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Improved Convergence Rate of Discontinuous Finite-Time Controllers ⋆

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Abstract: We propose a generalized homogeneous controller for a class of single-input uncertain dynamical system. The controllers are designed by means of Homogeneous Control Lyapunov Functions (HCLF's). The proposed controllers in feedback with the system make the close-loop system homogeneous of some degree. Depending on the selection of the parameters in the control law, the state trajectories reach the origin of the system asymptotically or in finite time. A class of homogeneous discontinuous controllers is also recovered. Furthermore, different discontinuous finite-time controllers with improved convergence rate can be synthesized.

Keywords: Lyapunov's methods; Discontinuous control; Sliding mode control.

1. INTRODUCTION

Continuous (discontinuous) finite-time controller's design has been a research area for decades. Finite-time reaching of the control target, high precision and better robustness and disturbance rejection properties are featured from this controllers, Bhat and Bernstein [2000], Levant [2005].

Homogeneity theory is widely exploited to construct systematically continuous and discontinuous feedback control laws. Homogeneous systems naturally appear as a local approximation to nonlinear systems. A linearization around the origin is, in fact, an approximation of a nonlinear system by a linear one, which is naturally homogeneous. The convergence rate of an asymptotically stable homogeneous system (homogeneous differential inclusion) is characterized by the homogeneity degree of its vector field. In general, (a) if the degree is negative, it is finite-time stable; (b) if the degree is zero, it is exponentially stable and (c) if the degree is positive, it is asymptotically stable, Hahn [1967], Bhat and Bernstein [2005], Bacciotti and Rosier [2005], Nakamura et al. [2002], Levant [2005]. Different convergence rates are obtained by changing the homogeneity degree of the vector field associated to the closed-loop system.

Generally, feedback stabilization of a control system is solved by using Control Lyapunov Functions (CLF's). Only few works establish methods for constructing CLF (see Freeman and Kokotovic [1996] and references there in). Lyapunov redesign or min-max method is one of the earliest frameworks for robust nonlinear control, Khalil [2002]. This method relies on a known CLF for the nominal system (system without uncertainty) to be used as a Robust CLF (RCLF) for the uncertain system under the well-known *matching condition*. Basically, a discontinuous control is introduced together with the nominal one in order to reject matched bounded uncertainties/disturbances that nominal control law is not able to compensate.

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An improvement on discontinuous controls can be provided by a suitable switching strategy between different suitable controllers. It allows to the controlled system, among other things, improving the regulation rates and the settling time, compared to the case where only a single controller is employed (see Adamy and Flemming [2004] and references there in).

In this paper a new family of homogeneous controllers is introduced for a single-input nonlinear perturbed dynamic systems. Some parameters are added to the control law to manipulate the homogeneity degree in order to ensure different stability properties for closed-loop system. It leads to synthesize a widely variety of robust controllers by changing the homogeneity degree. Therefore, we can recover homogeneous rational, exponential and continuous finite-time controllers from the same control law structure. Homogeneous discontinuous controllers are also recovered. Since, the continuous controllers are not robust in spite of matched disturbances, Lyapunov redesign is used to improve the robustness properties of the controller. A convergence rate improvement of the proposed controller is attained by switching different controllers with different stability proprieties. Therefore, fast and fixed-time discontinuous controllers can be synthesized, Yu and Man [2002], Cruz Zavala et al. [2012], Polyakov [2012]. Essentially, the control design is based on homogeneous CLF. They are obtained by using homogeneous properties and the standard backstepping technique. Besides, Lyapunov's design leads to estimate the convergence time and let us find sufficient conditions for gain tuning.

2. HOMOGENEOUS SYSTEMS AND CONTROL LYAPUNOV FUNCTIONS

We recall the concepts of continuous homogeneous functions and vector fields from Bacciotti and Rosier [2005]. The latter concept has been extended to (Filippov) Differential Inclusions (ID's) in Levant [2005].

Definition 1. Let $\Delta_\varepsilon^r x := (\varepsilon^{r_1}x_1, ..., \varepsilon^{r_n}x_n) = diag(\varepsilon^{r_i})x$ be the dilation operator for any $\varepsilon > 0$ and $\forall x \in \mathbb{R}^n$, where r_i are positive numbers (weights of the coordinates).

i) A function $V:\mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $m \in \mathbb{R}$ with respect to (w.r.t.) $\Delta_\varepsilon^r x$, if the identity $V(\Delta_\varepsilon^r x) = \varepsilon^m V(x), \forall x \in \mathbb{R}^n$, holds.
ii) A vector field $f:\mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous of degree $l \in \mathbb{R}$ w.r.t. $\Delta_\varepsilon^r x$, if the identity $f(\Delta_\varepsilon^r x) = \varepsilon^l \Delta_\varepsilon^r f(x)$, $\forall x \in \mathbb{R}^n$, holds.
iii) A vector-set field $F(x) \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, is called homogeneous of the degree $l \in \mathbb{R}$ w.r.t. $\Delta_\varepsilon^r x$, if the identity $F(\Delta_\varepsilon^r x) = \varepsilon^l \Delta_\varepsilon^r F(x)$ holds.

A DI, $\dot{x} \in F(x)$, is further called a Filippov DI if the vector set $F(x)$ is non-empty, closed, convex, locally bounded and upper-semicontinuous, Filippov [1988]. Some properties of homogeneous functions and vector fields are given in the following Corollary. We consider that the weights r_i and $\Delta_\varepsilon^r x$ are fixed.

Lemma 2. Bacciotti and Rosier [2005]. Let $\Delta_\varepsilon^r x$ be any family of dilations on $\mathbb{R}^n$, and let V_1, V_2 (respectively, f_1, f_2) be homogeneous functions (resp., vectors fields) w.r.t. $\Delta_\varepsilon^r x$ of degrees m_1,m_2 (resp., l_1, l_2). Then,

(1) $V_1 V_2$ (respectively, $V_1 f_1$, $[f_1, f_2]$ is homogeneous of degree $m_1 + m_2$ (respectively, $m_1 + l_1$, $l_1 + l_2$).
(2) $\dot{V} = L_{f_1} V$ is homogeneous of degree $m + l_1$.

Consider the input affine nonlinear system described by

$$\dot{x} = f(x) + g(x)u,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^1, \tag{1}$$

where $f(0) = 0$ and $f(x)$ and $g(x)$ are continuous mappings.

Definition 3. (Homogeneous System): System (1) is said to be homogenous of degree $l \in \mathbb{R}$ w.r.t. $\Delta_\varepsilon^r x$, if there exists $u(x)$ such that$f(\Delta_\varepsilon^r x) + g(\Delta_\varepsilon^r x)u(\Delta_\varepsilon^r x) = \varepsilon^l \Delta_\varepsilon^r (f(x) + g(x)u)$.

