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Control of Nonlinear PDEs based on Space Vectors Clustering reduced order systems ⋆

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Abstract: Nonlinear PDEs domain discretization yields finite but high dimensional nonlinear systems. Proper Orthogonal Decomposition (POD) is widely used to reduce the order of such systems but it assumes that data belongs to a linear space and therefore fails to capture the nonlinear degrees of freedom. To overcome this problem, we develop a Space Vector Clustering (SVC) POD and use the reduced order model to design the controller which will then be applied to the full order system. A space vector is the solution at a particular space location over all times. The solution space is grouped into clusters where the behavior has significantly different features, then local POD modes will be constructed based on these clusters. We apply our method to reduce and control a nonlinear convective PDE system governed by the Burgers' equation over 1D and 2D domains and show a significant improvement over global POD. We also design a reference tracking controller and compare the controlled systems. We show that the controller based on our SVC local POD reduced system yields more accurate tracking results over the one based on global POD.

1. INTRODUCTION

Large number of states is needed to accurately capture the dynamics of systems described by partial differential equations. It is computationally difficult to design control laws for such systems [17]. Conventionally the order of the system must be reduced before control law design can be done [2], [6].

Different model reduction approaches have been developed in literature, but only some of them are optimal in some sense. For linear systems, the balanced truncation based on singular value decompositions is one of them. The theory of balanced model reduction was initiated by B.C. Moore for controllable, observable and exponentially stable linear systems in state space form [1].

For nonlinear systems, Proper Orthogonal Decomposition (POD) is a model reduction technique that proved efficient performance when used to reduce models that approximate nonlinear infinite dimensional systems by high order finite dimensional systems, especially those who describe the dynamics of fluid flows [8], [4]. POD is a popular model reduction technique used to alleviate the computational expense required for very high dimensional systems. The POD snapshot method is usually used to create an ensemble of solutions with particular open loop control input data. The set is used to construct a set of POD basis modes [2]. It is well known that the modes maximize energy in mean square sense [8]. In our earlier paper [5], we showed that POD is optimal in a wider sense, which is of a distance minimization in spaces of Hilbert-Schmidt integral operators. These arguments are used to carry out model reduction, and determine its fidelity. From a given POD basis set, only the numbers of modes needed to capture a specified percentage of the total set energy are kept. This argument is problematic in the feedback control setting. Energy of POD modes generated from snapshots incorporating open-loop actuation might not correlate to the energy of the system under feedback control [2], [9].

POD fails to capture the nonlinear degrees of freedom in nonlinear systems, since it assumes that data belongs to a linear space and therefore relies on the Euclidean distance as the metric to minimize [7]. However, snapshots generated by nonlinear partial differential equations (PDEs) belong to manifolds for which the geodesics do not correspond in general to the Euclidean distance. A geodesic is a curve that is locally the shortest path between points. The global nonlinear manifold geodesic is difficult to be quantified in general but we show in this paper that it can be approximated efficiently by local linear Euclidean distances [16]. In [20], authors showed the poor performance of using global POD. They introduced a nonlinear model reduction approach via nonlinear projection framework. In [21], authors used the time domain partitioning approach and they presented a method for treating model reduction of evolution problems. This was realized by an adaptive partitioning of the time domain into several intervals and creating specialized reduced bases with limited size on each of the intervals.

In [18] the solution snapshots were partitioned into subregions that characterize the nonlinear features of the solutions of interest. This partitioning was performed by computing snapshots of the solution, clustering them according to their relative distances using the k-means algorithm, computing in each cluster a reduced-order basis using POD method, identifying each snapshot cluster with a sub-region of the solution space, and assigning to

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this sub-region the reduced-order basis computed in that cluster.

In this paper we introduce an improved POD technique and we call it Space Vectors Clustering (SVC) POD. We define the space vector to be the solution over all times at a particular space location. For infinite dimensional systems described by PDEs, the large number of states comes from the discretization of the space domain, not the time domain. This gives the advantage to space clustering over snapshot clustering presented in [18].

