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Passivity Analysis of Discrete Inverse Optimal Control Based on Control Lyapunov Functions CLF ⋆

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Abstract: In this paper, we present the analysis to demonstrate that the inverse optimal control, based on a CLF, is passive. In order to do so, a storage function and a supply rate are established using the construction of such control and the properties of passive systems; these functions allow to state that this inverse optimal control is indeed passive.

Keywords: Inverse Optimal control, Control Lyapunov Function, Passivity, Dissipativity.

1. INTRODUCTION

The energy concept is very useful for analysis of physical systems. Considering gain and loss of energy, intuitively, a dissipative system can not store all its input energy. A dissipative system dissipates energy and does not produce it; any increase of stored energy is only due to external sources. This definition implies the existence of three energy-like functions: the storage function (representing the stored energy), the supply function(the injected energy to the system from an external source) and the dissipation function. The supply function is interpreted as an input power, denomination inherited from circuit theory. Depending on the supply function, different kinds of dissipativity are obtained; passivity is the one which has attracted more attention. Dissipativity and passivity concepts by means of the notion of the storage, the supply rate and the dissipation rate functions appears in the "earlys 70" (Willems (1972a),Willems (1972b)).
There are already publicated works which defined passivity properties in discrete time (López (2002),Byrnes and Lin. (1994),Byrnes and Lin. (1993)).
We focus on inverse optimality because it avoids to solve the HJB partial differential equations and still allows to obtain Kalman-type stability margins (Krstic and Deng (1998)). Due to the structure of the inverse optimal control in discrete time and the properties of passive systems, in theorem (7), we establish that such control is indeed passive.

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2. FUNDAMENTALS

2.1 Discrete time passivity definitions

In this section, we introduce for discrete-time, basic definitions and concepts related to the notions of dissipativity and passivity (Byrnes and Lin. (1994)). Assume that a discrete-time dynamic system Σ of the form

$$\Sigma : \begin{array}{ll} x(k+1) & = f(x(k)) + g(x(k))u(k) \\ y(k) & = h(x(k)) \end{array} \qquad (1)$$

is given together with a real-valued function $W \in \Re^m \times \Re^m$, named the supply rate. We suppose that for any $u \in \Re^m$ and for any $x(0) \in \Re^n$, the output $y(k)$ of (1) is such that $W(k) = W(u(k), y(k))$ satisfies $\Sigma_{i=0}^{k}|W(i)| < \infty$ for all $k \geq 0$.

Definition 1. A dynamic system Σ with supply W is said to be dissipative if there exists a nonnegative function $S : \Re^m \to \Re$, with $S(0) = 0$, called the storage function, such that for all $u \in \Re^m$ and all $k \in Z_+ := \{0, 1, 2, \dots\}$,

$$S(x(k+1)) - S(x(k)) \leq W(y(k), u(k)) \qquad (2)$$

Note that the above inequality holds if and only if

$$S(x(k+1)) - S(x(0)) \leq \Sigma_{i=0}^{k} W(y(i), u(i)) \quad \forall k, \forall u(k) \ \text{and} \ \forall x(0) \qquad (3)$$

which is called the dissipation inequality in discrete-time setting. The equivalence of (2) and (3) can be proven as follows: (2)⇒(3) is obvious. Conversely, since (3) holds $\forall k$, $\forall u(k) \in \Re^m$ and arbitrary initial state $x(0) = x \in \Re^n$, setting $x(0) = x = x(j)$,$y(0) = y(j)$, $u(0) = u(j)$, and $k = 1$, from (3)

$$S(x(j+1)) - S(x(j)) \leq W(y(j), u(j)) \quad \forall j, \forall u(j) \tag{4}$$

which is nothing else but (2).

For convenience, we denote (2) as the passivity condition for further discussions. It is worth to note that by definition, the storage function S is always nonnegative. Later on, among the dissipative systems, we shall be interested in studying a special class of dissipative systems with supply rate $W(y(k), u(k)) = y^T(k)u(k)$, which results in the following definition.