Asymptotic stability of homogenous systems (and homogeneous DI's) can be studied by means of homogeneous Lyapunov functions (HLF's), Bacciotti and Rosier [2005], Bhat and Bernstein [2005], Nakamura et al. [2002]. For a homogeneous continuous vector field f of degree l with locally asymptotically stable equilibrium point, a C^p HLF of degree m exists if $m > p \cdot \max_i\{r_i\}$ for any $p \in \mathbb{N}$, Bacciotti and Rosier [2005].

Theorem 4. Nakamura et al. [2002]. Assume that the origin of a homogeneous Filippov DI's, $\dot{x} \in F(x)$ is uniformly globally asymptotically stable (AS). Then, there exists a C^∞ homogeneous strong Lyapunov function.

It is worth pointing out that it is possible to assert the existence of continuous (but not necessarily C^1) HLF's.

Definition 5. Sontag [1998]. A C^1 proper positive-definite (homogeneous) function $V:\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a (homogeneous) CLF for system (1) if $\inf_{u \in \mathbb{R}^1}\{L_f V + L_g V \cdot u\} < 0, \forall x \in \mathbb{R}^n \backslash \{0\}$, where $L_f V = \frac{\partial V(x)}{\partial x} \cdot f(x)$ and $L_g V = \frac{\partial V(x)}{\partial x} \cdot g(x)$.

The definition guarantees that the origin is the only stationary point of a CLF. Finally, we introduce the notion on Fixed-time (FxT) stability. Consider the following dynamical system

$$\dot{x} = f(x),\ x_0 = x(0), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $f:\mathbb{R}^n \to \mathbb{R}^n$ is the associated vector field. Assume that the origin is an AS equilibrium point, i.e.,$f(0) = 0$. Solutions of (2) are understood in the sense of Filippov [1988]. Define an open ball centered at the origin with radius $\mu > 0$ by $B_\mu = \{x \in \mathbb{R}^n : \|x\| < \mu\}$.

Definition 6. Bacciotti and Rosier [2005]. The origin of system (2) is said to be

i) rationally stable if there exists positive constants $r, b_1, b_2 > 0$ and $0 < \eta \leq 1$ such that for any $x_0 \in B_\mu$, the solution $x(t, x_0)$ is defined on $[0, +\infty)$ and satisfies

$$\|x(t, x_0)\| \leq b_1 (1 + \|x_0\|^{b_2} t)^{-\frac{1}{b_2}} \|x_0\|^\eta,\ \forall t \geq 0,$$

ii) exponentially stable if there exists positive constants r, b_1, $b_2 > 0$ and $0 < \eta \leq 1$ such that for any $x_0 \in B_\mu$, the solution $x(t, x_0)$ is defined on $[0, +\infty)$ and satisfies

$$\|x(t, x_0)\| \leq b_1 \exp(-b_2 t)\|x_0\|,\ \forall t \geq 0,$$

iii) finite-time stable (FTS) if it is AS and for every $x_0 \in B_\mu \setminus \{0\}$, any solution $x(t, x_0)$ of (2) reaches $x(t, x_0) = 0$ at some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0)$, where $T:\mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

The notions are global if they satisfy Definition 6 with $B_\mu = \mathbb{R}^n$.

Definition 7. Polyakov [2012]. The set B_μ is said to be

i) globally finite-time attractive (GFTA) for system (2) if any solution $x(t, x_0)$ of (2) reaches B_μ at some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0)$. $T:\mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.
ii) FxT attractive for system (2) if it is GFTA and the settling-time $T(x_0)$ is bounded by $T_{\max} > 0$, i.e., $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

The upper bound $T_{\max}$ is a positive constant independent on initial conditions and called, here, *fixed-time constant*.

Definition 8. Polyakov [2012]. The origin $x = 0$ is said to be FxT stable if it is globally FTS and the settling-time $T(x_0)$ is uniformly bounded by $T_{\max}$, i.e., $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

3. GENERALIZED HOMOGENEOUS CONTROLLER

We introduce the first result of this paper. Consider the uncertain dynamical system given by

$$\begin{aligned} \dot{x}_i &= x_{i+1} + w_i(x,t), \qquad \forall i = 1, ..., n-1, \\ \dot{x}_n &= g(x,t)u + w_n(x,t), \end{aligned} \tag{3}$$

where $x = [x_1, x_2, ..., x_n] \in \mathbb{R}^n$ defines the states, $u \in \mathbb{R}^1$ is the control input, $w_i(x,t)$ $(i = 1, ..., n-1)$ represents a set of unmatched uncertainties/disturbances which are uniformly bounded by known functions in t, $\forall t \geq 0$, and satisfy the following growth condition.

Assumption 9. For every $i = 1, ..., n$, and some $p \in [0,1]$, $q \in [1,2)$, the perturbations belong to the class

$$\mathscr{W} = \{w_i : |w_i(x,t)| \leq \rho_i |\sigma_i|^{\frac{\alpha_i}{2-p}\frac{\beta_{i+1}}{\beta_i}}, \rho_i \geq 0\}, \tag{4}$$

where $\sigma_1 = x_1$ and $\sigma_i = \lceil x_i \rfloor^{\frac{2-p}{\alpha_i}} + k_{i-1}^{\frac{2-p}{\alpha_i}} \lceil \sigma_{i-1} \rfloor^{\frac{\alpha_{i-1}}{\alpha_i} \varsigma_{i-1}}$ with $\alpha_i = (i-2)p - (i-3)$, $\varsigma_i = \frac{\beta_{i+1}}{\beta_i} = \frac{(i-1)q + (2-i)}{(i-2)q + (3-i)}$, $\forall i = 1, ..., n$. [1]

This condition characterizes a wide variety of perturbations. For example, the disturbances are bounded by smooth functions for any $q \geq 1$ and $p \in [0,1)$. In the case $1 > q > p > 0$, the uncertainties are bounded by continuous functions. In the case $q = p$, the bounds of $w_i(x,t)$, $i = 1, ..., n-1$, are still continuous but the matched perturbation $w_n(x,t)$ is bounded by a positive constant, then it does not necessarily vanish at the origin. Also, for some known positive constants K_m, K_M,

$$K_m \leq g(x,t) \leq K_M,\ \forall x \in \mathbb{R}^2,\ \forall t \geq 0. \tag{5}$$

[1] Along this paper the operator $\lceil \cdot \rfloor^m := |\cdot|^m \operatorname{sign}(\cdot)$, $z \in \mathbb{R}$, $m \geq 0$, is used. This operator preserves the sign of the value of the functions. Note that for any odd integer m, $\lceil \cdot \rfloor^m = (\cdot)^m$ and $\lceil \cdot \rfloor^0 = \operatorname{sign}(\cdot)$.

We design a state feedback controller such that the close-loop system is homogeneous of some degree. For positive homogeneity degrees, the system's trajectories converge asymptotically to a vicinity of the origin but also, this vicinity is FxT attractive. For negative homogeneity degrees, the system's trajectories reach the origin in finite time. Under the Assumption 9, we design a class of concrete state feedback controllers with guaranteed convergence rate in spite of unmatched perturbations.