This paper is organized as follows. In section (2), the Burgers' equation in both 1D and 2D domains are numerically solved using finite difference techniques to construct the full order model for comparison purposes. In section (3) we reduce the full order system using global POD. In section (4) we show the performance when SVC POD reduced order bases are constructed based on space clustering. In section (5) we design a tracking controller based on the reduced order system and apply it to the full order system, and finally section (6) is the conclusion and future work.

2. THE BURGERS' EQUATION

The Burgers' equation is a nonlinear PDE with a quadratic type nonlinearity. POD assumes Euclidean distance minimization which is not the case in the nonlinear Burgers' equation, and that is the reason why POD fails to reduce the order of the system efficiently. The 1-D Burgers' equation is given by [19]:

$$\frac{du}{dt} + u\frac{du}{dx} = \frac{d^2u}{dx^2} \tag{1}$$

where $x \in [a, b]$ and $t \in [0, T]$ for some initial condition $u(x, 0) = u_0(x)$ and boundary conditions $u(a, t) = u_a(t)$ and $u(b, t) = u_b(t)$. Discretization of the space domain into N points and using finite difference approximations for the space derivatives yields:

$$\frac{du_i}{dt} + u_i\frac{u_i - u_{i-1}}{\Delta x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \tag{2}$$

where $\lim_{\Delta x \to 0} \Delta x = \frac{1}{N}$ and $i = 0 \cdots N$.
Writing (2) in the matrix form we have:

$$\dot{u}(t) = Au + N(u) + Bu_{a,b} := f(u, u_{a,b}) \tag{3}$$
$$u(0) = u_0$$

where Au is the linear term $\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}$ and $N(u)$ is the nonlinear term $u_i\frac{u_i-u_{i-1}}{\Delta x}$, $Bu_{a,b}$ is the boundaries term which will be used in the boundary control process and u_0 is system the initial value. Full order system is solved in Matlab with $N = 500$, $x \in [0, 100]$, $t \in [0, 50]$, Dirichlet boundary condition $u_0(t) = 2$ and Newman boundary condition $\frac{du_{100}(t)}{dx} = 0$.

For the 2D Burgers' equation system, (1) becomes:

$$\frac{du}{dt} + u(\frac{du}{dx} + \frac{du}{dy}) = \frac{1}{R_e}(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) \tag{4}$$

where x and y are the spacial variables, t is the time variable and R_e is a constant that is analogues to the Reynolds number that appears in the Navier Stokes equations. The 2D Burgers' equation PDE shares the same nonlinearity as the Navier stokes PDE. It has the same quadratic nonlinearity and can be used to model incompressible fluid flows. 2D Burgers' equation full order system is solved in Matlab with $N = 2000$ and $R_e = 300$, on the space domain shown in Fig. (1) which shows the solution at $t = 30$ seconds. This domain models the velocity of a fluid with a constant Dirichlet parabolic velocity at the left boundary that is maximum in the middle and zero in the top and bottom. The fluid passes around an obstacle to show the velocity behavior that the 2D Burgers' equation models.

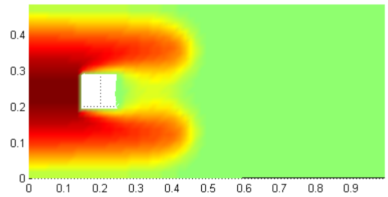

Fig. 1. 2D Full order solution at t=30

3. GLOBAL PROPER ORTHOGONAL DECOMPOSITION

Let the number of snapshots be M, the $M \times M$ correlation matrix L is defined by [2]:

$$L_{i,j} = \langle S_i, S_j \rangle \tag{5}$$

is constructed, where $\langle , \rangle$ denotes the Euclidean inner product of snapshots S. With R denotes the number of POD modes to be constructed, the first R eigenvalues of largest magnitude, $\{\lambda\}_{i=1}^{R}$, of L are found. They are sorted in descending order, and their corresponding eigenvectors $\{v\}_{i=1}^{R}$ are calculated. Each eigenvector is normalized so that

$$\|v_i\|^2 = \frac{1}{\lambda_i} \tag{6}$$

The orthonormal POD basis set $\{\phi_i\}_{i=1}^{R}$ is constructed according to [8]:

$$\phi_i = \sum\nolimits_{j=1}^{M} v_{i,j} S_j \tag{7}$$

where $v_{i,j}$ is the j^{th} component of v_i. The first four global POD modes are shown for the 1D and 2D domains in Fig. (2) and (3) respectively.

