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On the Linear Constrained Regulation Problem for Continuous-Time Systems

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Abstract: In this paper the regulation problem for linear continuous-time systems by linear state-feedback under linear state and/or control constraints is investigated. This problem, named the Linear Constrained Regulation Problem, has been extensively studied when the regulation concerns an equilibrium situated in the interior of the domain of admissible states. In this paper the case when the desired equilibrium state is on the boundary of the domain of admissible states is considered. The tools used for the analysis and design of this kind of control problems are the conditions of positive invariance of polyhedral sets, Lyapunov-like polyhedral functions, LMI methods and eigenstructure assignment techniques.

Keywords: Constrained control, polyedral sets, positive invariance, stability, linear systems.

1. INTRODUCTION

The Linear Constrained Regulation Problem (LCRP) (Bitsoris and Vassilaki, 1990), namely the regulation of linear systems by linear state-feedback under linear state and/or control constraints has been the object of intensive research work for both continuous-time and discrete-time systems since the early publications on this subject (Gutman and Hagander (1985), Vassilaki et al. (1988), Benjaouia and Burgat (1988), Blanchini (1991)). For the case of continuous-time systems, the problem has been faced by applying optimization methods (Vassilaki and Bitsoris, 1989), eigenstructure assignment approaches (Castelan and Hennet (1993), Tarbouriech and Burgat (1994)) or Lyapunov function based methods (Gutman and Hagander, 1985). For the stability analysis, both quadratic and polyhedral Lyapunov functions has been used (Bitsoris (1991), Castelan and Hennet (1994)). In all these publications, the regulation is made around an equilibrium state situated in the interior of the region of the set where the state constraints are respected. In many engineering problems however, the regulation around an equilibrium lying on the boundary of this set is necessary. For this kind of problems the classical methods cannot be applied and design control methods are missing. The object of this paper is to present new results on the LCRP for continuous-time systems concerning the regulation around an equilibrium situated on the boundary of domain defined by the state constraints. The tools used for the analysis and design of this kind of control problems are the conditions of positive invariance of polyhedral sets, Lyapunov-like polyhedral functions, LMI methods and eigenstructure assignment techniques.

The paper is organized as follows: In Section 2, the notations adopted in this paper and the problem statement are presented. In Section 3, conditions guaranteeing the existence of a state-feedback control making the whole region defined by the state constraints an admissible domain of attraction are established. It is shown that if such a control exists then it can be determined by solving a linear programming problem. In the following sections, we investigate the case when a control resulting to the maximal admisssible domain of attraction does not exists. Two particular cases are considered: In section 4, we consider the case when the cone on which the equilibrium is situated can become positively invariant and in Section 5. the case when no linear state-feedback control making this cone positively invariant exists. For both cases design techniques for the determination of a solution to the LCRP are proposed.

2. THE LINEAR CONSTRAINED REGULATION PROBLEM

In this paper, capital letters denote real matrices, lower case letters denote column vectors or scalars, $\mathcal{T}$ denotes the time set $\mathcal{T} = [0, \infty)$, $\mathbb{R}^n$ denotes the real n-space, $\mathbb{R}^p_+$ ($\mathbb{R}^p_-$) is the nonnegative orthant (non positive orthant) of the real p-space, $\mathbb{R}^{n\times p}$ the set of real $n \times p$ matrices. I_p denotes the $p \times p$ identity matrix, $0_{s\times q}$ denotes the $s \times p$ matrix with zero elements and $e_p \in \mathbb{R}^p$ is the vector $e_p = [1\ 1\ \cdots\ 1]^T$. For two real vectors $x = [x_1 \quad x_2\ ...\ x_n]^T$ and $y = [y_1 \quad y_2\ ...\ y_n]^T$, $x < y \quad (x \leq y)$ is equivalent to $x_i < y_i \quad (x_i \leq y_i) \quad i = 1, 2, ..., n$. Similar notation is applied for real matrices. A matrix $H = (h_{ij})$ with nonnegative elements, that is $h_{ij} \geq 0$ for all i and j, is said to be a nonnegative matrix while a square matrix

$H = (h_{ij})$ with nonnegative off-diagonal elements, that is $h_{ij} \geq 0$ for all $i \neq j$, is said to be a Metzler matrix. Finally, for square matrices $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ ($P \succeq 0$) means that P is positive definite (positive semi-definite).