Theorem 10. Assume that (4) holds. Then, the homogeneous controller

$$u = \upsilon_n = -k_n \lceil \sigma_n \rfloor^{\frac{\alpha_n}{2-p} \frac{\beta_{n+1}}{\beta_n}}, \tag{6}$$

stabilizes the origin $x = 0$ of system (3) if the gains are selected

- for $n = 1$, $k_1 > \rho_1$,
- for $n = 2$, $k_1^2 > 2^{\frac{2(1-p)}{2-p}} \frac{q-p}{2+q-p} + 2^{\frac{1-p}{2-p}} \frac{1+q-p}{2+q-p} \rho_1 + \rho_1$ and $k_2 > 2^{\frac{1-p}{2-p}} \frac{2-p}{2-q} k_1^{2-p} [2^{\frac{1-p}{2-p}} \frac{2}{2+q-p} + \frac{\rho_1}{2+q-p}] + \rho_2$,
- for $n \geq 3$, $k_1^2 > 2^{\frac{2(1-p)}{2-p}} \frac{q-p}{2+q-p} + 2^{\frac{1-p}{2-p}} \frac{1+q-p}{2+q-p} \rho_1 + \rho_1$, $k_2 > 2^{\frac{1-p}{2-p}} \frac{2-p}{2-q} k_1^{2-p} [2^{\frac{1-p}{2-p}} \frac{2}{2+q-p} + \frac{\rho_1}{2+q-p}] + \rho_2$, $k_i > 2^{\frac{(i-2)(1-p)}{2-p}} \Lambda_{i-1} + \rho_i, \forall i = 3, .., n-1$, and $K_m k_i > 2^{\frac{(i-2)(1-p)}{2-p}} \Lambda_{i-1} + \rho_i, \forall i = n$, where

$$\begin{aligned} \Lambda_{i-1} = {} & c_{(i-1)1} \left(\frac{d_i}{\alpha_{i-1}} \frac{\lambda_{(i-1)1}}{\varpi_i} \right)^{\frac{\varpi_i}{2\alpha_i\beta_i}} k_{i-1}^{\frac{d_i}{\alpha_i} - \frac{\nu_{(i-1)1}}{2\alpha_i\beta_i}} \\ & + c_{(i-1)2} \left(\frac{d_i}{\alpha_{i-1}} \frac{\lambda_{(i-1)1}}{\varpi_i} \right)^{\frac{\varpi_i}{\alpha_i\beta_i}} k_{i-1}^{\frac{1-q}{\beta_i}}, \end{aligned} \tag{7}$$

where the parameters $\lambda_{(i-1)1}, \lambda_{(i-1)2}$ depend on the k_{i-1}'s and $\rho'_{i-1}s$, $c_{(i-1)1} = 2 \cdot 2^{\frac{2(i-1)(1-p)}{2-p} \frac{\varpi_i}{2\alpha_i\beta_i}} \beta_i^2 \alpha_{i-1} [\theta_{(i-1)1} \nu_{(i-1)1}]^{\frac{\nu_{(i-1)1}}{2\alpha_i\beta_i}} / \beta_{i-1}$, $c_{(i-1)2} = 2^{\frac{(i-1)(1-p)}{2-p} \frac{\varpi_i}{\alpha_i\beta_i}} \beta_i^2 \alpha_{i-1} [\theta_{(i-1)2} \nu_{(i-1)2}]^{\frac{\nu_{(i-1)2}}{\alpha_i\beta_i}} / \beta_{i-1}$, $d_i = i - (i-1)p$ and $\theta_{(i-1)1}\theta_{(i-1)2} > \theta_{(i-1)1} + \theta_{(i-1)2} > 0$. Furthermore, $\forall p \in [0, 1)$, if $n = 1$, and $\forall p \in [\frac{i-2}{i-1}, 1)$ if $i = 2, ..., n$, the following statements are true for the closed-loop system (3) with (6): i) if $q \in (1, 2)$, the origin is rationally stable; ii) if $q = 1$, the origin is exponentially stable; and iii) if $q \in [p, 1)$, the origin is finite-time stable.

The generalized homogeneous controller (GHC) (6) provides different stability properties to the closed-loop system. It is directly determined by the value of the parameter q w.r.t. $p \in [0, 1)$. The vector field of the closed-loop system with controllers (6) becomes homogeneous of degree $l = q - 1$, then, for different values of q, the vector field is homogeneous of some degree. This is a distinguishing feature, since from the same control structure, we recover discontinuous, finite-time, exponential and rational controllers. For example, a family of continuous finite-time controllers is recovered when $q = p$. However, they are not robust under nonvanishing matched perturbations. A family of discontinuous controllers can deal with this problem and it is recovered from the particular selection $q = p = (n-1)/(n-2)$. A class of rational controllers is recovered fixing $p = 1$ and $q \geq 1$.

Modeling errors and disturbances are not precisely known. Only a bound of them is needed. If Assumption 9 is locally satisfied, then local stability is ensured. The gains $k_1, ..., k_i$ are designed large enough in the index order. The previous result only gives sufficient conditions on the gains and it is possible that they are far away from the necessary stability conditions. In following, some controllers are listed for $i \leq 4$,

(1) $\upsilon_1 = -k_1 \lceil x_1 \rfloor^{\frac{1}{2-q}}$, $k_1 > 0$, $\forall q \in [0, 2)$;
(2) $\upsilon_2 = -k_2 \lceil \sigma_2 \rfloor^{\frac{q}{2-p}}$, $\sigma_2 = \lceil x_2 \rfloor^{2-p} + k_1^{2-p} \lceil x_1 \rfloor^{\frac{2-p}{2-q}}$, $p \in [0, 1)$ and $\forall q \in (p, 2)$;
(3) $\upsilon_3 = -k_3 \lceil \sigma_3 \rfloor^{\frac{p}{2-p} \frac{2q-1}{q}}$, $\sigma_3 = \lceil x_3 \rfloor^{\frac{2-p}{p}} + k_2^{\frac{2-p}{p}} \lceil \sigma_2 \rfloor^{\frac{q}{p}}$, $p \in [1/2, 1)$ and $\forall q \in (p, 2)$;
(4) $\upsilon_4 = -k_4 \lceil \sigma_4 \rfloor^{\frac{2p-1}{2-p} \frac{3q-2}{2q-1}}$, $\sigma_4 = \lceil x_4 \rfloor^{\frac{2-p}{2p-1}} + k_3^{\frac{2-p}{2p-1}} \lceil \sigma_3 \rfloor^{\frac{p(2q-1)}{(2p-1)q}}$, $\forall p \in [2/3, 1)$, $\forall q \in (p, 2)$.

In what follows, particular control laws derived from Theorem 10 are presented. Of course, once the control input is chosen, system (3) is robust to certain kind of matched and unmatched uncertainties/disturbances.