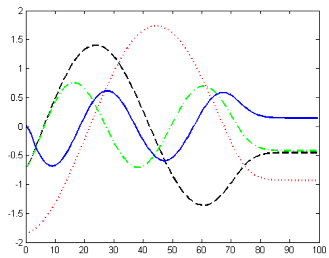

Fig. 2. First Four 1D Global POD Modes

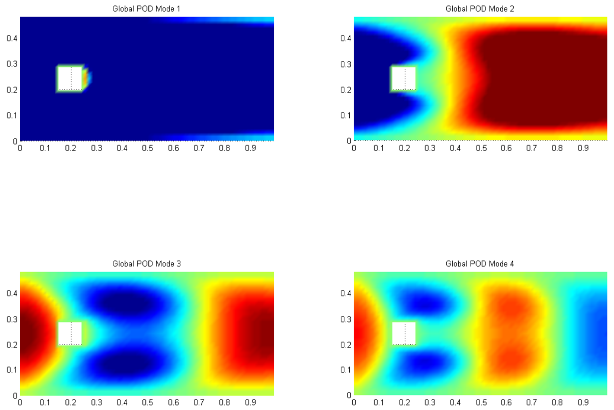

Fig. 3. First Four 2D Global POD Modes

With a POD basis in hand, the solution U of the distributed parameter model is approximated as a linear combination of POD modes, i.e.,

$$U \approx \sum_{i=1}^{R} \alpha_i \phi_i \tag{8}$$

This shows that POD finds a low dimensional embedding of the snapshots that preserve most of the energy as measured in a much higher dimensional solution space.

Comparison between reduced order system using Global POD and full order system for the 1D Burgers' equation is shown in Fig. (4) with full number of states $N = 500$ is reduced to $R = 15$ where the space domain is $x \in [0, 100]$ and the time domain is $t \in [0, 50]$. The Figure shows the comparison at $t = 30$. The error norm between the full order and the reduced order systems is found to be 0.3080

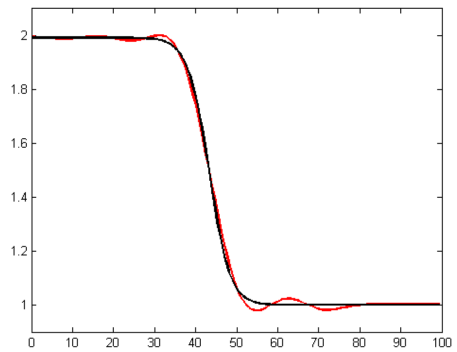

Fig. 4. Full order solution Vs Global POD at t=30

For the 2D problem, the reduced order system is shown in Fig. (5) with full number of states $N = 2000$ is reduced to $R = 50$. The Figure shows the comparison at $t = 30$. The error norm between the full order and the reduced order systems is found to be 0.001. With this small error, full and reduced order models look identical because the reduced number of states ($R = 50$) is still relatively high. R can be reduced even more using our SVC POD method discussed in the next section.

4. SPACE VECTOR CLUSTERING (SVC) POD

The solution space domain is partitioned into clusters where the solution exhibits significantly different features.

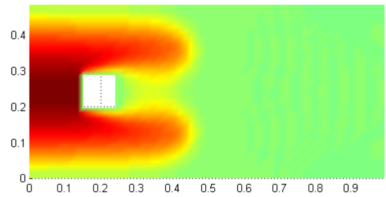

Fig. 5. 2D Global POD Reduced system at t=30

A cluster is a group that contains states which are close in some defined distance. Local bases are pre-computed and assigned to each cluster. The set of pre-computed solution space domain is partitioned into T clusters using K-means clustering algorithm discussed in the next subsection.

4.1 Clustering Using K-Means Algorithm

K- means algorithm groups together nearby locations according to their relative clustering distances. Note that the distance here is not spacial, it refers to the solution difference between two locations for all times. Therefore, it is possible that two locations separated by a large spacial distance might be SVC neighbors and they will be grouped in the same cluster if their solution values are close. The clustering distance is defined as follows:

$$d(W_i, W_j) = \sqrt{(W_i - W_j)^T (W_i - W_j)} \tag{9}$$

where d is the 2- norm distance between two vectors W_i and W_j. These vectors contain the solution at locations i and j respectively for all times.

