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A proportional integral extremum-seeking control approach

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Abstract: This paper proposes an alternative extremum seeking control design technique for the solution of real-time optimization control problems. The technique considers a proportional-integral approach that avoids the need for a time-scale separation in the formulation of the ESC. It is assumed that the equations describing the dynamics of the nonlinear system and the cost function to be minimized are unknown and that the objective function is measured. The dynamics are assumed to be asymptotically stable and relative order one with respect to the objective function. The extremum-seeking problem is solved using a time-varying parameter estimation technique.

Keywords: Extremum-seeking control, Real-time optimization, Time-varying systems

1. INTRODUCTION

Extremum-seeking control (ESC) has been the subject of considerable research effort over the last decade. This approach, which dates back to the 1920s Leblanc [1922], is an ingenious mechanism by which a system can be driven to the optimum of a measured variable of interest Tan et al. [2010]. The revived interest in the field was primarily sparked by Krstic and co-workers who provided an elegant proof of the convergence of a standard perturbation based extremum seeking scheme for a general class of nonlinear systems. The main drawback of ESC is the lack of transient performance guarantees. As highlighted in the proof of Krstic and Wang Krstic and Wang [2000], the stability analysis relies on two components: 1) an averaging analysis of the persistently perturbed ESC loop and 2) a time-scale separation of ESC closed-loop dynamics between the fast transients of the system dynamics and the slow quasi steady-state extremum-seeking task.

Over the last few years, many researchers have considered various approaches to overcome the limitations of ESC. In Krstic [2000], the performance limitations associated with ESC were considered in detail. The non-local properties on ESC was studied in Tan et al. [2006]. This work extends the work in Krstic and Wang [2000] by considering the case where the fast dynamics can be assumed to be uniformly global asymptotically stable along the equilibrium manifold. In Adetola and Guay [2007], Guay et al. [2004] and Cougnon et al. [2011], an alternative ESC algorithm is considered where an adaptive control and estimation approach is used. The key aspect of this approach is that the equilibrium map is parameterized and the parameters are estimated with the help of a tailored adaptive estimation technique. The results in Nesic et al. [2010] unify the approaches based on singular perturbation and parameter estimation by considering the case where the objective function is parameterized in a known fashion. A three-time scale approach is proposed to establish the combined adaptive estimation and extremum seeking control algorithms. Recent work reported in Ghaffari et al. [2012] and Moase et al. [2010] have proposed a Newton-based extremum-seeking technique that provides an estimate of the inverse of the Hessian of the cost function. This technique can effectively alleviate the convergence problems associated with the increase of the gain of the Newton update. Other alternative techniques such as proposed Zhang and Ordóñez [2009] and Zhang and Ordóñez [2012] make use of sampled gradient measurements to improve the convergence properties of ESC techniques that implement numerical optimization techniques. A sliding-mode approach is presented in Fu and Özgüner [2011].

Although the limitations associated with the tuning of ESC is generally well understood, the limitations associated with the two time-scale approach to ESC remains problematic. Under the two time-scale assumption, the optimization operates at a quasi steady-state, or slow, time-scale such that the search for optimal operating conditions does not affect the process dynamics. To overcome the time-scale separation, one must incorporate some knowledge of the transient behaviour of the process dynamics. In the case where a model is available, one can use adaptive extremum seeking technique as proposed in Guay and Zhang [2003] to stabilize a nonlinear system to the unknown optimum of a known but unmeasured cost function. If a model is not available but similar systems are available, the use of multi-unit extremum seeking control techniques Srinivasan [2007] can be used to steer both systems in a neighbourhood of the unknown optimum. Both classes of techniques can solve the steady-state optimization ESC problem without the need for time-scale separation. In Scheinker and Krstic [2013], Lie bracket averaging techniques are considered to stabilize unknown dynamical systems using ESC. The approach does not explicitly rely on the need for time-scale separation but it requires a known CLF of the unknown control system.