If $G \in \mathbb{R}^{s\times n}$ and $w \in \mathbb{R}^s$ then $\mathcal{P}(G, w)$ denotes the polyhedral set

$$\mathcal{P}(G, w) \triangleq \{x \in \mathbb{R}^n : Gx \leq w\}$$

and $\mathcal{C}(G)$ denotes the polyhedral proper cone

$$\mathcal{C}(G) \triangleq \{x \in \mathbb{R}^n : Gx \leq 0\}$$

In the case when $G \in \mathbb{R}^{n\times n}$ and $\det G \neq 0$, $\mathcal{C}(G)$ is said to be a simplicial proper cone. If $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix and d is a positive real number, then $\mathcal{Q}(G, d)$ denotes the ellipsoidal set

$$\mathcal{Q}(P, d) \triangleq \{x \in \mathbb{R}^n : x^T Px \leq d\}$$

Finally, if $v(x)$ is a continuous function $v : \mathbb{R}^n \to \mathbb{R}_+$, and $d \in \mathbb{R}$, then $\mathcal{R}(v, d)$ denotes the set

$$\mathcal{R}(v, d) \triangleq \{x \in \mathbb{R}^n : v(x) \leq d\}$$

We consider linear continuous-time systems described by differential equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $t \in \mathcal{T}$ is the time variable and $A \in \mathbb{R}^{n\times n}$. $B \in \mathbb{R}^{n\times m}$.

The state vector has to satisfy linear constraints of the form

$$Gx \leq w \tag{2}$$

where $G \in \mathbb{R}^{s\times n}$ and $w \in \mathbb{R}^s_+$. The control input u has also to satisfy linear constraints of the form

$$Du \leq \rho \tag{3}$$

where $D \in \mathbb{R}^{q\times m}$ and $\rho \in \mathbb{R}^q_+$.

The Linear Constrained Regulation Problem (LCRP) consists in the determination of a linear state feedback control law $u = Fx$ and of a domain of attraction $\mathcal{D} \subseteq \mathcal{P}(G, w)$ such that for all initial states $x_0 \in \mathcal{D}$ the corresponding trajectories $x(t; x_0)$ of the resulting closed-loop system

$$\dot{x}(t) = (A + BF)x(t) \tag{4}$$

converge to the equilibrium asymptotically while respecting the linear state and/or control constraints (2) and/or (3) respectively. Such a set $\mathcal{D}$ is said to be an *admissible domain of attraction.*

The notions of positively invariant and linearly controlled invariant sets defined below play an important role in the investigation of the LCRP.

Definition 1: The subset $\mathcal{D} \subset \mathbb{R}^n$ of the state space of the autonomous system $x(t + 1) = Ax(t)$ is *positively invariant* if all trajectories $x(t; x_0)$ starting from $\mathcal{D}$ remain in it, that is $x(t; x_0) \in \mathcal{D}$ for all $x_0 \in \mathcal{D}$ and $t \in \mathcal{T}$.

Definition 2: The subset $\mathcal{D} \subset \mathbb{R}^n$ of the state space of system $\dot{x}(t) = Ax(t)+Bu(t)$ is *linearly controlled invariant* if there exists a linear state-feedback control $u = Fx$ such that $\mathcal{D}$ is a positively invariant set of the resulting closed-loop system $\dot{x}(t) = (A + BF)x(t)$.

Set conditions for a linear state feedback control law together with a domain $\mathcal{D} \subset \mathbb{R}^n$ to be a solution to the LCRP are given by the following theorem:

Theorem 1: The control law $u = Fx$ is a solution of the LCRP if and only if there exists a positively invariant set $\mathcal{D} \subset \mathbb{R}^n$ of the resulting closed-loop system (4) such that

$$\mathcal{D} \subseteq \mathcal{P}(G, w) \tag{5}$$

$$\mathcal{D} \subseteq \mathcal{P}(DF, \rho) \tag{6}$$

$$\lim x(t; x_0) = 0 \quad \forall x_0 \in \mathcal{D}$$

In the case when the origin is an interior point of the set $\mathcal{P}(G, w)$ and the pair (A, B) is stabilizable, this problem has always a solution because any stabilizing control together with a sufficiently small positively invariant set (e.g. an ellipsoidal set $\mathcal{D} = \mathcal{Q}(P, d)$) constitute a solution to the LCRP. Thus the interest is to derive the control law $u = Fx$ that results to the largest admissible domain of attraction $\mathcal{D}$, or/and to an admissible domain of attraction $\mathcal{D}$ with guaranteed performance. This problem has been extensively investigated. However, in the case when the desired equilibrium $x_e = 0$ is on the boundary of the set $\mathcal{P}(G, w)$ the stabilizability of the pair (A, B) does not guarantee the existence of a solution to the LCRP. In this case, the methods developed when desired equilibrium is an interior point of set $\mathcal{P}(G, w)$ cannot be applied. The aim of this paper is to develop methods for solving the LCRP when the desired equilibrium $x_e = 0$ is on the boundary of the set $\mathcal{P}(G, w)$.