Suppose we want to group N space vectors $\{W_i\}_{i=1}^{N}$ into T clusters $\{\chi_j\}_{j=1}^{T}$, we first randomly choose T space vectors as centroids $\{W_{c_j}\}_{j=1}^{T}$. Then the distance between each space vector and the centroid is calculated as:

$$d(W_i, W_{c_j}) = \sqrt{(W_i - W_j)^T (W_i - W_{c_j})} \tag{10}$$

Let c_i be the argument of the minimum distance between W_i and W_{c_j}:

$$c_i = \arg \min_{j=1,\cdots T} d(W_i, W_{c_j}) \tag{11}$$

Then the new centroids would be:

$$W_{c_j} = \frac{\sum_{i=1}^{N} 1_{c_i=j} W_i}{\sum_{i=1}^{N} 1_{c_i=j}} \tag{12}$$

where $j = 1, \cdots T$ and $1_{c_i=j} = 1$ if $c_i = j$ and zero otherwise. Then the last step is to assign each space vector W_i to the cluster χ_{c_j}.

4.2 Construction of SVC POD Bases

In the previous subsection, space vectors were grouped into clusters. Reduced order bases are now computed for each cluster as follows:

Let the number of space vectors in cluster k be N^k, the $N^k \times N^k$ correlation matrix L^k is defined by [2]:

$$L_{i,j}^k = \langle W_i^k, W_j^k \rangle \tag{13}$$

is constructed, where $\langle , \rangle$ denotes the Euclidean inner product of space vectors W^k. With R^k denotes the number of SVC POD modes to be constructed for cluster k, the first R^k eigenvalues of largest magnitude, $\{\lambda\}_{i=1}^{R^k}$, of L^k are found. They are sorted in descending order, and their corresponding eigenvectors $\{v^k\}_{i=1}^{R^k}$ are calculated. Each eigenvector is normalized so that

$$\left\|v_i^k\right\|^2 = \frac{1}{\lambda_i^k} \tag{14}$$

The orthonormal SVC POD basis set $\{\phi_i^k\}_{i=1}^{R^k}$ is constructed according to [8]:

$$\phi_i^k = \sum_{j=1}^{N^k} v_{i,j}^k W_j^k \tag{15}$$

where $v_{i,j}^k$ is the j^{th} component of v_i^k . The 1-D Burgers' equation solution space vectors were grouped into 4 clusters and the first four modes of each cluster are shown in figure (6).

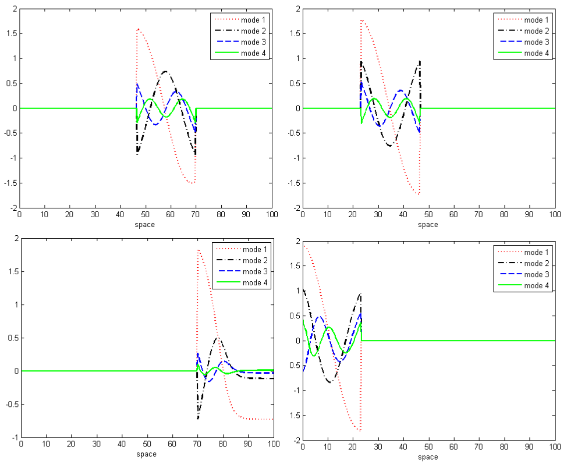

Fig. 6. First four modes of space vectors Local POD in four clusters

The 2-D Burgers' equation solution space vectors were grouped into 8 clusters. The first four modes of cluster 4 are shown in Fig.(7).