ESC problems cannot be currently solved in the absence of time-scale separations if explicit process models or multiple identical units are not available. This paper attempts to bridge this gap in the application of ESC. It proposes a proportional-integral ESC design technique. This technique can be interpreted as a generalization of the standard approach where the integral action corresponds to the standard ESC control task used to identify the steady-state optimum. The proportional control action is designed to ensure that the measured cost function is optimized instantaneously. The approach considers an alternative parameterization of the ESC problem in which the rate of change of the output is parameterized directly without the need to invoke a time-scale separation argument. Under suitable assumption on the dynamics of the system, this action can be shown to minimize the cost over short times while reaching the optimum steady-state conditions.

The paper is organized as follows. A brief description of the ESC problem is given in section 2. In section 3, the proposed ESC formulation is presented for a known cost function and process dynamics. The proposed proportional-integral ESC controller is described in section 4. A simulation example is presented in 5 followed by brief conclusions in 6.

2. PROBLEM DESCRIPTION

Consider a nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, u is the vector of input variables taking values in $\mathcal{U} \subset \mathbb{R}^p$ and $y \in \mathbb{R}$ is the variable to be minimized. It is assumed that $f(x)$ and $g(x)$ a smooth vector valued functions of x and that $h(x)$ is a smooth function of x.

The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$. The equilibrium (or steady-state) map is the n dimensional vector $\pi(u)$ which is such that:

$$f(\pi(u)) + g(\pi(u))u = 0.$$

The equilibrium cost function is given by:

$$y = h(\pi(u)) = \ell(u) \tag{3}$$

Thus, at equilibrium, the problem is reduced to finding the minimizer u^* of $y = \ell(u^*)$. Let $\mathcal{D}(u)$ be a neighbourhood of the steady-state $x = \pi(u)$.

Some additional assumptions are required concerning the cost function $h(x)$.

Assumption 1. The cost $h(x)$ is such that

(1) $\frac{\partial h(x^*)}{\partial x} = 0$
(2) $\frac{\partial^2 h(x)}{\partial x \partial x^T} > \alpha I, \ \forall x \in \mathbb{R}^n$

where α is a strictly positive constant.

Note that, in contrast to standard ESC, convexity of the cost function $h(x)$ is required. We also require the following properties for the dynamics:

Assumption 2. The dynamics (1) are such that:

(1) the cost function $h(x)$ decreases in the direction of $f(x)$:
$$\frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)u \leq -\alpha \|x - \pi(u)\|^2, \ \forall x \in \mathcal{D}(u),$$
(2) the matrix valued function $g(x)$ is full rank $\forall x \in \mathcal{D}(u)$,

$\forall u \in \mathcal{U}$.

Assumption 2 states that h is non-decreasing in along the vector field $f(x) + g(x)u$ over some neighbourhood of the steady-state manifold $x = \pi(u)$ at a fixed value of the input u. It also states that the cost function is relative order 1 in a neighbourhood of the origin.

Finally, we will require the following additional assumption concerning the steady-state cost function $\ell(u)$.

Assumption 3. The equilibrium steady-state map $\ell(u)$ is such that

$$\nabla_u \ell(u)(u - u^*) \geq \alpha_u \|u - u^*\|^2$$

for some positive constant α_u $\forall u \in \mathcal{U}$.

3. EXTREMUM SEEKING CONTROLLER WITH FULL INFORMATION

In this section, we propose the extremum-seeking control approach that will form the basis of the development in later sections. Let us first consider the cost function $y = h(x)$ and compute its time derivation:

$$\dot{y} = L_f h + L_g h u \tag{4}$$

where $L_f h$ and $L_g h$ are the Lie derivatives of $h(x)$ with respect to $f(x)$ and $g(x)$, respectively. The Lie derivative is the directional derivative of the function $h(x)$ given by:

$$L_f h = \frac{\partial h}{\partial x} f, \ L_g h = \frac{\partial h}{\partial x} g.$$

By the relative order assumption it follows that $L_g h \neq 0$ in a neighbourhood of the unknown optimum x^*.