Definition 2. A system Σ is said to be passive if it is dissipative with supply rate $W(k) = y^T(k)u(k)$ and the storage function S satisfies $S(0) = 0$. In other words, a system Σ is passive if there is a nonnegative function $S : \Re^n \to \Re$, with $V(0) = 0$, satisfying

$$S(x(k+1) - S(x(k))) \leq y^T(k)u(k) \quad u(k) \in \Re^m \ \ and \ \forall k. \tag{5}$$

Similarly, it can also be shown that the above inequality is equivalent to

$$S(x(k+1)) - S(x(0)) \leq \Sigma_{i=0}^{k} y^T(i)u(i) \quad \forall u(k), \forall x(0), \forall k. \tag{6}$$

Now we state the definition in continuous time (Khalil (2002)) that define passivity for a dynamical system represented by the state model

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \tag{7}$$

where $f : \Re^n \times \Re^p \to \Re^n$ is locally Lipschitz, $h : \Re^n \times \Re^p \to \Re^p$ is continuos, $f(0,0) = 0$, and $h(0,0) = 0$. In this case the system has the same number of inputs and outputs.

Definition 3. System (7) is said to be passive if there exists a continuously differentiable positive semidefinite function $V(x)$ (called the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \quad \forall (x,u) \in \Re^n \times \Re^p \tag{8}$$

Moreover, it is said to be

- *lossless* if $u^T y = \dot{V}$
- *input-feedforward passive* if $u^T y \geq \dot{V} + u^T \varphi(u)$ for some function φ.
- *input strictly passive* if $u^T y \geq \dot{V} + u^T \varphi(u)$ and $u^T \varphi(u) > 0$, $\forall u \neq 0$.
- *output-feedback passive* if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ.
- *output strictly passive* if $u^T y \geq \dot{V} + y^T \rho(y)$ and $y^T \rho(y) > 0$, $\forall y \neq 0$.
- *strictly passive* if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ.

In all cases, the inequality should holds for all (x, u)

We extend the above definition to nonlinear discrete time systems. Again we assume that the system has the same number of inputs and outputs.

Definition 4. System (1) is said to be passive if there exists a continuously differentiable positive semidefinite function $S(x(k))$ (called the storage function) such that

$$\begin{aligned} u^T(k)y(k) &\geq S(x(k+1)) - S(x(k)) \\ &= \frac{\partial S(x(k))}{\partial x(k)} f(x(k), u(k)) \\ &\quad \forall (x(k), u(k)) \in \Re^n \times \Re^p \end{aligned} \tag{9}$$

Moreover, (1) is said to be

- *lossless* if $u^T(k)y(k) = S(x(k+1)) - S(x(k))$
- *input-feedforward passive* if $u^T(k)y(k) \geq S(x(k+1)) - S(x(k)) + u^T(k)\varphi(u(k))$ for some function φ.
- *input strictly passive* if $u^T y \geq S(x(k+1)) - S(x(k)) + u^T(k)\varphi(u(k))$ and $u^T(k)\varphi(u(k)) > 0$, $\forall u(k) \neq 0$.
- *output-feedback passive* if $u^T(k)y(k) \geq S(x(k+1)) - S(x(k)) + y^T(k)\rho(y(k))$ for some function ρ.
- *output strictly passive* if $u^T(k)y(k) \geq S(x(k+1)) - S(x(k)) + y^T(k)\rho(y(k))$ and $y^T(k)\rho(y(k)) > 0$, $\forall y(k) \neq 0$.
- *strictly passive* if $u^T(k)y(k) \geq S(x(k+1)) - S(x(k)) + \psi(x(k))$ for some positive definite function ψ.

In all cases, the inequality should holds for all $(x(k), u(k))$

2.2 Optimal Control

In this section,the discrete-time optimal control approach is presented. We consider the following cost functional (Lewis and Syrmos (1995)):

$$J = \sum_{k=0}^{\infty} L(x(k), u(k)) \tag{10}$$

associated with (1), to be minimized by the control law $u(k)$, where $L(x(k), u(k)) = l(x(k)) + u^T(k)u(k)$ and $l(x(k))$ is a positive semidefinite function. Similar to the continuous-time case, and considering that there exists a positive definite function $V : \Re^n \to \Re$, the discrete-time Hamiltonian becomes

$$H(x(k), u(k)) = L(x(k), u(k)) + V(x(k+1)) - V(x(k)) \tag{11}$$

which is used to obtain the control law $u(k)$ by calculating

$$\min_{u_k} H(x(k), u(k)) \tag{12}$$

The value of $u(k)$ which achieves this minimization is a feedback law denoted as $u(k) = \overline{u}(x(k))$, then

