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Passivity and Passive Feedback Stabilization for a Class of Mixed Potential Systems

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Abstract: This paper studies the input-output properties of a class of control affine systems where the drift dynamics is generated by a metriplectic structure. Those systems, related to generalized (or dissipative) Hamiltonian systems, are generated by a conserved component and a dissipative component and appear, for example, in non-equilibrium thermodynamics. In non-equilibrium thermodynamics, the two potentials generating the dynamics are interpreted as generalized energy and generalized entropy, respectively. In this note, passivity and passive feedback stabilization of this class of systems are studied, with the output function taken as the gradient of the conserved component of the dynamics, and the proposed storage function is computed using the dissipative (metric) component of the dynamics.

Keywords: Nonlinear control systems, metriplectic systems, thermodynamic systems, passive systems, feedback passive systems.

1. INTRODUCTION

We consider control affine systems of the form

$$\begin{cases} \dot{x} = X_0(x) + X_1(x)u \\ y = h(x) \end{cases} \tag{1}$$

with states $x \in \mathbb{R}^n$, and control $u \in \mathbb{R}^1$, where the drift vector field $X_0(x)$ is of the special form

$$X_0(x) = J(x)\nabla^T E(x) - R(x)\nabla^T S(x), \tag{2}$$

under degeneracy constraints

$$J(x) \cdot \nabla^T S(x) = 0 \tag{3}$$

$$R(x) \cdot \nabla^T E(x) = 0, \tag{4}$$

and such that $J(x)$ is antisymmetric ($J(x) = -J^T(x)$) and $R(x)$ is symmetric positive-definite ($R(x) = R^T(x) \succ 0$). We assume that the generating functions $E(x)$ and $S(x)$ are of class $\mathcal{C}^k(\mathbb{R}^n;\mathbb{R})$, with $k \geq 2$ and that the drift dynamics $X_0(x)$ has an isolated equilibrium x_s. It is assumed that the output map $h(x)$ is a certain function of the gradient of the conserved quantity $E(x)$,

$$h(x) = X_1^T(x)\nabla^T E(x). \tag{5}$$

The objective of the present note is to study passivity and passive stabilizing feedback control design for this class of systems, following the approach surveyed for example in (Byrnes et al., 1991; van der Schaft, 2000; Sepulchre et al., 1997).

Systems of the form (2) are known in the literature as metriplectic systems (Kaufman, 1984; Morrison, 1986; Guha, 2007). These systems are generated by a conserved quantity $E(x)$ and a metric quantity $S(x)$. Under the degeneracy conditions (3-4), their dynamics can be re-expressed as dissipative Hamiltonian systems (van der Schaft, 2000), *i.e.*, systems of the form

$$\dot{x} = [J(x) - R(x)]\nabla^T H(x) + g(x)u,$$

studied extensively in the literature, for example in (van der Schaft, 2000; Ortega et al., 2002; Cheng et al., 2002), if one pick $H(x)$ to be the free energy at unit temperature, $H(x) = E(x) - S(x)$ (Favache et al., 2010). However, the problem proposed above differs by the nature of the output, which is usually taken as