We propose the following controller:

$$u = -k L_g h + \hat{u} \tag{5}$$

where $\hat{u}$ is a steady-state bias term to be estimated. Let the optimal steady-state input be given by u^*. The error in the deviation bias is denoted by $\tilde{u} = u^* - \hat{u}$. Pose the function

$$V = y + \frac{1}{2}\tilde{u}^T \tilde{u}$$

Its time derivative is given by:

$$\dot{V} = L_f h - k\|L_f g\|^2 + L_g h \hat{u} - \tilde{u}\dot{\hat{u}}.$$

Let $\dot{\hat{u}} = -L_g h$. Upon substitution of $\tilde{u} = u^* - \hat{u}$, one obtains:

$$\dot{V} = L_f h - k\|L_f g\|^2 + L_g h u^*$$

By assumption, it follows that:

$$\dot{V} \leq -\alpha \|x - \pi(u^*)\|^2 - k\|L_g h\|^2$$

Since $g(x)$ is everywhere full rank and x^* is the unique point where $\nabla_x h(x^*) = 0$. Thus the system reaches the largest invariant on the set $L_g h = 0$ which occurs at the point x^* with corresponding input u^*.

As a result, the ESC closed-loop system converges to the steady-state optimum of the cost $h(x)$. The main feature of the proposed ESC is the combination of a proportional component $u - \hat{u} = -kL_g h$ with integral action given by $\dot{\hat{u}} = -L_g h$.

4. EXTREMUM-SEEKING CONTROLLER

In this section, the minimization of y is performed in real-time. Since the dynamics of the system are unknown, one must consider an adaptive control approach to implement the ESC (5). In the following, we parameterize the unknown dynamics (4). Defining $\theta_0 = L_f h$ and $\theta_1 = L_g h$, we propose the following parameterization:

$$\dot{y} = \theta_0 + \theta_1 u = \phi^T \theta \tag{6}$$

where $\phi = [1,\, u^T]^T$ and $\theta = [\theta_0, \theta_1^T]^T$. The design of the extremum seeking routine is based on the dynamics (6). The first step consists in the estimation of the time-varying parameters θ_0 and θ_1. In the second step, we define a controller, based on the controller (5), that achieves the extremum-seeking task.

4.1 Parameter estimation

The first element of the proposed time-varying parameter estimation scheme is the following output prediction model for (6). Let $\hat{y}$ represent the predicted output for a given value of the parameter estimates $\hat{\theta} = [\hat{\theta}_0,\, \hat{\theta}_1^T]^T$. The output prediction error is denoted by $e = y - \hat{y}$ and the parameter estimation error is given by $\tilde{\theta} = \theta - \hat{\theta}$. We consider the following prediction dynamics:

$$\dot{\hat{y}} = \phi^T \hat{\theta} + Ke + c^T \dot{\hat{\theta}}, \tag{7}$$

where K is a positive constant to be assigned and where the time varying parameter $c \in \mathbb{R}^p$ is the solution of the differential equation:

$$\dot{c}^T = -Kc^T + \phi^T \tag{8}$$

with initial conditions $c(0) = 0$. The prediction error dynamics are given by:

$$\dot{e} = \dot{u}^T \tilde{\theta} - Ke - c^T \dot{\hat{\theta}} \tag{9}$$

with initial conditions $e(0) = y(0) - \hat{y}(0)$. Following standard arguments, we define the auxiliary variable $\eta = e - c^T \tilde{\theta}$. The dynamics of η are as follows:

$$\dot{\eta} = -K\eta - c^T \dot{\theta}, \qquad \eta(0) = e(0) \tag{10}$$

A filtered estimate, $\hat{\eta}$, of η is also defined. The $\hat{\eta}$ dynamics are given by:

$$\dot{\hat{\eta}} = -K\hat{\eta}. \tag{11}$$

As a result, the dynamics of the estimation error $\tilde{\eta} = \eta - \hat{\eta}$ are

$$\dot{\tilde{\eta}} = -K\tilde{\eta} - c^T \dot{\theta}, \qquad \tilde{\eta}(0) = 0. \tag{12}$$