$$\min_{u_k} H(x(k), u(k)) = H(x(k), \overline{u}(k)) \tag{13}$$

A necessary condition, which this feedback optimal control law $\overline{u}(x(k))$ must satisfy (Kirk (1970)), is

$$H(x(k), \overline{u}(k)) = 0 \tag{14}$$

Then the optimal control law $\overline{u}(x(k))$ is obtained by calculating the gradient of (11) right-hand side with respect to $u(k)$ (Al-Tamimi and Lewis (2008))

$$0 = 2u(k) + \frac{\partial V(x(k+1))}{\partial u(k)} \tag{15}$$

which results in

$$0 = 2u(k) + g^T(x(k)\frac{\partial V(x(k+1))}{\partial x(k+1)} \tag{16}$$

Therefore, the feedback optimal control law is formulated as

$$\begin{aligned} u^*(k) &:= \overline{u}(k) \\ &= -\frac{1}{2}g^T(x(k))\frac{\partial V(x(k+1))}{\partial x(k+1)} \end{aligned} \tag{17}$$

with the boundary condition $V(0) = 0$. $u^*(k)$ is used to emphasize that $u(k)$ is optimal.

2.3 Inverse Optimal Control via CLF

In this section the discrete time inverse optimal control and its solution by proposing a quadratic CLF (Control Lyapunov Function) is established (Sanchez and Ornelas-Tellez (2013)). For the inverse optimal control approach, a candidate CLF is used to construct an optimal control law directly without solving the associated HJB equation (Freeman and Kokotovic (1996)). We focus on inverse optimality because it avoids to solve the HJB partial differential equations and still allows to obtain Kalman-type stability margins (Krstic and Deng (1998)).

We establish the following assumptions and definitions which allow the inverse optimal control solution via the CLF approach.

Assumption 5. The full state of system:

$$x(k+1) = f(x(k)) + g(x(k))u(k) \tag{18}$$

is measurable.

Definition 6. (Inverse Optimal Control Law)
Let define the control law (Ornelas-Tellez et al. (2012), Sanchez and Ornelas-Tellez (2013))

$$u(k) = -\frac{1}{2}R^{-1}(x(k))g^T(x(k))\frac{\partial V(x(k+1))}{\partial x(k+1)} \tag{19}$$

to be inverse optimal (globally) stabilizing if:

(1) It achieves (global) asymptotic stability of $x = 0$ for system (18)
(2) $V(x(k))$ is (radially unbounded) positive definite function such that inequality

$$\begin{aligned} \overline{V} &:= V(x(k+1)) - V(x(k)) \\ &\quad + u(k)^T R(x(k))u(k) \le 0 \end{aligned} \tag{20}$$

is satisfied.

When we select $l(x(k)) := -\overline{V} \ge 0$ then $V(x(k))$ is a solution for the HJB equation

$$\begin{aligned} & l(x(k)) + V(x(k+1)) - V(x(k)) \\ & + \frac{1}{4}V^{T*}R^{-1}(x(k))g^T(x(k))V^* = 0 \end{aligned} \tag{21}$$

where

$$V^{T*} = \frac{\partial V^T(x(k+1))}{\partial x(k+1)} \text{ y } V^* = \frac{\partial V(x(k+1))}{\partial x(k+1)}$$

We can establish the main conceptual differences between optimal control and inverse optimal control as follows:

- For optimal control, the cost indexes $Q(x(k)) \ge 0$ and $R(x(k)) > 0$ are given a priori; then, they are used to calculate $u(x(k))$ and $V(x(k))$ by means of the HJB equation solution.
- For inverse optimal control, a candidate CLF $V(x(k))$ and the meaningful cost index $R(x(k))$ are given a priori, and then these functions are used to calculate the inverse control law $u(k)$ and the cost index $l(x(k))$, defined as $l(x(k)) := -\overline{V}(x(k))$.