Homogeneous discontinuous controller. Fix $q = p$ and $p = \frac{n-2}{n-1}$, then, $\sigma_i = \lceil x_i \rfloor^{\frac{2-p}{\alpha_i}} + k_{i-1}^{\frac{2-p}{\alpha_i}} \sigma_{i-1}$, where $\sigma_1 = x_1$ and $\alpha_i = (i-2)p - (i-3)$, $\forall i = 2, ..., n$. One obtains a family of discontinuous controllers with a particular feature. The set where the controllers are discontinuous, defines a finite-time stable continuous manifold. These discontinuous controllers differ from those proposed in Levant [2005]. Solutions of closed-loop system (3) in feedback with such discontinuous controllers are understood in the Filippov's sense. Moreover, the associated DI has negative homogeneity degree.

Corollary 11. Assume that (4) holds with $q = p = (n-1)/(n-2)$, $\forall n \geq 2$. If the gains $k'_i s$ are selected large enough as in Theorem (10), then, the origin $x = 0$ of system (3) is stabilized in *finite-time* by the homogeneous discontinuous controller

$$u_d = \upsilon_n = -k_n \, \mathrm{sign}(\lceil x_n \rfloor^{\frac{2-p}{\alpha_n}} + k_{n-1}^{\frac{2-p}{\alpha_n}} \sigma_{i-1}), p = \frac{n-2}{n-1}. \tag{8}$$

Homogeneous controller with Exponential convergence. Fixing $q = p = 1$, the standard linear state feedback controller is recovered. There exists other controller which does not have linear structure and still provides exponential convergence. Fix $q = 1$ and $p \in [\frac{i-2}{i-1}, 1)$, then, $\sigma_i = \lceil x_i \rfloor^{\frac{2-p}{\alpha_i}} + k_{i-1}^{\frac{2-p}{\alpha_i}} \lceil \sigma_{i-1} \rfloor^{\frac{\alpha_{i-1}}{\alpha_i}}$, where $\sigma_1 = x_1$ and $\alpha_i = (i-2)p - (i-3)$, $\forall i = 2, ..., n$.

Corollary 12. Assume that (4) holds with $q = 1$ and $p \in [\frac{i-2}{i-1}, 1)$. If the gains $k'_i s$ are selected large enough as in Theorem (10), then, the origin $x = 0$ of system (3) is stabilized *exponentially* by a homogeneous controller

$$u_{exp} = \upsilon_n = -k_n \lceil \sigma_n \rfloor^{\frac{\alpha_{n+1}}{2-p}}. \tag{9}$$

3.1 Lyapunov's analysis

This Section is devoted to prove Theorem 10. First, stability of the closed-loop system with the controller is proved by using HCLF's and the dynamical behavior of the state trajectories is characterized by the proposed HCLF's. Finally, an estimation of the convergence time is obtained.

Proposition 13. The continuously differentiable function

$$\begin{aligned} V_i = {} & \frac{\alpha_i}{3-p} |x_i|^{\frac{3-p}{\alpha_i}} + k_{i-1}^{\frac{d_i}{\alpha_i}} \lceil \sigma_{i-1} \rfloor^{\frac{d_i}{2-p} \frac{\alpha_{i-1}}{\alpha_i} \varsigma_{i-1}} x_i \\ & + \frac{d_i}{3-p} k_{i-1}^{\frac{3-p}{\alpha_i}} |\sigma_{i-1}|^{\frac{3-p}{2-p} \frac{\alpha_{i-1}}{\alpha_i} \varsigma_{i-1}} + \delta_{i-1} V_{i-1}^{\frac{\alpha_{i-1}}{\alpha_i} \varsigma_{i-1}}, \end{aligned} \tag{10}$$

$\delta_{i-1} = k_{i-1}^{\frac{3-p}{\alpha_i}}$, $i = 2,\dots,n$, is a global RCLF for system (3). Moreover, the time derivative $\dot V_i$ of the RCLF along the trajectories of the system satisfy

$$\dot V_i \le -\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}\delta_{i-1}b_{i(i-1)}\mathscr{V}_{i-1} - b_{ii}|\sigma_i|^{\varpi_i/(2-p)\beta_i}, \qquad (11)$$

where $\mathscr{V}_{i-1} = -V_{i-1}^{\frac{q-p}{\alpha_i\beta_{i-1}}}\dot V_{i-1} \ge 0$, $\varpi_i = (2i-3)q-(2(i-3)+p)$, $b_{i(i-1)} = 1-\sum_{j=1}^{2}\theta_{(i-1)j}^{-1}$, $\forall\theta_{(i-1)j} > 0$, $b_{ii} = \frac{k_i-\rho_i}{2^{\frac{(i-2)(1-p)}{2-p}}} - \Lambda_{i-1}$, $\forall i = 3,\dots,n-1$, and $b_{nn} = \frac{K_m k_n-\rho_n}{2^{\frac{(n-2)(1-p)}{2-p}}} - \Lambda_{n-1}$, $\forall i = n$, with Λ_{i-1} as in (7). Moreover, the time derivative of the RCLF satisfies

$$\dot V_i \le -\kappa_{i\min}V_i^{\varpi_i/(3-p)\beta_i}(x), \qquad (12)$$

where $\kappa_{i\min}$ is a scalar depending on the k_i's and ρ_i's.

Remark 14. In the case $i = 1$, the time derivative $\dot V_1$ satisfies $\dot V_1 \le -(\frac{3-p}{2-p})^{\frac{2-q}{3-p}}a_{11}V_1^{\frac{2+q-p}{3-p}}$, with $a_{11} = k_1-\rho_1$. Note that $\frac{2+q-p}{3-p} > 1$, $\forall p \in [0,1]$, if $q > 1$, then, the time derivative $\dot V_1$ shows rational stability, Bacciotti and Rosier [2005]. Besides, $\frac{2+q-p}{3-p} < 1$, $\forall p \in [0,1]$, if $p \le q < 1$, then, the time derivative $\dot V_1$ shows finite-time stability.

Remark 15. In the case $i = 2$, $b_{21} = k_1-\rho_1-2^{\frac{2(1-p)}{2-p}}\frac{q-p}{2+q-p}\frac{1}{k_1} - 2^{\frac{1-p}{2-p}}\frac{1+q-p}{2+q-p}\frac{\rho_1}{k_1}$, $b_{22} = k_2-\rho_2-2^{\frac{1-p}{2-p}}\frac{2-p}{2-q}k_1^{2-p}[2^{\frac{1-p}{2-p}}\frac{2}{2+q-p}+\frac{\rho_1}{2+q-p}]$.

The parameters $\theta_{(i-1)1}, \theta_{(i-1)2} > 0$ are introduced to enforce the negative definiteness of the time derivative of the RCLF. The control input u, the RCLF V_i and its time derivative $\dot V_i$ are homogeneous w.r.t. $\Delta_\varepsilon^r x = (\varepsilon x_1, \varepsilon^{\frac{1}{2-q}}x_2, \varepsilon^{\frac{q}{2-q}}x_3,\dots,\varepsilon^{\frac{\beta_i}{2-q}}x_i)$, for any $\varepsilon > 0$, i.e., $\upsilon_i(\Delta_\varepsilon^r x) = \varepsilon^{\frac{\beta_{i+1}}{2-q}}\upsilon_i(x)$, $V_i(\Delta_\varepsilon^r x) = \varepsilon^{\frac{3-p}{\alpha_i}\frac{\beta_i}{2-q}}V_i(x)$, $\dot V_i(\Delta_\varepsilon^r x) = \varepsilon^{\frac{\varpi_i}{\alpha_i(2-q)}}\dot V_i(x)$. From the Lyapunov's inequality (12) an estimation of the convergence time can be obtained. It lets us show how the state trajectories reach the origin.