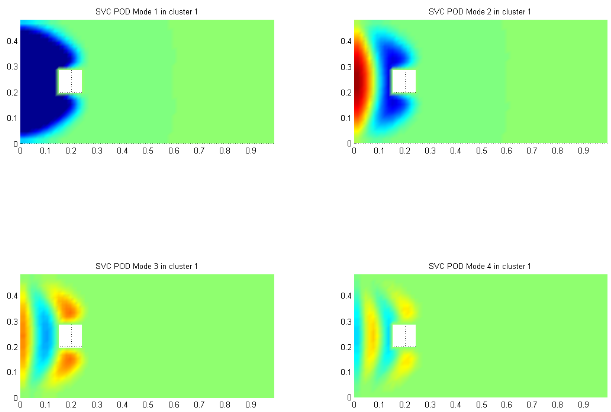

Fig. 7. 2D First four modes of SVC POD in cluster 1

The SVC modes in Fig. (7) are different from the modes of the global POD in Fig. (3) because they are based only on one cluster that includes relatively close states. These states are clearly the ones on the far left of the domain that share a relatively higher fluid velocities most of the time. We should notice from the modes of this cluster that locations which are close to the top and bottom of the left side do not appear to belong to this cluster. This is due to the fact that they have lower velocities most of the time so they are grouped in some other clusters other than the one shown in Fig. (7).

Now we have computed the clusters with their local reduced order bases. The last step is the projection to the full solution. It is important to record the original ordering of snapshots because we need this in the projection process. The constructed local reduced order bases are projected to their corresponding locations in the full solution as follows:

$$\{U\}_{k=1}^T \approx \{\sum_{i=1}^{R^k} \alpha_i^k \phi_i^k\}_{k=1}^T \tag{16}$$

Comparison between Reduced order system using SVC POD reduced order system and full order system for the 1D Burgers' equation is shown in Fig. (8) with the full number of states $N = 500$ is reduced to $R = 15$ where the space domain is $x \in [0, 100]$ and the time domain is $t \in [0, 50]$. The Figure shows the comparison at $t = 30$. The error norm for this case is found to be 0.0012. The error norms at $t = 30$ for the two methods when reducing the order from 500 states to 15 states are shown below. A third method based on snapshots clustering in [18] is added to the comparison, our method still shows significant improvement.

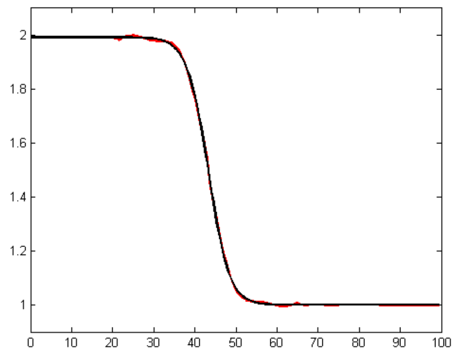

Fig. 8. Full order solution Vs Space vectors Local POD reduced order at t=30

Method	Error
Global POD	0.3080
Snapshot Clustering POD	0.0352
SVC POD	0.0012

The reduced order system for the 2D domain using the SVC POD method is shown in Fig. (9) with full number of states $N = 2000$ is reduced to $R = 10$ states. The Figure shows the comparison at $t = 30$. The error norm between the full order and the reduced order systems is found to be 0.0012 which is almost the same amount of error we got using global POD with reduction to 50 states in the last section, which is a significant improvement.

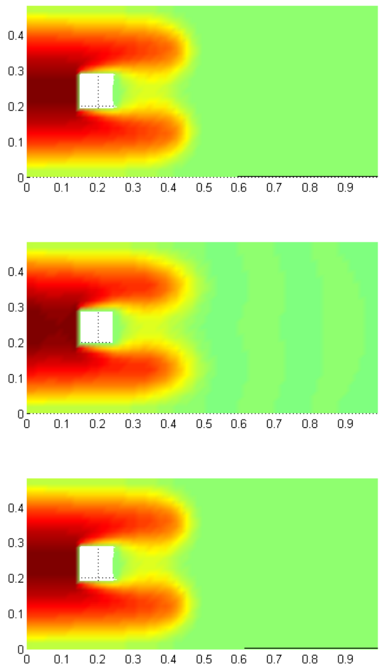

Fig. 9. Full order solution (top) Vs Global POD reduced model (middle) VS SVC POD reduced model (bottom)

5. CONTROL

We will solve a tracking control problem for the nonlinear system in (3). Desired output should track a given fixed reference signal u_{ref}. Linearization around u_{ref} is required. The nonlinear term in (3) is given by:

$$N(u) = diag(u)Mu$$
$$= \begin{pmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_N \end{pmatrix} \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & \ddots & \vdots \\ m_{N1} & \cdots & m_{NN} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$
$$= \begin{pmatrix} u_1 \sum_{j=1}^{N} m_{1j} u_j \\ \vdots \\ u_N \sum_{j=1}^{N} m_{Nj} u_j \end{pmatrix}$$