As established in definition (6), the inverse optimal control approach is based on the knowledge of $V(x(k))$. Thus, we propose a CLF $V(x(k))$, such that (1) and (2) are guaranteed. That is, instead of solving (21) for $V(x(k))$, we propose a control Lyapunov function $V(x(k))$ with the form:

$$V(x(k)) = \frac{1}{2}x^T(k)Px(k) \tag{22}$$

for control law (19), in order to ensure stability of the equilibrium point $x(k) = 0$ of system (18). Moreover, it is established that control law (19) with (22), which is referred to as the inverse optimal control law, optimizes a cost functional of the form:

$$J(x(k)) = \sum_{0}^{\infty}(l(x(k)) + u^T(k)R(x(k))u(k)) \tag{23}$$

Consequently, by considering $V(x(k))$ as in (22), the control law takes the following form:

$$\begin{aligned} \alpha(x(k)) &:= u(k) \\ &= -\frac{1}{2}(R(x(k)) + P_1(x(k)))^{-1}P_2(x(k)) \end{aligned} \tag{24}$$

where $P_1(x(k)) = \frac{1}{2}g^T(x(k))Pg(x(k))$ y $P_2(x(k)) = g^T(x(k))Pf(x(k))$. It is worth to point out that P and $R(x(k))$ are positive definite and symmetric matrices; thus, the existence of the inverse in (24) is ensured.

3. PASSIVITY ANALYSIS

Considering the inverse optimal control law based on CLF (24), with the associated cost functional (23), where $l(x(k)) := -\overline{V}$ in (21)

$$\begin{aligned} \overline{V} &:= V(x(k+1)) - V(x(k)) \\ &\quad + u^T(k)R(x(k))u(k) \le 0 \end{aligned} \tag{25}$$

satisfies the HJB equation, then we can formulate the following theorem

Theorem 7. (The Inverse Optimal Control law is passive)
Suppose there exists an inverse optimal control law $\alpha(x(k))$ (24), in which the term

$$-\overline{V} = -\left(V(x(k+1)) - V(x(k)) + u^T(k)R(x(k))u(k)\right) \ge 0 \tag{26}$$

is part of its structure. Then a storage function $S(x(k))$ and a supply rate $W = y^T(k)u(k)$ exist for discrete time systems (4) such that the passivity property (5) is fulfilled.

Proof
Let denote

$$-V(x(k+1)) + V(x(k)) - u^T(k)R(x(k))u(k) \geq 0 \tag{27}$$

as part of the inverse optimal control structure in discrete-time. From (26)

$$-V(x(k+1)) + V(x(k)) \geq u^T(k)R(x(k))u(k) \tag{28}$$

Then multiply by -1, this inequality is inverted

$$V(x(k+1)) - V(x(k)) \leq -u^T(k)R(x(k))u(k) \tag{29}$$

Consider $\lambda_{min}R(x(k))$, then

$$V(x(k+1)) - V(x(k)) \leq -\lambda_{min}R(x(k))u^T(k)u(k) \tag{30}$$

Finally, with $\lambda_{min}R(x(k)) = \sigma$, we reformulate (30) as

$$V(x(k+1)) - V(x(k)) \leq -\sigma u^T(k)u(k) \tag{31}$$

Moreover a storage function for the system is proposed as

$$S(x(k)) = V(x(k)) \tag{32}$$

with $V(x(k))$ the CLF function; then the following equivalence is stated

$$\begin{aligned} \triangle S(x(k)) &= \triangle V(x(k)) \\ S(x(k+1)) - S(x(k)) &= V(x(k+1)) - V(x(k)) \end{aligned} \tag{33}$$

Hence, (31) is rewritten as

$$S(x(k+1)) - S(x(k)) \leq -\sigma u^T(k)u(k) \tag{34}$$

Under the assumption that there exists a supply rate $W = y^T(k)u(k)$ and a storage function

$$S(x(k+1)) - S(x(k)) \leq y^T(k)u(k) \tag{35}$$

Then, if the following condition is satisfied

$$-\sigma u^T(k)u(k) \leq y^T(k)u(k) \tag{36}$$

we can establish

$$S(x(k+1)) - S(x(k)) \leq -\sigma u^T(k)u(k) \leq y^T(k)u(k) \tag{37}$$

Hence, we conclude that the inverse optimal control in discrete time is passive. □