Proposition 16. Select the control input (6). Then, any trajectory of system (3) starting at any initial state $x_i(0) \in \mathbb{R}^i$ reaches the origin

i) asymptotically, for $q \in (1,2)$. Moreover, any trajectory converges to a neighborhood of the origin of radius $0 < \mu_i < V_i(x_i(0))$ from any $x_i(0)$ in a finite time smaller than

$$T_i(v_0,\mu_i) \le \frac{(3-p)\beta_i}{(q-1)\alpha_i\kappa_{i\min}}[\mu_i^{-\frac{(q-1)\alpha_i}{(3-p)\beta_i}} - V_i^{-\frac{(q-1)\alpha_i}{(3-p)\beta_i}}(x_i(0))], \qquad (13)$$

where $\kappa_{i\min}$ is a constant depending on k_i's and ρ_i's. Furthermore, the convergence time is uniformly bounded by

$$T_{\mu_i} = \frac{(3-p)\beta_i}{(q-1)\alpha_i\kappa_{i\min}}\mu_i^{-[(q-1)\alpha_i]/[(3-p)\beta_i]}. \qquad (14)$$

ii) exponentially, for $q = 1$.

iii) in finite time, for $q \in [p,1)$. Moreover, the convergence time satisfies

$$T_{fi}(v_0) \le \frac{(3-p)\beta_i}{(1-q)\alpha_i\kappa_{i\min}}V_i^{[(1-q)\alpha_i]/[(3-p)\beta_i]}(x_i(0)), \qquad (15)$$

where $\kappa_{i\min}$ is a constant depending on k_i's and ρ_i's.

Proof. It follows from (12) and by using the comparison principle, Khalil [2002]. Then, for $V_{i0} = V_i(x_i(0))$, the solution $V_i(t)$ satisfies the differential inequality

i) $V_i(t) \le (v_{i0}^{-\frac{(q-1)\alpha_i}{(3-p)\beta_i}} + \frac{(q-1)\alpha_i}{(3-p)\beta_i}\kappa_{i\min}t)^{-\frac{(3-p)\beta_i}{(q-1)\alpha_i}}$, since $q \in (1,2)$ and $p \in [0,1]$ imply $\frac{\varpi_i}{(3-p)\beta_i} > 1$. This expression leads to estimate a bound of the convergence time. In fact, any trajectory starting at initial state $x_i(0)$ reaches a level set $V_i = \mu_i$, where $0 < \mu_i < v_{i0}$, in a time determined by (13). Moreover, since $\lim_{v_0\to\infty}T_i(v_0,\mu_i) = T_{\mu_i}$, the convergence time $T_i(v_0,\mu_i)$ of any trajectory is uniformly upper bounded by (14), i.e., $T_i(v_0,\mu_i) \le T_{\mu_i}$.

ii) $V_i(t) \le V_{i0}\exp(-\kappa_{i\min}t)$, since $q = 1$ and $p \in [0,1]$ imply $\frac{\varpi_i}{(3-p)\beta_i} = 1$. Exponential stability is concluded immediately.

iii) $V_i(t) \le (V_{i0}^{\frac{(1-q)\alpha_i}{(3-p)\beta_i}} - \frac{(1-q)\alpha_i}{(3-p)\beta_i}\kappa_{i\min}t)^{\frac{(3-p)\beta_i}{(1-q)\alpha_i}}$, $\forall q \in [p,1)$. Since, $q \in [p,1)$, with $p \in [0,1)$ if $i = 1$, and $p \in [\frac{i-2}{i-1},1)$ if $i \ge 2$, implies $\frac{\varpi_i}{(3-p)\alpha_i} < 1$. From this expression the inequality (15) is easily derived.

Remark 17. The convergence time estimation can be very cumbersome.

Both Proposition 13 and Proposition 16 guarantee: i) rational stability of the closed-loop system in spite of some $w_i \in \mathscr{W}$, but also the convergence time of any trajectory is bounded by a constant; ii) exponential stability of closed-loop system; iii) finite-time stability of the closed-loop system for some $w_i \in \mathscr{W}$.

Proof of Proposition 13. Asymptotic stability of the closed-loop system in feedback with the GHC is proved with a RCLF. The RCLF is given by (10). To stabilize the origin of the cascaded system (3), we proceed by induction using desingularizing functions (DsF's) and backstepping based approach (BBA).

Step 1: Consider the system $\dot x_1 = \upsilon_1 + w_1$, where $\upsilon_1 = -k_1\lceil x_1\rfloor^{1/(2-q)}$. The parameter q takes values on $q \in [0,2)$ which implies that $1/2 \le 1/(2-q) < \infty$. Hence

$$\dot x_1 = \upsilon_1 + w_1 = -k_1\lceil x_1\rfloor^{\frac{1}{2-q}} + w_1. \qquad (16)$$

System stability is studied by using the RCLF $V_1 = \frac{2-p}{3-p}|x_1|^{\frac{3-p}{2-q}}$, $p \in [0,1]$. Taking the time derivative of V_1 along the system trajectories of (16) and under the Assumption 9, one obtains

$$\dot V_1 = \frac{\partial V_1}{\partial x_1}(\upsilon_1 + w_1) \le -b_{11}|x_1|^{\frac{2+q-p}{2-q}},$$

where $b_{11} = k_1-\rho_1$. It is negative definite for $b_{11} > 0$.

Step $i-1$: Inspired from Praly et al. [1991], we use a DSF to construct the RCLF instead of the state variable $s_i = x_i - \upsilon_{i-1}$. The proposed DsF is given by $s_{id} = \lceil x_i\rfloor^{\frac{d_i}{\alpha_i}} + k_{i-1}^{\frac{d_i}{\alpha_i}}\lceil\sigma_{i-1}\rfloor^{\frac{d_i}{2-p}\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}}$, $d_i = i-(i-1)p$, $\forall i = 2,\dots,n$, where $\sigma_1 = x_1$ and $\sigma_i = \lceil x_i\rfloor^{\frac{2-p}{\alpha_i}} + k_{i-1}^{\frac{2-p}{\alpha_i}}\lceil\sigma_{i-1}\rfloor^{\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}}$, $\alpha_i = (i-2)p-(i-3)$, $\forall i = 2,\dots,n$. All these representations are equivalents when $s_i = s_{id} = \sigma_i = 0$. The DsF s_{id} is always continuously differentiable on $(x_1,\dots,x_i)$. It leads to construct the RCLF as