3. MAXIMAL DOMAINS OF ATTRACTION

If the desired equilibrium state $x_e = 0$ is on the boundary of the set $\mathcal{P}(G, w)$ then at least one of the boundary hyperplanes $g_j^T x = w_j$ of the set $\mathcal{P}(G, w)$ passes through the origin, that is $w_j = 0$. In order to simplify the notation, we assume that the desired equilibrium $x_e = 0$ of the closed-loop system (4) is situated on the boundary hyperplanes $g_j^T x = w_j \quad j = 1, 2, ..., p \quad p < s$. Then $w_j = 0 \quad j = 1, 2, ..., p$ and $w_j > 0 \quad j = p + 1, p + 2, ..., s$. Thus, the inequality $Gx \leq w$ which defines the polyhedral set $\mathcal{P}(G, w)$ is written as

$$G_1 x \leq 0$$

$$G_2 x \leq w_2$$

with

$$G_1 \triangleq \begin{bmatrix} g_{11}^T \\ g_{12}^T \\ \vdots \\ g_{1p}^T \end{bmatrix} = \begin{bmatrix} g_1^T \\ g_2^T \\ \vdots \\ g_p^T \end{bmatrix}, \; G_2 \triangleq \begin{bmatrix} g_{21}^T \\ g_{22}^T \\ \vdots \\ g_{2s}^T \end{bmatrix} = \begin{bmatrix} g_{p+1}^T \\ g_{p+2}^T \\ \vdots \\ g_s^T \end{bmatrix},$$

$$w_2 \triangleq \begin{bmatrix} w_{21} \\ w_{22} \\ \vdots \\ w_{2(s-p)} \end{bmatrix} = \begin{bmatrix} w_{p+1} \\ w_{p+2} \\ \vdots \\ w_s \end{bmatrix}$$

Thus,

$$\mathcal{P}(G, w) = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$$

where $\mathcal{C}(G_1)$ denotes the polyhedral proper cone defined by inequality $G_1 x \leq 0$.

We first investigate the case when there exists a state-feedback control $u = Fx$ making the whole region $\mathcal{P}(G, w) = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ an admissible domain of attraction.

Theorem 2: If the set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is bounded and for a matrix $F \in \mathbb{R}^{m\times n}$ there exist a real number ε, two nonnegative matrices $H_{21} \in \mathbb{R}^{(s-p)\times p}$ and $K \in \mathbb{R}^{q\times s}$ and two Metzler matrices $H_{11} \in \mathbb{R}^{p\times p}$ and $H_{22} \in \mathbb{R}^{(s-p)\times(s-p)}$ satisfying the relations

$$G_1(A+BF) = H_{11}G_1 \tag{7}$$

$$G_2(A+BF) = H_{21}G_1 + H_{22}G_2 \tag{8}$$

$$H_{22}w_2 \leq -\varepsilon w_2 \tag{9}$$

$$\varepsilon > 0 \tag{10}$$

$$K\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = DF \tag{11}$$

$$K\begin{bmatrix} 0 \\ w_2 \end{bmatrix} \leq \rho \tag{12}$$

$$K \geq 0 \tag{13}$$

then the set $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is an admissible domain of attraction of the resulting closed-loop system (4).

Proof: It is sufficient to prove that all the hypotheses of Theorem 1 are satisfied. By virtue of Farkas lemma, conditions (11)-(13) are equivalent to the set relation $\mathcal{P}(G_1, 0) \cap \mathcal{P}(G_2, w_2) \subseteq \mathcal{P}(DF, \rho)$ or, equivalently, to the set relation $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2) \subseteq \mathcal{P}(DF, \rho)$. To complete the proof, we shall prove that $u = Fx$ is a stabilizing control in $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ and $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is positively invariant. Condition (7) together with the hypothesis that matrix H_{11} is Metzler imply the positive invariance of the polyhedral cone $\mathcal{C}(G_1)$. Therefore, $x_0 \in \mathcal{C}(G_1)$ implies that $x(t; x_0) \in \mathcal{C}(G_1) \quad \forall t \in \mathcal{T}$. Let $v(x)$ be the continuous function defined by relation

$$v(x) \triangleq \max_{1\leq i\leq s-p}\left\{\frac{(G_2x)_i}{w_{2i}}\right\} \triangleq \max_{p+1\leq i\leq s}\left\{\frac{g_i^T x}{w_i}\right\} \tag{14}$$

We define the total-time derivative $\dot{v}(x)_{(4)}$ of function $v(x)$ with respect to system (4) as

$$\dot{v}(x(t))_{(4)} = \lim_{\tau\to 0^+}\sup\frac{v[x(t+\tau)] - v[x(t)]}{\tau}$$

where $x(t)$ denotes the trajectory of system (4). The function $v(x)$ is positive definite in $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$, that is $v(0) = 0$ and $v(x) > 0$ for all $x \in \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ and $x \neq 0$. This holds because otherwise there would exist a $x \in \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$, $x \neq 0$ such that $G_2x \leq 0$. Then for any $r > 0$ it would follow that $G_1(rx) \leq 0$ and $G_2(rx) \leq 0 < w_2$ which would contradict the hypothesis that set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is bounded.