We can now define the proposed parameter estimation update. Let $\Sigma \in \mathbb{R}^{p \times p}$ be the solution to the following matrix differential equation

$$\dot{\Sigma} = cc^T - k_T \Sigma + \delta I \tag{13}$$

with initial conditions $\Sigma(0) = \alpha_1 I \succ 0$, where α_1, δ and k_T are strictly positive constants to be assigned. The inverse of Σ is then given as the solution to the matrix differential equation:

$$\dot{\Sigma}^{-1} = -\,\Sigma^{-1} cc^T \Sigma^{-1} + k_T \Sigma^{-1} - \delta \Sigma^{-2} \tag{14}$$

with initial condition $\Sigma^{-1}(0) = \frac{1}{\alpha} I$. Based on (7),(8) and (11), one considers the following parameter update law proposed in Adetola and Guay [2009]:

$$\dot{\hat{\theta}} = \mathrm{Proj}(\Sigma^{-1}(c(e - \hat{\eta}) - \sigma\hat{\theta}), \hat{\theta}), \qquad \hat{\theta}(0) = \theta^0 \in \Theta^0, \tag{15}$$

where σ is a positive constant. $\mathrm{Proj}\{\tau, \hat{\theta}\}$ denotes a Lipschitz projection operator Krstic et al. [1995] such that

$$-\mathrm{Proj}\{\tau, \hat{\theta}\}^T \tilde{\theta} \le -\tau^T \tilde{\theta}, \tag{16}$$

$$\hat{\theta}(0) \in \Theta^0 \implies \hat{\theta} \in \Theta, \forall t \ge 0 \tag{17}$$

where $\Theta \triangleq B(\hat{\theta}, z_\theta)$, where $B(\hat{\theta}, z_\theta)$ is the ball centered at $\hat{\theta}$ with radius z_θ. Following standard arguments from adaptive control, the filter parameter, $c(t)$, must satisfy the following assumption.

Assumption 4. There exists constants $\alpha_2 > 0$ and $T > 0$ such that

$$\int_t^{t+T} c(\tau)c(\tau)^T d\tau \ge \alpha_2 I \tag{18}$$

$\forall t > 0$.

4.2 Controller design

The input space $\mathcal{U}$ is defined as $\mathcal{U} = \{u \,|\, \|u\| \le z_u\}$ where z_u is a positive constant that identifies the upper limit on the size of the norm the control input u. Let $P(u) = \|u\|^2 - z_u^2$ and define $\tau = -k\hat{\theta} + d(t)$ where $d(t)$ is a bounded dither signal with $\|d(t)\| \le D$ and $k > 0$.

The extremum-seeking controller considered is given by:

$$u = -k_g \hat{\theta}_1 + \hat{u} + d(t) \tag{19}$$

$$\dot{\hat{u}} = \frac{1}{\tau_I} \hat{\theta}_1 \tag{20}$$

where $d(t)$ is a dither signal to be determined, k_g and τ_I are tuning parameters taken as positive constants.

Theorem 1. Let Assumptions 1 to 4 hold. Consider the extremum-seeking controller (19) and the parameter estimation algorithm (14) and (15). Then there exists tuning parameters k_g, k_T, K and τ_I^* such that for all $\tau_I > \tau_I^*$. the system converges exponentially to an $\mathcal{O}(D/\tau_I)$ neighbourhood of the minimizer x^* of the measured cost function y.