$$V_i = W_i + \delta_i V_{i-1}, \quad \forall i = 2,\dots,n, \qquad (17)$$

where $W_i = \int_{\upsilon_{i-1}}^{x_i}(\lceil\tau_i\rfloor^{\frac{d_i}{\alpha_i}} + k_{i-1}^{\frac{d_i}{\alpha_i}}\lceil\sigma_{i-1}\rfloor^{\frac{d_i}{2-p}\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}})d\tau_i$ and δ_i is some positive constant. Function (17) is positive definite but also C^1 by construction and its explicit form is given by (10). The function s_{id} is chosen such that V_i is homogeneous of degree $m_{V_i} = \frac{(3-p)\beta_i}{\alpha_i(2-q)}$ w.r.t. $\Delta_\varepsilon^r x = (\varepsilon x_1, \varepsilon^{\frac{1}{2-p}}x_2,\dots,\varepsilon^{\frac{\alpha_i}{2-p}}x_i)$. Now, we assume that the controller υ_{i-1} stabilizes the $(i-1)^{th}$-order system (3) and it is guaranteed by a RCLF V_{i-1} (which is positive definite by construction) whose time derivative $\dot{V}_{i-1}$ is definite negative.

Step i: Taking the time derivative of the RCLF V_i defined in (10) along the system trajectories of (3), we obtain

$$\dot{V}_i = s_{id}[\upsilon_i + w_i] + \frac{d_i}{2-p}\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}k_{i-1}^{\frac{d_i}{\alpha_i}} s_i |\sigma_{i-1}|^{\frac{d_i}{2-p}\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}-1}\dot{\sigma}_{i-1} + \frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}\delta_{i-1}V_{i-1}^{\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}-1}\left(\sum_{j=2}^{i}\frac{\partial V_{i-1}}{\partial x_{j-1}}\dot{x}_{j-1}\right).$$

By straightforward calculation, it is deduced that $\sum_{j=2}^{i}\frac{\partial V_{i-1}}{\partial x_{j-1}}x_j = \dot{V}_{i-1} + \frac{\partial V_{i-1}}{\partial x_{i-1}}s_i$, where $\frac{\partial V_{i-1}}{\partial x_{i-1}} = s_{(i-1)d}$. Since $x_i = s_i + \upsilon_{i-1}$, it is not difficult to show that $\dot{\sigma}_{i-1} = \sum_{j=2}^{i}\frac{\partial \sigma_{i-1}}{\partial x_{j-1}}(x_j + w_{j-1}) = \Psi_{(i-1)n} + \frac{2-p}{\alpha_{i-1}}|x_{i-1}|^{\frac{(i-2)(1-p)}{\alpha_{i-1}}} s_i + \Psi_{(i-1)\rho}$, where the functions

$$\Psi_{(i-1)\rho} = \sum_{j=2}^{i}\frac{\partial \sigma_{i-1}}{\partial x_{j-1}}w_{j-1},\ \Psi_{(i-1)n} = \sum_{j=2}^{i-1}\frac{\partial \sigma_{i-1}}{\partial x_{j-1}}x_j + \frac{2-p}{\alpha_{i-1}}|x_{i-1}|^{\frac{(i-2)(1-p)}{\alpha_{i-1}}}\upsilon_{i-1}.$$

According to Assumption 9, we have $\Psi_{(i-1)\rho} \le \Delta_{(i-1)\rho}$, where the function $\Delta_{(i-1)\rho} = \sum_{j=2}^{i}\frac{\partial \sigma_{i-1}}{\partial x_{j-1}}w_{j-1}$, evaluated on $w_{j-1} = \rho_{j-1}|\sigma_{j-1}|^{\frac{\alpha_j}{2-p}\varsigma_{j-1}}$. Taking into account these facts, and applying the control input υ_i as in (6), follows that

$$\dot{V}_i \le -(k_1-\rho_i)|s_{id}||\sigma_i|^{\frac{\alpha_{i+1}}{2-p}\varsigma_i} - \frac{d_i}{2-p}\frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}k_{i-1}^{\frac{3-p}{\alpha_i}}\mathscr{V}_{i-1} + \frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}\cdot k_{i-1}^{\frac{d_i}{\alpha_i}}\left[\frac{d_i}{\alpha_{i-1}}|x_{i-1}|^{\frac{(i-2)(1-p)}{\alpha_{i-1}}}|s_i|^2|\sigma_{i-1}|^{\frac{(2-p)(q-p)+(i-2)(1-p)\beta_i\alpha_{i-1}}{(2-p)\alpha_i\beta_{i-1}}} + s_i\Upsilon_{i-1}\right],$$

where $\Upsilon_{i-1} = \frac{d_i}{2-p}|\sigma_{i-1}|^{\frac{(2-p)(q-p)+(i-2)(1-p)\beta_i\alpha_{i-1}}{(2-p)\alpha_i\beta_{i-1}}}(\Psi_{(i-1)n}+\Delta_{(i-1)\rho}) + k_{i-1}V_{i-1}^{\frac{q-p}{\alpha_i\beta_{i-1}}}\sigma_{i-1}$ and $\mathscr{V}_{i-1} = -V_{i-1}^{\frac{q-p}{\alpha_i\beta_{i-1}}}\dot{V}_{i-1} \ge 0$, since $\dot{V}_{i-1} \le 0$. From Lemma 24, the following inequalities are derived

$$|x_{i-1}|^{\frac{(i-2)(1-p)}{\alpha_{i-1}}}|\sigma_{i-1}|^{\frac{(2-p)(q-p)+(i-2)(1-p)\beta_i\alpha_{i-1}}{(2-p)\alpha_i\beta_{i-1}}} \le \lambda_{(i-1)1}\mathscr{V}_{i-1}^{\frac{\nu_{(i-1)1}}{\varpi_i}},$$

$$s_i\Upsilon_{i-1}(x_1,...,x_{i-1}) \le \frac{d_i}{\alpha_{i-1}}\lambda_{(i-1)2}|s_i|\mathscr{V}_{i-1}^{\frac{\nu_{(i-1)2}}{\varpi_i}},$$

where $\lambda_{(i-1)2} = \max_{x\mathscr{V}_{i-1}(x)=1} = |\frac{\alpha_{i-1}}{d_i}\Upsilon_{i-1}(\cdot)|$ and $\lambda_{(i-1)1} = \max_{x:\mathscr{V}_{i-1}(x)=1}|x_{i-1}|^{\frac{(i-2)(1-p)}{\alpha_{i-1}}}|\sigma_{i-1}|^{\frac{(2-p)(q-p)+(i-2)(1-p)\beta_i\alpha_{i-1}}{(2-p)\alpha_i\beta_{i-1}}}$. With help of Lemma 23 (see Appendix) we express s_i and s_{id} in terms of σ_i, i.e., $|s_i| \le 2^{\frac{(i-1)(1-p)}{2-p}}|\sigma_i|^{\frac{\alpha_i}{2-p}}$ and $|\sigma_i| \le 2^{\frac{(i-2)(1-p)}{d_i}}|s_{id}|^{\frac{2-p}{d_i}}$. It results in