For a $x \in \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ let $k_i \quad i = 1, 2, .., \bar{p}$ be the indices $1 \leq k_i \leq s - p$ for which

$$v(x) \triangleq \max_{1\leq i\leq s-p}\left\{\frac{(G_2x)_i}{w_{2i}}\right\} = \frac{(G_2x)_{k_i}}{w_{2k_i}}$$

Then,

$$(G_2x)_{k_i} = v(x)w_{2k_i} \quad i = 1, 2, .., \bar{p} \tag{15}$$

$$(G_2x)_j < v(x)w_{2j} \quad j = 1, 2, .., \bar{p}, \quad j \neq k_i \tag{16}$$

$$G_1x \leq 0 \tag{17}$$

and

$$\dot{v}(x(t))_{(4)} = \max_{1\leq i\leq \bar{p}}\left\{\frac{(G_2\dot{x}(t))_{k_i}}{w_{2k_i}}\right\}$$

Therefore, from (8)-(9) it follows that

$$\begin{aligned}\dot{v}(x(t))_{(4)} &= \max_{1\leq i\leq \bar{p}}\left\{\frac{(G_2\dot{x}(t))_{k_i}}{w_{2k_i}}\right\} \\ &= \max_{1\leq i\leq \bar{p}}\left\{\frac{(G_2(A+BF)x)_{k_i}}{w_{2k_i}}\right\} \\ &= \max_{1\leq i\leq \bar{p}}\left\{\frac{((H_{21}G_1 + H_{22}G_2)x)_{k_i}}{w_{2k_i}}\right\} \\ &\leq \max_{1\leq i\leq \bar{p}}\left\{\frac{(H_{22}G_2x)_{k_i}}{w_{2k_i}}\right\}\end{aligned}$$

because $G_1x \leq 0$ and $H_{21} \geq 0$. Furthermore, from (15) and (16) it follows that

$$(H_{22}G_2x)_{k_i} \leq v(x)(H_{22}w_2)_{k_i}$$

because the matrix has nonnegative off-diagonal elements. Therefore,

$$\begin{aligned}\dot{v}(x(t))_{(4)} &\leq \max_{1\leq i\leq \bar{p}}\left\{\frac{v(x)(H_{22}w_2)_{k_i}}{w_{2k_i}}\right\} \\ &\leq \max_{1\leq i\leq \bar{p}}\left\{\frac{(-\varepsilon w_{2k_i}}{w_{2k_i}}\right\}v(x) = -\varepsilon v(x)\end{aligned}$$

Thus, $\dot{v}(x(t))_{(4)}$ is negative definite in $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ because $\varepsilon > 0$. This, in turn, implies that $\lim_{t\to\infty} x(t; x_0) = 0$ for all $x_0 \in \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ because the function $v(x)$ is continuous. In addition, from $\dot{v}(x(t))_{(4)} \leq -\varepsilon v(x)$ for all $x_0 \in \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ and the fact that $\mathcal{C}(G_1)$ is positively invariant it follows that $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is also positively invariant. Thus, all the hypotheses of Theorem 1 are satified. Consequently, the set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is an admissible domain of attraction. ■

Remark 1: A polyhedral set can be viewed as the intersection of translated polyhedral proper cones. If these cones are positively invariant then the polyhedral set is also positively invariant. This condition is sufficient but not necessary. In Theorem 2 however, it has been shown that in the case when the origin is a vertex of a polyhedral set then the positive invariance of the cone $\mathcal{C}(G_1)$ corresponding to this vertex is a necessary condition for the positive invariance of the polyhedral set. This is expressed by condition (7) and the hypothesis that H_{11} is a Metzler matrix. □

Using the result established in Theorem 1, we can determine a control law $u = Fx$ corresponding to a maximal admissible domain of attraction $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$. If such a control law exists, it can be determined by solving the linear programming problem

$$\max_{H_{11}, H_{21}, H_{22},, K, F, \varepsilon}\{\varepsilon\} \tag{18}$$

under constraints (7)-(13).

a) If $\arg\max\{\varepsilon\} > 0$ and the set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is bounded, then the so obtained control $u = Fx$ is a stabilizing one and $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is an admissible domain of attraction. This is also true in the case when the set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is unbounded provided that the resulting closed-loop matrix $A + BF$ is Hurwitz. In both cases, the so obtained control law provides the greatest rate of convergence if the distance from the origin of a state $x \in \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is measured by $v(x)$, $v(x)$ being the scalar function defined by (14).

b) If $\arg\max\{\varepsilon\} = 0$ then the set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ is positively invariant and is also an admissible domain of attraction provided that matrix $A + BF$ is Hurwitz.

c) Finally, if the optimization problem (18) is not feasible or is feasible but $\arg\max\{\varepsilon\} < 0$ then there does not exist any control law making the set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ positively invariant and as a result neither an admissible domain of attraction. This means that the maximal set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ cannot be an admissible domain of attraction. Therefore, in these cases, if the LCRP has a solution, then the admissible domain of attraction will be a strict subset the polyhedral set $\mathcal{P}(G, w) = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$. These cases are investigated in the following sections of the paper.

4. DOMAINS OF ATTRACTION OF THE FORM $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{D}_2$, $\mathcal{D}_2 \subset \mathcal{P}(G_2, W_2)$

We first consider the case when the maximal set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ cannot be an admissible domain of attraction but a control $u = Fx$ rendering the cone $\mathcal{C}(G_1)$ positively invariant exists. Then a set of the form $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{D}_2$ with $\mathcal{D}_2 \subset \mathcal{P}(G_2, w_2)$ may be an admissible domain of attraction.