$$y = g^T(x)\nabla^T H(x)$$

in the case of dissipative Hamiltonian systems. In the present case, we assume no direct measure of the metric quantity $S(x)$, which is however central to the construction of the storage function candidate in the sequel. Systems of the form (2) are interesting from the point of view of non-equilibrium thermodynamic systems, since the so-called GENERIC formulation of thermodynamics, proposed originally in (Grmela and Öttinger, 1997; Öttinger and Grmela, 1997) and reviewed extensively in (Öttinger, 2005), is based on the development of metriplectic systems proposed originally in (Kaufman, 1984; Morrison, 1986). It should be noted that under the degeneracy conditions given above, the quantity $F(x) = E(x) - S(x)$ (or more generally, $F(x) = E(x) - TS(x)$, where T is the temperature) can be interpreted as generalized free energy (Morrison, 1986; Guha, 2007). Obviously, mixed potentials, that is, potentials combining energy and entropy, are known from classical thermodynamics, see for example the availability potential used in (Ydstie and Alonso, 1997) in the context of passive systems theory and control of process systems. Moreover, this class of systems is related to the representation of smooth nonlinear dynamical systems as the sum of a gradient system, given by the metric part of the system generated by the generalized entropy $S(x)$, and $(n-1)$ Hamiltonian systems, given by the symplectic part of the system generated by the generalized energy $E(x)$, as presented for example in (Roels, 1974). It can also be related to a recent construction presented in (Guay et al., 2012, 2013), where nonlinear control systems are decomposed using the Hodge decomposition theorem (Warner, 1983) to obtain potential-driven representations, such as gradient systems, generalized Hamiltonian systems, and systems described by Brayton–Moser equations. It should be noted that asymptotic stability of metriplectic systems was studied in (Birtea et al., 2007) using Lasalle's invariance principle and in (Birtea and Comănescu, 2009) in relation to the energy-Casimir method (Aeyels, 1992). In particular, the contribution by (Birtea et al., 2007) considered a class of metriplectic systems where the metric part of the system is generated by Casimir functions.

In this note, we consider a local decomposition of the drift vector field to generate a potential $V(x)$ to be used as a storage function candidate for the system. Stability and state feedback stabilization of metriplectic systems using a radial homotopy decomposition were recently studied in (Hudon et al., 2013a). A similar construction using a homotopy decomposition based on the gradient of the generalized entropy of the system was proposed in (Hudon et al., 2013b). Essentially, the decomposition in (Hudon et al., 2013b) amounts to a projection of the metriplectic dynamics on its metric part. This particular construction is used in the present note to construct a storage function candidate, as an extension to input-output characterization of metriplectic systems. In particular, applying passive systems theory following (Byrnes et al., 1991) leads to a straightforward output feedback control strategy for metriplectic systems, or more generally, in the context of systems composed of conserved elements and dissipative elements.

The paper is organized as follows. Preliminaries on passive systems theory are briefly reviewed in Section 2. Construction of a storage function candidate for the metriplectic using a homotopy decomposition approach, characterization of passive properties and passive feedback stabilizing controller design are given in Section 3. An illustration of the proposed approach is given in Section 4. Conclusion and future areas for investigation are outlined in Section 5.

2. REVIEW OF PASSIVE SYSTEMS THEORY

We first recall elements of passive systems theory, following the contribution from Byrnes et al. (1991). Consider the control affine system

$$\Sigma : \quad \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x), \end{cases} \tag{6}$$

$x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^1$, $y \in \mathcal{Y} \subset \mathbb{R}^1$, where f, g are smooth vector fields and $h(x)$ is a smooth map. It is assumed that the drift vector field f has at least one isolated equilibrium x_s, *i.e.*, $f(x_s) = h(x_s) = 0$.

We first recall the classical definition of dissipative systems (Sepulchre et al., 1997; van der Schaft, 2000).

Definition 1. A control affine system Σ is said to be dissipative with respect to the supply rate $s(u, y) : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ if there exists a storage function $V : \mathcal{X} \to \mathbb{R}^+$, and that for all $t_1 \geq t_0$, and all input functions $u(\cdot)$, the following inequality holds

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t))dt, \tag{7}$$

with $x(t_0) = x_0$ and $x(t_1)$ is the state resulting, at time t_1 from the solution of (6) taking x_0 as initial condition and $u(t)$ as control input the function. If $V(\cdot)$ is differentiable with respect to time for all $x \in \mathcal{X}$ and u, the inequality (7) is equivalent to

$$\dot{V}(x) \leq s(u(t), y(t)). \tag{8}$$

The system is said to be lossless if inequalities (7) or (8) are equalities.

In the present note, we focus on the special case of passive systems, *i.e.*, single-input single-output dissipative systems with the special form of supply rate $s(u, y) = u \cdot y$. Following the developments in (Byrnes et al., 1991), we are interested in the sequel in a fundamental property of passive systems, the Kalman–Yacubovich–Popov property.