$$\dot{V}_i \le -\frac{k_i-\rho_i}{2^{\frac{(i-2)(1-p)}{2-p}}}|\sigma_i|^{\frac{\varpi_i}{(2-p)\beta_i}} - \frac{\alpha_{i-1}}{\alpha_i}\varsigma_{i-1}k_{i-1}^{\frac{3-p}{\alpha_i}}\mathscr{V}_{i-1} + 2^{\frac{(i-1)(1-p)}{2-p}}\frac{d_i}{\alpha_i}\varsigma_{i-1}\cdot k_{i-1}^{\frac{d_i}{\alpha_i}}\left(2^{\frac{(i-1)(1-p)}{2-p}}\lambda_{(i-1)1}|\sigma_i|^{\frac{2\alpha_i}{2-p}}\mathscr{V}_{i-1}^{\frac{\nu_{(i-1)1}}{\varpi_i}} + \lambda_{(i-1)2}|\sigma_i|^{\frac{\alpha_i}{2-p}}\mathscr{V}_{i-1}^{\frac{\nu_{(i-1)2}}{\varpi_i}}\right).$$

We use Lemma 22 to deal with the crossed terms. Then,

$$|\sigma_i|^{\frac{2\alpha_i}{2-p}}\mathscr{V}_{i-1}^{\frac{\nu_{(i-1)1}}{\varpi_i}} \le \frac{\nu_{(i-1)1}}{\varpi_i}\gamma_{(i-1)1}^{-\frac{\varpi_i}{\nu_{(i-1)1}}}\mathscr{V}_{i-1} + \frac{2\alpha_i\beta_i}{\varpi_i}\gamma_{(i-1)1}^{\frac{\varpi_i}{2\alpha_i\beta_i}}|\sigma_i|^{\frac{\varpi_i}{(2-p)\beta_i}},$$

$$|\sigma_i|^{\frac{\alpha_i}{2-p}}\mathscr{V}_{i-1}^{\frac{\nu_{(i-1)2}}{\varpi_i}} \le \frac{\nu_{(i-1)2}}{\varpi_i}\gamma_{(i-1)2}^{-\frac{\varpi_i}{\nu_{(i-1)2}}}\mathscr{V}_{i-1} + \frac{\alpha_i\beta_i}{\varpi_i}\gamma_{(i-1)2}^{\frac{\varpi_i}{\alpha_i\beta_i}}|\sigma_i|^{\frac{\varpi_i}{(2-p)\beta_i}},$$

where $\nu_{(i-1)1} = \varpi_i - 2\alpha_i\beta_i$ and $\nu_{(i-1)2} = \varpi_i - \alpha_i\beta_i$. From the previous inequalities, it follows that the time derivative $\dot{V}_i$ satisfies (11) with

$$b_{ii} = \frac{k_i-\rho_i}{2^{\frac{(i-2)(1-p)}{2-p}}} - 2^{\frac{(i-1)(1-p)}{2-p}}\frac{d_i}{\alpha_i}\varsigma_{i-1}k_{i-1}^{\frac{d_i}{\alpha_i}}\cdot \left(2^{\frac{(i-1)(1-p)}{2-p}}\lambda_{(i-1)1}\frac{2\alpha_i\beta_i}{\varpi_i}\gamma_{(i-1)1}^{\frac{\varpi_i}{2\alpha_i\beta_i}} + \lambda_{u(i-1)2}\frac{\alpha_i\beta_i}{\varpi_i}\gamma_{(i-1)2}^{\frac{\varpi_i}{\alpha_i\beta_i}}\right), \tag{18}$$

$$b_{i(i-1)} = 1 - 2^{\frac{(i-1)(1-p)}{2-p}}\frac{d_i}{\alpha_{i-1}}\cdot \left(2^{\frac{(i-1)(1-p)}{2-p}}\frac{\lambda_{(i-1)1}}{k_{i-1}}\frac{\nu_{(i-1)1}}{\varpi_i}\gamma_{(i-1)1}^{-\frac{\varpi_i}{\nu_{(i-1)1}}} + \frac{\nu_{(i-1)2}}{\varpi_i}\frac{\lambda_{(i-1)2}}{k_{i-1}}\gamma_{(i-1)2}^{-\frac{\varpi_i}{\nu_{(i-1)2}}}\right), \tag{19}$$

where, we fix $\gamma_{(i-1)1} = [\frac{2^{\frac{2(i-1)(1-p)}{2-p}}d_i}{\alpha_{i-1}}\frac{\lambda_{(i-1)1}\theta_{(i-1)1}}{k_{i-1}}\frac{\nu_{(i-1)1}}{\varpi_i}]^{\frac{\nu_{(i-1)1}}{\varpi_i}}$ and $\gamma_{(i-1)2} = [\frac{2^{\frac{(i-1)(1-p)}{2-p}}d_i}{\alpha_{i-1}}\frac{\lambda_{(i-1)2}\theta_{(i-1)2}}{k_{i-1}}\frac{\nu_{(i-1)2}}{\varpi_i}]^{\frac{\nu_{(i-1)2}}{\varpi_i}}$. Finally, the time derivative $\dot{V}_i$ is negative definite if $1 > \theta_{(i-1)1}^{-1} + \theta_{(i-1)2}^{-1}$, and $k_i > 2^{\frac{(i-2)(1-p)}{2-p}}\Lambda_{i-1} + \rho_i, \forall i = 3,..,n-1$ with Λ_{i-1} as in (7). Choosing k_i larger enough renders $\dot{V}_i$ negative definite.

Step $i=n$: In this case, by induction argument, we get (11) with $b_{nn} = \frac{(K_m k_n - \rho_n)}{2^{\frac{(n-2)(1-p)}{2-p}}} - \Lambda_{n-1}$.

Finally, for an arbitrary i, the continuous functions $V_i(x)$ and $\dot{V}_i(x)$ are homogeneous of degrees $m_{V_i} = \frac{(3-p)\beta_i}{\alpha_i(2-q)}$ and $m_{\dot{V}_i} = \frac{\varpi_i}{\alpha_i(2-q)}$ for any $i = 1,...,n$, w.r.t. $\Delta_\varepsilon^r x$. Respectively, it follows from Lemma 24 that (12) holds $\forall x \in \mathbb{R}^i$, where $\kappa_{i\min} = \min_{x:V_i(x)=1}\{-\dot{V}_i(x)\}$. Note that $\kappa_{i\min} > 0$ since $-\dot{V}_i(x)$ is positive definite. It shows that the time derivative $\dot{V}_i$ can be bounded by a function of V_i.

4. LYAPUNOV REDESIGN

In this Section, we present the second result. In general, the controller (6) is continuous for any $p < q$, then, the closed-loop system is not robust under matched bounded perturbations. Lyapunov redesign can be applied for system (3) in order to enhance the robustness in spite of matched bounded perturbations. For that, suppose that $w_i(x,t) = 0, i = 1,...,n-1, \forall t \ge 0$, i.e, unmatched disturbances are not present. Then, system (3) is described by

$$\begin{aligned} \dot{x}_i &= x_{i+1}, & \forall i = 1,...,n-1,\\ \dot{x}_n &= g(x,t)u_{l_2} + w_n(x,t). \end{aligned} \tag{20}$$

Moreover, the matched disturbance satisfies $|w_n(x,t)| \le \rho_m$.