Conditions for the existence of a stabilizing control $u = Fx$ rendering the cone $\mathcal{C}(G_1)$ positively invariant are established in the following theorem:

Theorem 4: There exists a control $u = Fx$ that stabilizes the system (1) and renders the cone $\mathcal{C}(G_1)$ positively invariant if and only if there exist a Metzler matrix $H_{11} \in \mathbb{R}^{pp\times}$ and two matrices $Q \in \mathbb{R}^{n\times n}$, $Q^T = Q$ and $Y \in \mathbb{R}^{m\times n}$, satisfying the relations

$$G_1AQ + G_1BY = H_{11}G_1Q \tag{19}$$

$$-Q \prec 0 \tag{20}$$

$$QA^T + Y^TB^T + AQ + BY \prec 0 \tag{21}$$

Proof: a) Sufficiency: If relation (20) is satisfied, then $detQ \neq 0$. Thus, setting

$$F = YQ^{-1}, \tag{22}$$

from (19) it follows that $G_1(A + BF) = H_{11}G_1$ which implies the positive invariance of the cone $\mathcal{C}(G_1)$ with respect to the closed-loop system $\dot{x}(t) = (A+BF)x(t)$ because, by hopotheses, H_{11} is a Metzler matrix. Moreover, taking into account that matrix Q is symmetric, from (21) it follows that $Q(A^T + Q^{-1}Y^TB^T) + (A+BYQ^{-1})Q \prec 0$ or equivalently

$$Q(A^T + F^TB^T) + (A + BF)Q \prec 0 \tag{23}$$

because, by (22), $YQ^{-1} = F$. Finally, from (20) it follows that matrices Q and Q^{-1} are positive definite and thus relation (23) is equivalently written as $(A + BF)^TQ^{-1} + Q^{-1}(A + BF) \prec 0$. This means that $v(x) = x^TQ^{-1}x$ is a Lyapunov function for the system $\dot{x}(t) = (A + BF)x(t)$. Therefore $F = YQ^{-1}$ is the gain matrix of a stabilizing linear state-feedback control for system (1).

b) Necessity: If there exists a stabilizing control $u = Fx$ then there also exists a symmetric positive definite matrix P that satisfies the Lyapunov matrix inequality

$$(A + BF)^TP + P(A + BF) \prec 0 \tag{24}$$

Since matrix P is positive definite their inverse exists and is also positive definite. Therefore, there exists a matrix Y such that $F = YP$ and thus relation (24) is written as $P(AP^{-1}+BY)^TP+P(AP^{-1}+BY)P \prec 0$ or, equivalently, $(AP^{-1} + BY)^T + (AP^{-1} + BY) \prec 0$. Setting $Q = P^{-1}$ we obtain condition (21).

If, in addition, the control law $u = YPx = YQ^{-1}$ renders the cone $\mathcal{C}(G_1)$ positively invariant, then by virtue of Theorem 2, there exists a Metzler matrix H_{11} such that $G_1(A + BYP) = H_{11}G_1$ or $G_1(A + BYQ^{-1}) = H_{11}G_1$ or, finally, $G_1(AQ + BY) = H_{11}G_1Q$.■

By solving relations (19)-(21), we obtain not only a stabilizing control $u = YQ^{-1}x$ that renders the cone $\mathcal{C}(G_1)$ positively invariant, but also a Lyapunov function $v(x) = x^TQ^{-1}x$ for the resulting unconstrained closed-loop system. This or any other quadratic Lyapunov function $v(x) = x^TPx$ for the resulting closed-loop systen can then be used for the construction of an admissible domain of attraction of the form $\mathcal{D} = \mathcal{C}(G_1)\cap\mathcal{D}_2$ where $\mathcal{D}_2 = \mathcal{Q}(P, d)$, d being a positive scalar such that $\mathcal{C}(G_1)\cap\mathcal{Q}(P, d) \subset \mathcal{C}(G_1)\cap \mathcal{P}(G_2, w_2)$. Admiisible domains of attraction can also be obtained by simply determining a polyhedral positively invariant set $\mathcal{P}(G_2^*, w_2^*)$ for the resulting closed-loop system such that $\mathcal{C}(G_1) \cap \mathcal{P}(G_2^*, w_2^*) \subset \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$.These approaches require the determination of a solution of a nonlinear problem which, generally is not easy to be solved. In the following subsection we show how this can be done in the case when the equilibrium is situated on one boundary hyperplane of the state constraint set $\mathcal{P}(G, w)$.