Definition 2. A system Σ has the Kalman–Yacubovich–Popov (KYP) property if there exists a C^1 nonnegative function $V : \mathcal{X} \to \mathbb{R}$, with $V(0) = 0$ such that

$$\mathcal{L}_f V(x) \leq 0 \tag{9}$$

$$\mathcal{L}_g V(x) = h(x) \tag{10}$$

for each $x \in \mathcal{X}$.

The following proposition has been proved in (Byrnes et al., 1991).

Proposition 3. A system Σ which has the KYP property is passive, with storage function V. Conversely, a passive system having a C^1 storage function has the KYP property.

In a previous contribution (Hudon et al., 2013b), we studied the problem of designing a damping state feedback controller of the form

$$u = -(\mathcal{L}_g V(x)) \tag{11}$$

for metriplectic systems. This approach is obviously equivalent to a passive feedback design with unit gain if the output for the system is $y = (\mathcal{L}_g V(x))$. This property was exploited in the case of dissipative Hamiltonian systems (van der Schaft, 2000; Ortega et al., 2002) under mild conditions on the generating Hamiltonian function. In the present note, we seek to study under which conditions metriplectic systems (1) with drift dynamics (2) can be rendered passive with the output function (5), which is only a function of the conserved quantity $E(x)$.

Following (Byrnes et al., 1991), we assume that the system is zero-state detectable, *i.e.*, we assume that for $y \equiv 0$

(and $u \equiv 0$), $x \to x_s$ as $t \to \infty$, with x_s the desired equilibrium (Sepulchre et al., 1997). In the following, our interest is to develop an output feedback control strategy for metriplectic systems using essentially the following result proved in (Byrnes et al., 1991).

Theorem 4. Suppose Σ is passive with storage function V which is positive definite. Suppose Σ is locally zero-state detectable. Let $\phi : \mathcal{Y} \to \mathcal{U}$ be any smooth function such that $\phi(0) = 0$ and $y \cdot \phi(y) > 0$ for each nonzero y. The control law $u = -\phi(y)$ asymptotically stabilizes the equilibrium $x = x_s$.

3. PASSIVITY OF METRIPLECTIC SYSTEMS

This section presents the main construction of the paper. Following the previous approach proposed in (Hudon et al., 2013b), a storage function candidate for the metriplectic system is constructed using homotopy in Section 3.1. The KYP property is studied in Section 3.2. Properties of metriplectic systems in closed-loop with passive output feedback is studied in Section 3.3.

3.1 Construction of a Storage Function Candidate

We first construct a storage function $V(x)$ for the metriplectic drift vector field (2) by using homotopy decomposition of an exterior differential form $\omega(x)$ associated to the system. We first briefly recall the exterior differential notation (Lee, 2006). We denote a smooth vector field in $\Gamma^\infty(\mathbb{R}^n)$ as $X(x) = \sum_{i=1}^n v^i(x)\frac{\partial}{\partial x_i}$ and a smooth differential one-form in $\Lambda^1(\mathbb{R}^n)$ as $\omega(x) = \sum_{i=1}^n \omega_i(x)dx_i$, where $v^i(x)$ and $\omega_i(x)$ are smooth functions on $\mathbb{R}^n$. The standard basis for vectors in $\Gamma^\infty(\mathbb{R}^n)$ and one-forms in $\Lambda^1(\mathbb{R}^n)$ are denoted by $\frac{\partial}{\partial x_i}$ and dx_i, respectively. The interior product of a differential form ω with respect to a vector field X is denoted by $X \lrcorner \omega$.

The derivation of a differential one-form associated to the drift vector field $X_0(x)$ relies on the canonical Riemannian metric in $\mathbb{R}^n$, given as $g = dx_1 \otimes dx_1 + \ldots + dx_n \otimes dx_n$ with its associated volume form in $\Lambda^n(\mathbb{R}^n)$, expressed as $\mu = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. For a given drift vector field $X_0(x) = \sum_{i=1}^n X_{0,i}(x)\frac{\partial}{\partial x_i}$, we seek to construct a representation with an associated natural generating potential function. The proposed construction consists in computing a differential one-form $\omega \in \Lambda^1(\mathbb{R}^n)$ that encodes the divergence of the drift vector field $X_0(x)$. Such a one-form is obtained by using the Hodge star operator $\star$ of a $(n-1)$ form j, *i.e.*,

$$\omega = \star j = \star(X_0(x) \lrcorner \mu) = (-1)^{n-1}\sum_{i=1}^n X_{0,i}(x)dx_i. \quad (12)$$