Theorem 18. The controller

$$u_{l_2} = \upsilon_n = -k_d\,\mathrm{sign}(\sigma_n) - k_n\lceil\sigma_n\rfloor^{\frac{\alpha_n}{2-p}\frac{\beta_{n+1}}{\beta_n}}, \tag{21}$$

robustly stabilizes system (20) if the gains $k_i's$ are selected as in Theorem 10 with $\rho_i = 0, \forall i = 1,...,n$, and $k_d > \rho_m$.

Proof. It follows from Proposition 13 assuming that $w_i(x,t) = 0, i = 1,...,n-1, \forall t \ge 0$, and $|w_n(x,t)| \le \rho_m$.

Remark 19. Actually, controller (21) can still deal with unmatched disturbances given in the Assumption 9.

This controller enforces to reach the origin of the closed-loop system in some manner. The parameter $l_2 = q-1$ determines the convergence rate. This controller can only provide: i) robust finite-time stability of the origin when $p \le q < 1$; ii) robust exponential stability if $q = 1$ and iii) robust FxT stability if $q \in (1,2)$. In general, the closed-loop system is not homogeneous in feedback with controller (21).

In the next section, we define $V_i(x) = V_{i,l_2}(x)$, where $V_i(x)$ is the RCLF defined by (10). Particularly, if $q = 1$, we define $V_{i,0}(x) = V_{i,exp}(x)$, and if $p = q = (n-1)/(n-2)$, we define $V_{i,l_2}(x) = V_{i,d}(x)$, which is the RCLF for the discontinuous finite-time controller.

5. CONVERGENCE RATE IMPROVEMENT

Finally, this section is devoted to show how the GHC is combined in a hybrid control algorithm in order to improve the convergence rate of the discontinuous finite-time controllers of Subsection 3.0.1. We combine a discontinuous finite-time and a robust rational (exponential) controller. Consider the finite-time stabilization problem of system

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \qquad \forall i = 1, ..., n-1, \\ \dot{x}_n &= g(x,t)u + w_n(x,t). \end{aligned} \tag{22}$$

where $|w_n(x,t)| \le \rho_m$, such that the convergence rate need to be improved. To achieve our goal, let us define the set

$$\varepsilon(l_2, A) = \min_{V_{i,d}(x)=A} V_{i,l_2}(x), \tag{23}$$

it ensures that the level set $V_{i,d}(x) = A$ is contained *inside* the level set $V_{i,l_2}(x) = \varepsilon(l_2, A)$.The idea of combining the controllers u_d and u_{l_2} is simple. For that, we will use of the proposed RCLF. Define the hybrid control input as $u = \vartheta(u_d, u_{l_2}, A, \varepsilon)$, where u is chosen using the following algorithm:

(i) Apply the control input $u = u_{l_2}$ until the equality $V_{n,d}(x) = A > 0$ holds. Note that u_{l_2} is the controller as in (21) which is robust in spite of matched bounded perturbations.

(ii) Once the level set A is reached, we apply the control input $u = u_d$, which enforces to reach the origin in finite-time, for any $V_{i,l_2}(x) < \varepsilon^*$, where ε^* is a positive constant such that $\varepsilon^* > \varepsilon(l_2, A)$. It avoids that a level set $V_{n,d}(x) = A$ is a positive invariant set. Note that u_d is the homogeneous discontinuous finite-time controller as in (11) . This controller is robust to the same matched bounded perturbation as the controller u_{l_2}.

(iii) If $V_{i,l_2}(x) > \varepsilon^*$, then we return to the step 1.

This strategy ensures that the origin is the unique positive invariant set.

An arbitrary fast discontinuous finite-time controller. The following controller provides faster convergence to the origin than the controllers proposed in Levant [2005].

Proposition 20. Consider $q = 1$ and $A > 0$. Then, the controller $u = \vartheta(u_d, u_{l_2}, A, \varepsilon)$ stabilizes system (20) in finite time $\forall n \ge 2$.

An arbitrary Fixed-time controller. FxT attractivity property in the controller (21) is due to the parameter $q > 1$ and $p \in [\frac{i-2}{i-1}, 1)$. Therefore, the convergence time to any ball centered at the origin is guaranteed within FxT regardless if the state trajectory starts at very large initial conditions. For sake of simplicity, we take $p = \frac{n-1}{n-2}$, then, $\sigma_i = \lceil x_i \rfloor^{\frac{2-p}{\alpha_i}} + k_{i-1}^{\frac{2-p}{\alpha_i}} \lceil \sigma_{i-1} \rfloor^{\frac{\alpha_{i-1}}{\alpha_i}} \varsigma_{i-1}$, where $\sigma_1 = x_1$ and $\alpha_i = (i-2)p - (i-3)$, $\forall i = 2, ..., n$.

Proposition 21. Consider $q > 1$ and $B > 0$. Then, the controller $u = \vartheta(u_d, u_{l_2}, B, \varepsilon)$ stabilizes system (20) within fixed-time constant $T = T_{fi} + T_{\mu_i}$, $\forall n \ge 2$.

6. CONCLUDING REMARKS

A generalization of a homogeneous controller has been proposed. In general, the proposed controllers enforce the trajectories of the system to reach the origin in some way. It is reached asymptotically, exponentially or in finite time, depending on how the parameters of the controllers are selected. Robust discontinuous finite-time controllers with improved convergence rate are also synthesized.

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Appendix A. TECHNICAL LEMMAS

Lemma 22. Moreno [2011]. For every real numbers $a > 0$, $b > 0$, $\gamma > 0$, $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality is satisfied $ab \le \gamma^p a^p/p + \gamma^{-q} b^q/q$.

Lemma 23. For $x_1, x_2 \in R$, and p, q nonzero real numbers, such that $0 < p \le q$, the inequality $|\lceil x_2 \rfloor^p + \lceil x_1 \rfloor^p|^{1/p} \le 2^{\frac{1}{p}-\frac{1}{q}} |\lceil x_2 \rfloor^q + \lceil x_1 \rfloor^q|^{1/q}$ holds. Furthermore, equality holds if and only if either $p = q$ or $x_1 = x_2$.

Lemma 24. Bhat and Bernstein [2005]. Suppose V_1 and V_2 are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with respect to ν of degree $l_1 > 0$ and $l_2 > 0$, respectively, and V_1 is positive definite. Then, for every $x \in \mathbb{R}^n$,

$$[\min_{z:V_1(z)=1} V_2(z)][V_1(x)]^{\frac{l_2}{l_1}} \le V_2(x) \le [\max_{z:V_1(z)=1} V_2(z)][V_1(x)]^{\frac{l_2}{l_1}}.$$