Proof: We consider the Lyapunov function:

$$W = \frac{1}{2}\tilde{\eta}^T\tilde{\eta} + \frac{1}{2}\tilde{\theta}^T\Sigma\tilde{\theta}.$$

Taking the derivative of W yields:

$$\begin{aligned}\dot{W} \leq & -\tilde{\eta}^T K\tilde{\eta} + \tilde{\eta}^T c^T\dot{\theta} + \tilde{\theta}^T\Sigma\dot{\theta} - \frac{k_T}{2}\tilde{\theta}^T\Sigma\tilde{\theta} \\ & + \frac{\delta}{2}\tilde{\theta}^T\tilde{\theta} - \tilde{\theta}^T c(e-\hat{\eta}) + \frac{1}{2}\tilde{\theta}^T cc^T\tilde{\theta} + \sigma\tilde{\theta}^T\hat{\theta}\end{aligned} \tag{21}$$

where the property of the property (16) of the projection algorithm is invoked. Substituting for

$$c^T\tilde{\theta} = e + \eta = e - \hat{\eta} + \tilde{\eta}$$

and completing the squares yields the following inequality:

$$\begin{aligned}\dot{W} \leq & -\tilde{\eta}^T K\tilde{\eta} + \tilde{\eta}^T c^T\dot{\theta} + \tilde{\theta}^T\Sigma\dot{\theta} - \frac{k_T}{2}\tilde{\theta}^T\Sigma\tilde{\theta} \\ & + \frac{\delta}{2}\tilde{\theta}^T\tilde{\theta} - \frac{1}{2}(e-\hat{\eta})^T(e-\hat{\eta}) + \sigma\tilde{\theta}^T\hat{\theta} + \frac{1}{2}\tilde{\eta}^T\tilde{\eta}\end{aligned}$$

Noting that $\theta = \hat{\theta} + \tilde{\theta}$, one can rewrite the above inequality as follows:

$$\begin{aligned}\dot{W} \leq & -\tilde{\eta}^T\left(K - \frac{1}{2}I\right)\tilde{\eta} + \tilde{\eta}^T c^T\dot{\theta} + \tilde{\theta}^T\Sigma\dot{\theta} - \frac{k_T}{2}\tilde{\theta}^T\Sigma\tilde{\theta} \\ & - \left(\sigma - \frac{\delta}{2}\right)\tilde{\theta}^T\tilde{\theta} + \sigma\tilde{\theta}^T\theta\end{aligned}$$

By completing the squares, we remove the indefinite terms as follows:

$$\begin{aligned}\dot{W} \leq & -\tilde{\eta}^T\left(K - \frac{1}{2}I - \frac{k_1}{2}cc^T\right)\tilde{\eta} + \frac{1}{2k_1}\|\dot{\theta}\|^2 + \frac{k_2}{2}\tilde{\theta}^T\Sigma\tilde{\theta} \\ & + \frac{1}{2k_2}\dot{\theta}^T\Sigma\dot{\theta} - \frac{k_T}{2}\tilde{\theta}^T\Sigma\tilde{\theta} - \left(\sigma - \frac{\delta}{2}\right)\|\tilde{\theta}\|^2 \\ & + \frac{\sigma}{2}\|\tilde{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2\end{aligned}$$

where k_1 and k_2 are strictly positive constants.

The boundedness of the matrix Σ can be shown as follows. By integration, one gets:

$$\begin{aligned}\Sigma = & e^{-k_T t}\Sigma(0) + \int_0^t e^{-k_T(t-\tau)}\left(c(\tau)c(\tau)^T + \frac{\sigma}{2}I\right)d\tau \\ \geq & \int_{t-T}^t e^{-k_T(t-\tau)}\left(c(\tau)c(\tau)^T + \frac{\sigma}{2}I\right)d\tau \\ \geq & e^{-k_T T}(\alpha_2 + \frac{\sigma}{2})I = \gamma_1 I\end{aligned}$$

where Assumption 4 is invoked. By the boundedness of c, one can also write,

$$\begin{aligned}\Sigma \leq & \Sigma(0) + \left(\beta_2 + \frac{\sigma}{2}\right)\int_0^t e^{-k_T(t-\tau)}d\tau I \\ \leq & \left(\alpha_1 + \beta_2 + \frac{\sigma}{2}\right)I = \gamma_2 I.\end{aligned}$$