Then the Jacobian matrix $\frac{\partial N(u)}{\partial u}$ becomes:

$$\begin{pmatrix} u_1 m_{11} + \sum_{j=1}^{N} m_{1j} u_j & \cdots & u_N m_{1N} \\ \vdots & \ddots & \vdots \\ u_N m_{N1} & \cdots & u_N m_{NN} + \sum_{j=1}^{N} m_{Nj} u_j \end{pmatrix}$$
$$= diag(u)M + diag(Mu)$$

where

$$diag(u)M = \begin{pmatrix} u_1 m_{11} & \cdots & u_N m_{1N} \\ \vdots & \ddots & \vdots \\ u_N m_{N1} & \cdots & u_N m_{NN} \end{pmatrix}$$

and

$$diag(Mu) = \begin{pmatrix} \sum_{j=1}^{N} m_{1j} u_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} m_{Nj} u_j \end{pmatrix}$$

$$\frac{\partial N(u)}{\partial u}|_{u_{ref}} = diag(u_{ref})M + diag(Mu_{ref}) := A_N \quad (17)$$

Then the linearized system becomes:

$$\dot{u}(t) = f(u_{ref}) + (A + A_N)u + Bu_{a,b} \quad (18)$$
$$u(0) = u_0$$

where $f(u_{ref})$ is constant. The block diagram in figure (5) shows the state feedback tracking control problem. F and G are gains to be designed such that the output tracks u_{ref}. The cost function to be minimized to track the trajectory u_{ref} is given by:

$$J(u_0) = \int_0^T \{(u - u_{ref})^T Q(u - u_{ref}) + u_{a,b}^T R u_{a,b}\} \quad (19)$$

The reference input is chosen to be a sin function. Figure

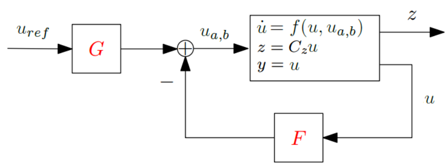

Fig. 10. The state feedback tracking problem

(11) shows the controlled system using the full order model while the reduced order controlled system is shown in figure (12) shows. In both results the output tracks the reference input efficiently. The full order controller result looks better but considering the huge computational savings provided by the reduced order controller, the latter shows very good tracking performance with significantly lower computational cost.

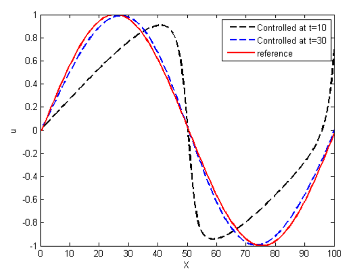

Fig. 11. Controlled system using the full order model at two different times

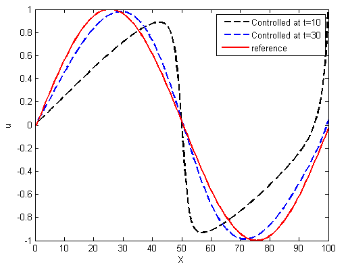

Fig. 12. Controlled system using the reduced order model at two different times

6. CONCLUSION AND FUTURE WORK

In this paper, the solution space of nonlinear PDEs is grouped into clusters where the behavior has significantly different features. We first solved the nonlinear problem for the full solution, we then grouped the space vectors into clusters where a space vector is the solution over all times at a particular space location. Then we computed the reduced order bases for the local clusters. We applied our method on the Burgers' equation for both 1D and 2D domains and we showed efficient and promising simulation results. It is important to note that this clustering method is different from methods found in the literature, all available clustering methods are based on snapshots clustering, not space clustering. Finally, the controller was designed for the 1D problem based on the reduced order system. Results showed very good tracking performance with significantly lower computational cost. For future work, we will work with a new space-time clustering approach where the $N \times M$ space-time solution is clustered according to relative distances, that is clusters will include the solution at different space locations and at different times.

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