spurgeon, S., Sliding Mode Control: Theory and Applications. Taylor and Francis, 1998.

[5] Faulkner, F.D., *The problem of Goddard and optimum thrust programming, Advances in the Astronautical Sciences vol I.* Plenum Press, 1957.

[6] Filippov, A.F., *Differential equations with discontinuous right hand-sides, Mathematics and its applications,*Kluwer Academic Publisher, 1983.

[7] Fridman, L. and Jimenez-Lizarraga, M. and Ibarra, E.,Integral Higher Order Sliding Mode and Singular Optimal Stabilization. *Proceedings of the European Control Conference 2013.*

[8] Ho, Y.C., Linear stochastic singular control problems. *J. Optim. Theory Appl, 9, 24-31* 1972.

[9] Isidori, A., *Nonlinear Systems.* Springer-Verlang, 1995.

[10] Johnson, C. D. and Gibson, J.E., Singular solutions in problems of optimal control, *IEEE Trans. Automat. Contr,* 8, 4-15, 1963.

[11] Kelley, H. J., A transformation approach to singular subarcs in optimal trajectory and control problems. *SIAM J. Control, Ser. A,* 2, 234-240, 1965.

[12] Lee, J.T. and Bien, Z.N., A Quadratic Regulator with Cheap Control for a Class of Nonlinear Systems. *Journal of Optimization Theory and Applications. Vol 55, 289-302, 1987.*

[13] Levant, A., Quasi-continuos high-order sliding-mode controllers. *IEEE Trans. Automat. Contr, 50, 1812-1186, 2006.*

[14] Levant, A. and Alelishvili, L., Integral High Order Sliding Modes. *IEEE Trans. Automat. Contr,* 1278-1282, 2007.

[15] Levant, A. and Michael, A., Adjustment of high-order sliding-mode controllers. *International Journal of Robust and Nonlinear Control, 19,* 1657-1672, 2008.

[16] Ramaswami, B. and Ramar, K., On the Transformation of Time-Variable Systems to the Phase-Variable Canonical Form. *IEEE Trans. Automat. Contr,* 44, 417-419, 1969.

[17] Ross, S., Minimality for problems in vcrtical and horizontal rocket flight. *Jet Propulsion,* 28, 55-56, 1958.

[18] Saberi, A. and Sannuti, P., Time-Scale Decomposition of a Class of Linear and Nonlinear Cheap Control Problems. *Proc. Am. Control Conf.* 1414-1421, 1985.

[19] Utkin, V., Slides Modes in Control and Optimization. Springer-Verlang, 1991.