4. EXAMPLE

Let consider the discrete-time nonlinear system:

$$\begin{aligned} x_1(k+1) &= \left(x_1^2(k) + x_2^2(k) + u(k)\right) cos\left\{x_2(k))\right\} \\ x_2(k+1) &= \left(x_1^2(k) + x_2^2(k) + u(k)\right) sin\left\{x_2(k))\right\} \\ y(k) &= \left(x_1^2(k) + x_2^2(k)\right) + \frac{u(k)}{x_1^2(k) + x_2^2(k) - R^2} \end{aligned} \tag{38}$$

which is open loop unstable as show in:

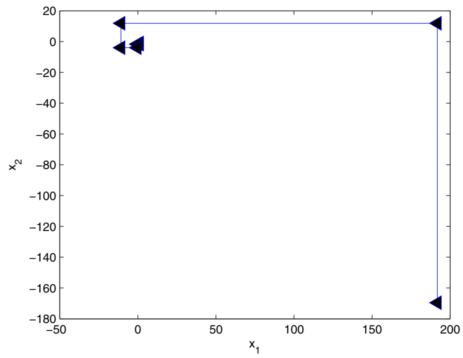

Fig. 1. Phase portrait of the open loop system

Then the inverse optimal control law (24), is defined as:

$$u(k) = -\frac{K(x_1^2(k) + x_2^2(k))(cos^2(x_2(k)) + sin^2(x_2(k))))}{2R + (cos^2(x_2(k)) + sin^2(x_2(k)))} \tag{39}$$

with the storage function $S(x(k)) = \frac{1}{2}\left(x_1^2(k) + x_2^2(k)\right)$ and a supply rate $w = y(k)u(k)$.
The simulation results with $K = 1$ and $R = 0.1$, are as follows:

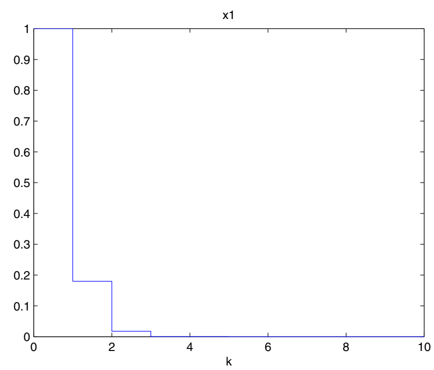

Fig. 2. x_1 Time evolution, with initial condition $x_{10} = 1$.

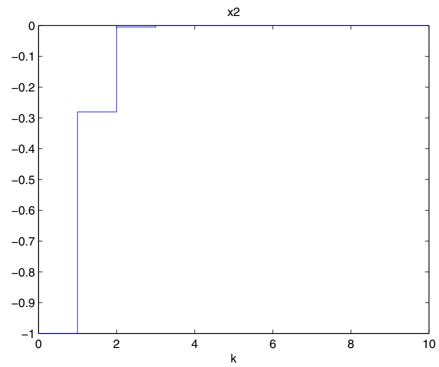

Fig. 3. x_2 Time evolution, with initial condition $x_{20} = -1$.

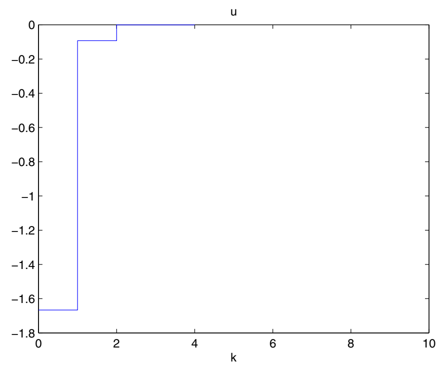

Fig. 4. Inverse optimal control law u.

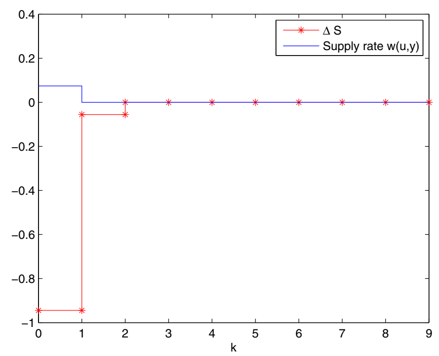

Fig. 5. Passivity condition: $\Delta S \leq w(u, y)$.

As can be seen, the passivity condition is fullfilled.

5. CONCLUSIONS

We establish a storage function $S(x(k))$ and a supply rate $W(u(k), y(k))$ from the construction of the discrete-time inverse optimal control based on CLF, to conclude that it is indeed passive.

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