If the one-form $\omega(x)$ is closed, *i.e.*, if $d\omega(x) = 0$, it can be shown that it is also locally exact, by virtue of the Poincaré Lemma, and the system is conservative (in particular, the dynamics is generated by the gradient of a potential function). However, if the one-form is not closed, $\omega(x)$ can be decomposed as the sum of an exact component and an anti-exact component. A geometric decomposition can be carried locally, using a homotopy operator, to distinguish both components. A homotopy operator $\mathbb{H}$, is a linear operator on elements of $\Lambda^k(\mathbb{R}^n)$ that satisfies the identity

$$\omega(x) = d(\mathbb{H}\omega)(x) + (\mathbb{H}d\omega)(x), \quad (13)$$

for a given differential form $\omega \in \Lambda^k(\mathbb{R}^n)$.

Using the notation from above, the drift vector field (2) is given as

$$X_0(x) = \sum_{i=1}^n \left(\sum_{j=1}^n J_{ij}(x)\frac{\partial E(x)}{\partial x_j} - R_{ij}(x)\frac{\partial S(x)}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \quad (14)$$

Following the approach depicted above, we first construct a differential one-form $\omega(x) \in \Lambda^1(\mathbb{R}^n)$ for $X_0(x)$. The one-form $\omega(x)$ for the metriplectic drift vector field (14) is thus given by:

$$\omega(x) = (-1)^{n-1}\sum_{i=1}^n X_{0,i}dx_i, \quad (15)$$

where

$$X_{i,0} = \sum_{j=1}^n J_{ij}(x)\frac{\partial E(x)}{\partial x_j} - R_{ij}(x)\frac{\partial S(x)}{\partial x_j}.$$

Following (Hudon et al., 2013b), we use a homotopy operator defined using the metric part of the system (2) to compute a storage function for the system. More precisely, we define a vector field $\mathfrak{X}(x)$ different from the radial vector field used in (Hudon et al., 2008).

First, we make the following assumption on the potentials $E(x)$ and $S(x)$ of (2).

Assumption 5. The potentials $E(x)$ and $S(x)$ are such that $J(x_s) \cdot \nabla^T E(x_s) = 0$ and $R(x_s) \cdot \nabla^T S(x_s) = 0$ at an isolated equilibrium x_s.

This assumption essentially states the fact that the gradients of both potentials are zero at the equilibrium x_s, which in the context of non-equilibrium thermodynamics can be interpreted as the fact that, at an equilibrium point x_s, the energy is minimized, while the entropy is maximized. In the sequel, we consider the locally-defined vector field defined by

$$\mathfrak{X} = (-1)^{n-1}\sum_{i=1}^n \frac{\partial S}{\partial x_i}(x)\frac{\partial}{\partial x_i}. \quad (16)$$

First, we note that by application of the homotopy operator constructed using the locally defined vector field $\mathfrak{X}$ from (16), we obtain the following.

Proposition 6. By projection of the metriplectic system (15) on the local vector field $\mathfrak{X}$ defined by (16), the potential $\psi(x)$ obtained by homotopy is given by

$$\psi(x) = -\int_0^1 [(\nabla S)(\lambda x)R(x_s + \lambda(x - x_s))(\nabla S)^T(\lambda x)]d\lambda.$$

Proof. Using a vector notation, we have $\omega(x)$ given as

$$\omega(x) = (-1)^{n-1}\left[J(x)\nabla^T E(x) - R(x)\nabla^T S(x)\right]\begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}^T.$$

Under the Assumption 5, denoting $\bar{x} = x_s + \lambda(x - x_s)$, we have

$$\omega(\bar{x}) = (-1)^{n-1}\left[J(\bar{x})\nabla^T E(\lambda x) - R(\bar{x})\nabla^T S(\lambda x)\right]\begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}^T.$$