4.1 Equilibrium on one boundary hyperplane

In the usual case when only one boundary hyperplane of the polyhedral set $\mathcal{P}(G, w) = \mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ passes through the origin, the matrix G_1 is a line vector g_{11}^T and thus $\mathcal{C}(G_1)$ degenerates to a half space defined by the relation $g_{11}^Tx \leq 0$. Then, the necessary and sufficient conditions (19)-(21) for the existence of a stabilizing control $u = YQ^{-1}x$ rendering the half-space $g_{11}^Tx \leq 0$ positively invariant become

$$g_{11}^TAQ + g_{11}^TBY = h_{11}g_{11}^TQ \tag{25}$$

$$QA^T + Y^TB^T + AQ + BY \prec 0 \tag{26}$$

$$h_{11} < 0 \tag{27}$$

$$-Q \prec 0$$

Conditions (25) and (27) mean that g_{11}^T is a left eigenvector of matrix $A + BYQ^{-1} = A + BF$ associated with a nonnegative eigenvalue h_{11}. Therefore, there exists an admissible domain of attraction of the form $\mathcal{C}(g_{11}^T) \cap D_2$ if and only if there exists a stabilizing control that assignes g_{11}^T as a left eigenvector of matrix $A+BF$. In such a case, $h_{11} < 0$. Consequently, if a stabilizing control $u = Fx$ that assigns g_{11}^Tas a left eigenvector of matrix $A + BF$ exists, then its gain matrix $F = YQ^{-1}$ can be determined by solving the parametrized convex problem (25) and (26) with a negative scalar parameter h_{11}.

Having computed a stabilizing control making the half-space $\mathcal{C}(g_{11}^T)$ positively invariant, the next step is the determination of an admissible domain of attraction. Two approaches are proposed:

4.1.1 Mixed polyhedral-ellipsoidal domains of attraction

We first establish a method for determining admissible domains of attraction of the form $\mathcal{D} = \mathcal{C}(g_{11}^T) \cap \mathcal{Q}(P, d)$, that is domains that are the intersection of the half space $\mathcal{C}(g_{11}^T)$ and of an ellipsoid defined by an inequality $x^T Px \leq d$. The elipsoidal set $\mathcal{Q}(P, d)$ is constructed by determining a matrix P so that $v(x) = x^T Px$ is a Lyapunov function for the stable closed-loop system. Such a matrix is the matrix $P = Q^{-1}$, where Q is the positive definite matrix resulting from the parametrized LMI problem (25)-(26). Any other positive definite matrix P satisfying the relation $(A+BF)^T P+P(A+BF) \prec 0$ may also be used. By Theorem 1, the value of parameter d must be chosen so that $\mathcal{C}(g_{11}^T) \cap \mathcal{Q}(P, d) \subset \mathcal{C}(g_{11}^T) \cap \mathcal{P}(G_2, w_2)$ and $\mathcal{C}(g_{11}^T) \cap \mathcal{Q}(P, d) \subset \mathcal{P}(DF, \rho)$. These relations are satisfied if $\mathcal{Q}(P, d) \subset \mathcal{P}(G_2, w_2)$ and $\mathcal{Q}(P, d) \subset \mathcal{P}(DF, \rho)$ or, equivalently (S. Boyd et al, 1994), if

$$d g_{2i}^T P^{-1} g_{2i}^T \leq w_{2i} \quad i = 1, 2, ..., s-p \tag{28}$$

and

$$d(DF)_i^T P^{-1}(DF)_i \leq \rho_i \quad i = 1, 2, ...q \tag{29}$$

Thus, by solving the linear programming problem

$$\max\{d\} \tag{30}$$

under constraints (28) and (29) we determine the maximal hyperellipsoid $\mathcal{Q}(P, d)$ included in the sets $\mathcal{P}(G_2, w_2)$ and $\mathcal{P}(DF, \rho)$. Since all sets $\mathcal{Q}(P, d)$ for $d > 0$ are attractive, the set $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{Q}(P, \hat{d})$ with $\hat{d} = \arg\max\{d\}$ is an admissible domain of attraction.

4.1.2 Polyhedral domains of attraction

The second approach consists in determining a polyhedral admissible domain of attraction of the form $\mathcal{D} = \mathcal{C}(g_{11}^T) \cap \mathcal{P}(G_2^*, w_2^*)$, that is a domain of attraction which is the intersection of the half space $\mathcal{C}(g_{11}^T)$ and of a polyhedral set $\mathcal{P}(G_2^*, w_2^*)$. To this end, by applying one of the well known methods of construction of polyhedral positively invariant sets for stable linear systems (Bitsoris (1991), Castelan and Hennet (1994), Tarbouriech and Burgat (1994), Blanchini and Miani (2007)) we determine a polyhedral positively invariant set $\mathcal{P}(G_2^*, e_{p*})$, $G_2^* \in \mathbb{R}^{p*\times n}$, for the resulting asymptotically stable system (4). Since all polyhedral sets $\mathcal{P}(G_2^*, re_{p*})$ with $r > 0$, by scaling the set $\mathcal{P}(G_2^*, e_{p*})$, are also positively invariant, by virtue of Theorem 1, for constructing an admissible domain of attraction it is sufficient to determine a r such that

$$\mathcal{C}(g_{11}^T) \cap \mathcal{P}(G_2^*, re_{p*}) \subset \mathcal{C}(g_{11}^T) \cap \mathcal{P}(G_2, w_2) \tag{31}$$

$$\mathcal{C}(g_{11}^T) \cap \mathcal{P}(G_2^*, re_{p*}) \subset \mathcal{P}(DF, \rho). \tag{32}$$