As a result, we get that: $\gamma_1 I \leq \Sigma \leq \gamma_2 I$ and $\gamma_2^{-1}I \leq \Sigma^{-1} \leq \gamma_1^{-1}I$. By the boundedness of Σ and Γ, one can write:

$$\begin{aligned}\dot{W} \leq & -\left(K - \frac{1}{2}I - \frac{k_1}{2}cc^T\right)\|\tilde{\eta}\|^2 \\ & - \left(\frac{k_T\gamma_1}{2} + \frac{\sigma}{2} - \frac{k_2\gamma_2}{2} - \frac{\delta}{2}\right)\|\tilde{\theta}\|^2 \\ & + \left(\frac{\gamma_2}{2k_2} + \frac{1}{2k_1}\right)\|\dot{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2\end{aligned}$$

We let the tuning constants of the ESC be such that:

$$K = k_{\eta_1}I + k_{\eta_2}c^T c, \ k_T\gamma 1 = k_{\tilde{\theta}},$$

where $k_{\eta_1} > 1/2$, $k_{\eta_2} > k/2$, $\sigma = \delta$ and $k_{\tilde{\theta}} > \frac{k_2\gamma_2}{2}$. Let

$$k_a = K - \frac{1}{2}I - \frac{k_1}{2}cc^T, \ k_b = \frac{k_T\gamma_1}{2} - \frac{k_2\gamma_2}{2}, \ k_c = \frac{\gamma_2}{2k_2} + \frac{1}{2k_1},$$

and rewrite the last inequality as follows:

$$\dot{W} \leq -k_a\|\tilde{\eta}\|^2 - k_b\|\tilde{\theta}\|^2 + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2.$$

Next we consider the proposed ESC at constant $\hat{u}$ using the Lyapunov function candidate: $\mathcal{V} = W + y$. Using the last inequality, its time derivative is given by:

$$\begin{aligned}\dot{\mathcal{V}} \leq & -k_a\|\tilde{\eta}\|^2 - k_b\|\tilde{\theta}\|^2 + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2 \\ & + \theta_0 + \theta_1^T u + \theta_1^T d.\end{aligned}$$

Substituting the proposed ESC:

$$\begin{aligned}\dot{\mathcal{V}} \leq & -k_a\|\tilde{\eta}\|^2 - k_b\|\tilde{\theta}\|^2 + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2 \\ & + \theta_0 - k_g\theta_1^T\hat{\theta}_1 + \theta_1^T\hat{u} + \theta_1^T d\end{aligned}$$

or,

$$\begin{aligned}\dot{\mathcal{V}} \leq & -k_a\|\tilde{\eta}\|^2 - k_b\|\tilde{\theta}\|^2 + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2 \\ & + \theta_0 + \theta_1^T\hat{u} - k_g\theta_1^T\theta_1 + k_g\theta_1^T\tilde{\theta}_1 + \theta_1^T d.\end{aligned}$$

By Assumption 2, we obtain:

$$\begin{aligned}\dot{\mathcal{V}} \leq & -k_a\|\tilde{\eta}\|^2 - k_b\|\tilde{\theta}\|^2 + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\|\theta\|^2 \\ & - \alpha\|x - \pi(\hat{u})\|^2 - k_g\theta_1^T\theta_1 + k_g\theta_1^T\tilde{\theta}_1 + \theta_1^T d.\end{aligned}$$