Taking the interior product $\mathfrak{X}\lrcorner\omega(\bar{x})$, with

$$\mathfrak{X} = (\nabla S)\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix},$$

we have

$$\mathfrak{X}\lrcorner\omega(\bar{x}) = \nabla S(\lambda x)J(\bar{x})\nabla^T E(\lambda x) - \nabla S(\lambda x)R(\bar{x})\nabla^T S(\lambda x).$$

By the degeneracy condition $J(\cdot)\nabla^T S(\cdot)$, and since $J(\cdot) = -J^T(\cdot)$, we have $\nabla S(\cdot)J(\cdot) = -\nabla S(\cdot)J^T(\cdot) = (J\nabla^T S)^T(\cdot) = 0$, and hence

$$\mathfrak{X}\lrcorner\omega(\bar{x}) = -\nabla S(\lambda x)R(\bar{x})\nabla^T S(\lambda x).$$

By homotopy, we obtain directly

$$\psi(x) = (\mathbb{H}\omega)(x) = -\int_0^1 [(\nabla S)(\lambda x)R(x_s + \lambda(x - x_s))(\nabla S)^T(\lambda x)]d\lambda.$$

Remark 7. At this point, one should remark that a projection on the conserved component of the dynamics $\nabla E(x)$ would not be useful to construct a suitable potential for the system. In particular, for

$$\mathfrak{X} = (-1)^{n-1}\sum_{i=1}^{n}\frac{\partial E}{\partial x_i}\frac{\partial}{\partial x_i}, \quad (17)$$

we have the interior product

$$\mathfrak{X}\lrcorner\omega(\bar{x}) = \nabla E(\lambda x)J(\bar{x})\nabla^T E(\lambda x) - \nabla E(\lambda x)R(\bar{x})\nabla^T S(\lambda x).$$

Since $J = -J^T$, the energy term $(\nabla E J\nabla^T E)(\cdot) = 0$ and by the degeneracy condition $R(x)\nabla^T E(x) = 0$ with symmetric $R(x) = R^T(x)$, we have $(\nabla E R\nabla^T E)(\cdot) = 0$. As a consequence $\psi(x) = \mathbb{H}\omega(x)$ computed along (17) would be trivially zero.

Since the symmetric matrix $R(x)$ is positive definite, the potential $\psi(x)$ is negative definite. Hence, a suitable storage function candidate for the system is therefore chosen as

$$\begin{aligned} V(x) &= -\psi(x) \\ &= \int_0^1 [(\nabla S)(\lambda x)R(x_s + \lambda(x - x_s))(\nabla S)^T(\lambda x)]d\lambda \quad (18) \\ &> 0. \end{aligned}$$

Moreover, under Assumption 5, $V(x)$ is bounded from below, *i.e.*, $V(x_s) = 0$. Passive properties of the metriplectic system (1)–(2) with output (5) is studied in the following section based on the storage function $V(x)$.

3.2 KYP property for Metriplectic Systems

In order to establish the passive properties of metriplectic systems, we seek to establish under which conditions the metriplectic system (1)–(2) with output (5) has the KYP property. The first part of Definition 2 is established in the following.

Proposition 8. For the metriplectic system (2), we have

$$\mathcal{L}_{X_0}V(x) = (X_0\lrcorner dV)(x) < 0, \quad x \neq x_s$$

for the storage function $V(x)$ derived in (18) for all x in a neighborhood of the equilibrium, with $\mathcal{L}_{X_0}V(x_s) = 0$.

Proof. As stated above, the exterior derivative of the potential $V(x)$ can be identified with the exact part of the one-form $\omega(x)$, which, given the construction above, is given by the metric part of $\omega(x)$, *i.e.*,

$$\begin{aligned} dV(x) &= \omega_e(x) \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_{ij}(x)\frac{\partial S}{\partial x_j}\right)dx_i. \end{aligned}$$

Taking the interior product, one obtains (dropping the arguments)

$$\begin{aligned} (X_0\lrcorner dV) = &\sum_{i=1}^{n}\left(\sum_{j=1}^{n}J_{ij}\frac{\partial E}{\partial x_j} - R_{ij}\frac{\partial S}{\partial x_j}\right)\frac{\partial}{\partial x_i} \\ &\lrcorner\sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_{ij}(x)\frac{\partial S}{\partial x_j}\right)dx_i. \end{aligned}$$