This can be achieved by using the following result:

Theorem 5: The set relations (31) and (32) are satisfied if and only if there exist matrices $M_1 \in \mathbb{R}^{(s-p)\times p}$, $M_2 \in \mathbb{R}^{(s-p)\times p*}$ and $L \in \mathbb{R}^{q\times(1+p*)}$ such that $\hat{r} = r^{-1}$,

$$M_1 g_{11}^T + M_2 G_2^* = G_2 \tag{33}$$

$$M_2 e_{p*} \leq \hat{r} w_2 \tag{34}$$

$$L \begin{bmatrix} g_{11}^T \\ G_2^* \end{bmatrix} = DF \tag{35}$$

$$L \begin{bmatrix} 0 \\ e_{p*} \end{bmatrix} \leq \hat{r}\rho \tag{36}$$

$$L \geq 0, \ M_i \geq 0 \quad i = 1, 2$$

Proof: The set relation (31) and (32) are equivalent written as

$$\begin{bmatrix} g_{11}^T \\ G_2^* \end{bmatrix} x \leq \begin{bmatrix} 0 \\ re_{p*} \end{bmatrix} \Rightarrow \begin{bmatrix} g_{11}^T \\ G_2 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} g_{11}^T \\ G_2^* \end{bmatrix} x \leq \begin{bmatrix} 0 \\ re_{p*} \end{bmatrix} \Rightarrow DFx \leq \rho$$

By Farkas Lemma, these relations are satisfied if and only there exist nonnegative real matrices $N \in \mathbb{R}^{s\times(1+p*)}$ and $L \in \mathbb{R}^{q\times(1+p*)}$ such that

$$N \begin{bmatrix} g_{11}^T \\ G_2^* \end{bmatrix} = \begin{bmatrix} g_{11}^T \\ G_2 \end{bmatrix} \text{ and } N \begin{bmatrix} 0 \\ re_{p*} \end{bmatrix} \leq \begin{bmatrix} 0 \\ w_2 \end{bmatrix} \tag{37}$$

$$L \begin{bmatrix} g_{11}^T \\ G_2^* \end{bmatrix} = DF \text{ and } L \begin{bmatrix} 0 \\ re_{p*} \end{bmatrix} \leq \rho \tag{38}$$

Partitioning matrix N as follows

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}$$

with $N_1 \in \mathbb{R}$, $N_2 \in \mathbb{R}^{1\times p*}$, $N_3 \in \mathbb{R}^{p\times p}$ and $N_4 \in \mathbb{R}^{p\times p*}$, relations (37) are equivalently written as

$$N_1 g_{11}^T + N_2 G_2^* = g_{11}^T$$
$$N_3 g_{11}^T + N_4 G_2^* = G_2$$
$$r N_2 e_{p*} \leq 0$$
$$r N_4 e_{p*} \leq w_2$$

These relations are satisfied for $N_2 = 0, N_1 = I$ and $N_3 g_{11}^T + N_4 G_2^* = G_2$ and $N_4 e_{p*} \leq \hat{r} w_2$, $\hat{r} = r^{-1}$. Thus, setting $M_1 = N_3$ and $M_2 = N_4$, we obtain conditions (33) and (34).■

According to this theorem, starting from a positively invariant set $\mathcal{P}(G_2^*, re_{p*}) = \mathcal{P}(G_2^*, \hat{r}^{-1}e_{p*})$ for the resulting closed-loop system we can obtain an admissible polyhedral domain of attraction $\mathcal{P}(G_2^*, re_{p*}) = \mathcal{P}(G_2^*, \hat{r}^{-1}e_{p*})$ by solving the linear programming problem

$$\min_{L, M_1, M_2, \hat{r}} \{\hat{r}\} \tag{39}$$

under constraints (33)-(28) and $L \geq 0$, $M_i \geq 0 \quad i = 1, 2$. It is clear that the so obtained admissible domain $\mathcal{D} = \mathcal{C}(G_1) \cap \mathcal{P}(G_2^*, re_{p*})$ is not unique because the asymptotically stable linear system (4) possesses many positively invariant polyhedral sets $\mathcal{P}(G_2^*, e_{p*})$. It is however possible to enlarge an initially determined admissible domain of attraction not by scaling but by using the recently established approach of enlargement of positively invariant set with specified complexity (Athanasopoulos et al. 2014).

5. DOMAINS OF ATTRACTION OF THE FORM $D = C(G_1^*) \cap D_2$, $C(G_1^*) \subset C(G_1)$

We consider now the case when there does not exist any stabilizing gain matrix F and nonnegative matrix H_{11} satisfying condition (7). This means that the cone $\mathcal{C}(G_1)$ cannot be positively invariant and thus its faces cannot be boundary hyperplanes of an admissible domain of attraction. A "quadratic" approach consisting in the determination of a paraboloidal positively invariant set $\mathcal{R}(v, 0) \subset \mathcal{C}(G_1)$ with $\mathcal{R}(v, 0)$ being a set defined by a second order polynomial inequality $v(x) \leq 0$ where $v(x) = x^T Px + l^T x$. is naturally excluded if $\mathcal{C}(G_1)$ is a proper cone. It can be shown that it is also excluded in the case when the cone $\mathcal{C}(G_1)$ is degenerated to a half space $g_{11}^T x \leq 0$. Therefore a natural candidate admissible domain of attraction will be of the form $\mathcal{D} = \mathcal{C}(G_1^*) \cap \mathcal{D}_2$ with $\mathcal{C}(G_1^*) \subseteq \mathcal{C}(G_1)$.