Distributing, we obtain

$$(X_0\lrcorner dV) = (J\nabla^T E)^T(R\nabla^T S) - (R\nabla^T S)^T(R\nabla^T S).$$

By using the degeneracy constraint $R(x)\nabla^T E(x) = 0$, and since $R(x) = R^T(x)$, we obtain directly

$$(X_0\lrcorner dV) = -(R\nabla^T S)^T(R\nabla^T S) < 0$$

and using Assumption 5, $\dot{V} = 0$ at the equilibrium x_s.

We now study the conditions under which metriplectic systems have the KYP property.

Proposition 9. (1)–(2) with output (5) has the KYP property with storage function (18) if

$$X_1^T(\nabla^T E - R\nabla^T S) = 0.$$

Proof. What is left to show is the second part of Definition 2, *i.e.*, $\mathcal{L}_g V(x) = h(x)$. With $h(x) = X_1^T\nabla^T E(x)$, we have, using exterior calculus notation:

$$X_1\lrcorner dV = X_1 dE. \quad (19)$$

From above, we have

$$\begin{aligned} dV(x) &= \omega_e(x) \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_{ij}(x)\frac{\partial S}{\partial x_j}\right)dx_i. \end{aligned}$$

Hence, we have the condition

$$X_1\lrcorner\left(dE - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}R_{ij}(x)\frac{\partial S}{\partial x_j}\right)dx_i\right) = 0.$$

In compact form, this is equivalent to $X_1^T(\nabla^T E - R\nabla^T S) = 0$.

Remark 10. In general, *i.e.*, for an arbitrary vector field X_1, the KYP condition boils down to

$$(\nabla^T E - R\nabla^T S) = 0.$$

The interpretation of this condition is equivalent to say that the generalized entropy function $S(x)$ is a Casimir of the generalized energy $E(x)$, see for example (Birtea et al., 2007).

3.3 Passive Output Feedback of Metriplectic Systems

We finally study the metriplectic system (1)–(2) with output (5) in closed-loop with the output feedback $u = -Ky$. The central element to consider here is the application of Theorem 4 in our context. The result is summarized in the following theorem.

Theorem 11. Consider the metriplectic system (1)–(2) with output (5) in closed-loop with the output feedback $u = -Ky$, with $K > 0$. Then, an isolated equilibrium x_s such that Assumption 5 is met, is an asymptotically stable equilibrium of the closed-loop if $S(x)$ is a Casimir function of $E(x)$.

Proof. First, by Proposition 3 and the development from the last Section, we have that if $X_1^T(\nabla^T E - R\nabla^T S) = 0$, since it is assumed that $S(x)$ is a Casimir of $E(x)$, then the system is passive, since the system has the KYP property with storage function $V(x)$ given in (18). By the Assumption 5, both conserved and metric components go to zero simultaneously, and as a consequence, the metriplectic system is zero-state detectable. Then, by Theorem 4, the control law $u = -\phi(y)$, and in particular the constant-gain control law $u = -Ky$, asymptotically stabilizes the equilibrium $x = x_s$.

4. EXAMPLE

To illustrate the above approach to the output stabilization of metriplectic systems, we consider the stabilization of the rigid body, an example considered in (Morrison, 1986; Bloch and Marsden, 1990; Aeyels, 1992). The control system is given by:

$$\dot{x}_1 = (I_2 - I_3)x_2x_3 \tag{20}$$

$$\dot{x}_2 = (I_3 - I_1)x_1x_3 \tag{21}$$

$$\dot{x}_3 = (I_1 - I_2)x_1x_2 + u, \tag{22}$$

where $I_1 < I_2 < I_3$ are positive constants. Following the discussion in (Morrison, 1986), we have two conserved quantities

$$E(x) = \frac{1}{2}(I_1x_1^2 + I_2x_2^2 + I_3x_3^2) \tag{23}$$

$$l^2(x) = x_1^2 + x_2^2 + x_3^2, \tag{24}$$

from which it is possible to recover a metriplectic representation (2), by setting $S(x) = \phi(l^2(x))$, with $\phi(\cdot)$, an arbitrary function, and