For a set $\mathcal{D} = \mathcal{C}(G_1^*) \cap \mathcal{D}_2$ to be an admissible domain of attraction it is necessary that the cone $\mathcal{C}(G_1^*)$ is positively invariant and $\mathcal{C}(G_1^*) \subseteq \mathcal{C}(G_1)$. This is equivalent to the existence of two matrices $H_{11}^* \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ such that

$$G_1^*(A+BF) = H_{11}^* G_1^* \tag{40}$$

$$h_{ij}^* \geq 0 \quad i,j = 1,2,...,p \quad i \neq j \tag{41}$$

$$KG_1^* = G_1 \tag{42}$$

$$K \geq 0 \tag{43}$$

Relations (40) and (41) guarantee the positive invariance of the proper cone $\mathcal{C}(G_1^*)$ and relations (42) and (43) are equivalent to the set relation $\mathcal{C}(G_1^*) \subseteq \mathcal{C}(G_1)$.

The determination of a gain matrix F and of a cone $\mathcal{C}(G_1^*)$ with G_1^* satisfying relations (40)-(43) is in general a nonlinear problem which however for some special but important cases can be solved by convenient eigenstructure assignment approaches. In the following subsection, the important particular case when only one boundary hyperplane of the set passes through the origin is considered.

5.1 Equilibrium on one boundary hyperplane

As already mentioned, when only one boundary hyperplane of set $\mathcal{C}(G_1) \cap \mathcal{P}(G_2, w_2)$ passes through the origin, the cone $\mathcal{C}(G_1)$ is degenerated to a half space defined by relation $g_{11}^T x \leq 0$.

If the pair (A, B) is controllable then by an eigenvalue assignment we can determine a gain matrix F such that all eigenvalues λ_i of matrix $A + BF$ are distinct and negative. Each eigenvalue is associated with a real left eigenvector g_{1i}^{*T}. The vectors g_{1i}^{*T} can be combined so that a nonnegative vector $k \in \mathbb{R}^n$, $k = [\, k_1 \; k_2 \; \cdots \; k_n \,]^T$ satisfying the relation

$$k_1 g_{11}^{*T} + k_2 g_{21}^{*T} + ... + k_n g_{1n}^{*T} = g_1^T$$

can be determined. Such a vector k exists because the vectors g_{1i}^{*T} $i - 1, 2, ..., n$ are linearly independent and if g_{11}^{*T} is a left eigenvector then this also true for the vector $-g_{11}^{*T}$. Then, setting $G_1^* = [\, g_{11}^* \; g_{12}^* \; \cdots \; g_{1n}^* \,]^T$, we get

$$G_1^*(A+BF) = H_{11}^* G_1^* \tag{44}$$

$$k^T G_1^* = g_1^T \tag{45}$$

where H_{11}^* is the Metzler matrix $H_{11}^* = diag(\lambda_1, \lambda_2, ..., \lambda_n)$. Then relation (44) guarantees the positive invariance of the cone $\mathcal{C}(G_1^*)$ and relation (45) together with $k \geq 0$ imply that $\mathcal{C}(G_1^*) \subseteq \mathcal{C}(g_1^T)$. Moreover, from (44) it follows that $u = Fx$ is a stabilizing control because by construction the matrix H_{11}^* has stable eigenvalues and $rank G_1^* = n$.

The next step is to determine a subset $\mathcal{D}_2^* \subset \mathcal{P}(G_2, w_2)$ such that $\mathcal{D} = \mathcal{C}(G_1^*) \cap \mathcal{D}_2^*$ is positively invariant and $\mathcal{C}(G_1^*) \cap \mathcal{D}_2^* \subset \mathcal{P}(DF, \rho)$. This can be done by applying one of the approaches established in subsections 4.1.1 and 4.1.2.

6. CONCLUSION

The Linear Constrained Regulation Problem around an equilibrium situated on the boundary of the polyhedral region where the state constraints are satisfied has been investigated. It has been shown that the control leading to the maximal admissible domain of attraction can be determined by solving a linear programming problem. For the cases when such a control does not exist, appropriate design approaches based on LMI and/or eigenstructure assignment methods for determining stabilizing linear state-feedback controllers and corresponding admissible domains of attraction have been proposed. These domains of attraction can also be viewed as the starting domains in the application of recently developed iterative approaches of enlargment of admissible domains of attraction (Athanasopoulos et al., 2014). It should also be noticed that all these results can be also established for discrete-time systems (Bitsoris and Olaru, 2013).

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