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{25}$$

$$R = \begin{bmatrix} I_2^2x_2^2 + I_3^2x_3^2 & -I_1I_2x_1x_2 & -I_1I_3x_1x_3 \\ -I_1I_2x_1x_2 & I_1^2x_1^2 + I_3^2x_3^2 & -I_2I_3x_1x_3 \\ -I_1I_3x_1x_3 & -I_2I_3x_1x_3 & I_1^2x_1^2 + I_2^2x_2^2 \end{bmatrix}. \tag{26}$$

The objective is to stabilize an isolated equilibrium located at $x_s = (x_{1d}, 0, 0)^T$, where x_{1d} depends on the initial conditions (Birtea and Comănescu, 2009). In the sequel, we consider the metriplectic system generated using J, $R(x)$, $E(x)$ and $S(x) = l^2(x)$, with the output $y = h(x) = X_1^T\nabla^T E(x)$. For this construction the system has the KYP property, with a storage function constructed as demonstrated above.

The output passive feedback controller $u = -Ky$ is given by

$$u(x) = -KX_1^T\nabla^T E(x) \tag{27}$$

$$= -K\frac{I_3}{2}x_3, \tag{28}$$

which is different from the controller proposed originally in (Bloch and Marsden, 1990). Figures 1 and 2 show the trajectory of the dynamics in closed-loop and the control action for the system with parameters $I_1 = 1/2$, $I_2 = 1$, $I_3 = 3/2$, initial conditions $x_0 = (1, 1, 1)^T$, and unit output controller gain $K = 1$.

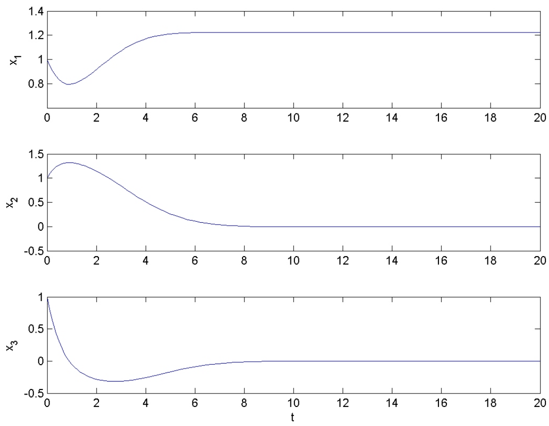

Fig. 1. Closed-loop state dynamics

5. CONCLUSION

We studied passive and passive feedback control design for a class of control affine systems where the drift is generated by two potentials, interpreted as generalized energy and generalized entropy, under some degeneracy constraints. This class of systems proved to be useful in studying dynamical properties of thermodynamical systems. In the present note, the output is taken as the gradient of the conserved quantity. A potential for the metriplectic system was derived using a homotopy operator centered at an isolated equilibrium of the system. Using the obtained potential, characterization of the passive character of the system was studied. Stability of the closed-loop system under unit output feedback was demonstrated. Future

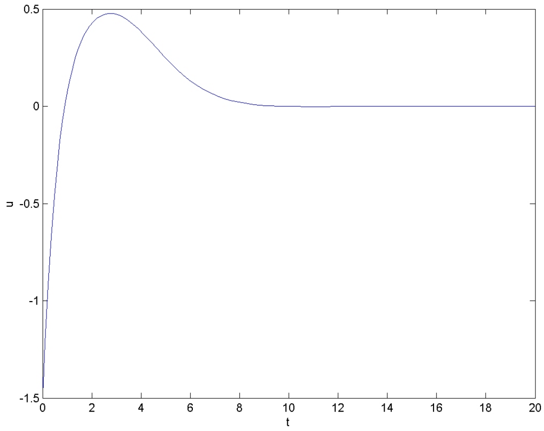

Fig. 2. Passive output feedback control signal

investigations would consider the problem of studying $\mathcal{L}_2$-gain and inverse optimality for this class of dynamical systems following the techniques developed in (van der Schaft, 2000).

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