Completing the squares using constants k_3 and k_4, we remove the indefinite terms and collect terms to obtain:

$$\begin{aligned}\dot{\mathcal{V}} \leq & -k_a\|\tilde{\eta}\|^2 - \left(k_b - \frac{k_g}{2k_3}\right)\|\tilde{\theta}\|^2 - \alpha\|x - \pi(\hat{u})\|^2 \\ & - \left(k_g - \frac{k_g k_3}{2} - \frac{k_4}{2} - \frac{\sigma}{2}\right)\|\theta_1\|^2 \\ & + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\theta_0^2 + \frac{1}{2k_4}\|d\|^2.\end{aligned}$$

As a result, we see that the proposed ESC in the absence of integral action can approach a neighbourhood of the unknown optimum whose depends on the distance of $\hat{u}$ from the true unknown steady-state optimum u^*. The next step of the proof focusses on proving that this distance can be minimized for some large enough τ_I.

To do this, we consider the equilibrium response, given by $x = \pi(\hat{u})$, of the system at a specific $\hat{u}$. The equilibrium map is such that:

$$f(\pi(\hat{u})) + g(\pi(\hat{u}))\hat{u} = 0.$$

The output trajectory along the equilibrium manifold, also called the quasi steady-state response, can be described by the differential equation:

$$\begin{aligned}\frac{dy}{d\tau} = & \dot{\hat{u}}^T\frac{\partial\pi^T}{\partial\hat{u}}\frac{\partial^2 h(\pi(\hat{u}))}{\partial x\partial x^T}f(\pi(\hat{u})) \\ & + \dot{\hat{u}}^T\frac{\partial\pi^T}{\partial\hat{u}}\frac{\partial^2 h(\pi(\hat{u}))}{\partial x\partial x^T}g(\pi(\hat{u}))\hat{u} \\ & + \dot{\hat{u}}^T\frac{\partial\pi^T}{\partial\hat{u}}\frac{\partial h(\pi(\hat{u}))}{\partial x}\left(\frac{\partial f(\pi(\hat{u}))}{\partial x} + \frac{\partial g(\pi(\hat{u}))}{\partial x}\hat{u}\right) \\ & + \frac{\partial h(\pi(\hat{u}))}{\partial x}g(\pi(\hat{u}))\frac{d\hat{u}}{d\tau}.\end{aligned}$$

where $d\tau = \tau_I dt$ represents the quasi steady-state time-scale. Along the steady-state manifold, it follows that:

$$\frac{dy}{d\tau} = \frac{\partial h(\pi(\hat{u}))}{\partial x} g(\pi(\hat{u})) \frac{d\hat{u}}{d\tau} = \theta_{1s}(\hat{u}) \frac{d\hat{u}}{d\tau}$$

where $\theta_{1s}(\hat{u})$ is the equilibrium value of θ_1.

The steady-state output map is $h(\pi(u)) = \ell(u)$. Differentiating with respect to τ along the steady-state yields:

$$\frac{dy}{d\tau} = \frac{\partial \ell}{\partial u} \frac{d\hat{u}}{d\tau}.$$

Thus following development above, one can always identify the gradient of the steady-state map $\ell(u)$ as follows:

$$\frac{\partial \ell}{\partial u} = \frac{\partial h(\pi(\hat{u}))}{\partial x} g(\pi(\hat{u})).$$

Remark 1. This observation suggests a unification of the proposed approach with the standard ESC which considers only the gradient of the steady-state objective function.

Consider the Lyapunov function candidate, $\mathcal{W} = \mathcal{V} + \frac{\epsilon}{2}\tilde{u}^T\tilde{u}$. Differentiating, one obtains:

$$\begin{aligned}\dot{\mathcal{W}} =& - k_a\|\tilde{\eta}\|^2 - \left(k_b - \frac{k_g}{2k_3}\right)\|\tilde{\theta}\|^2 - \alpha\|x - \pi(\hat{u})\|^2 \\ & - \left(k_g - \frac{k_g k_3}{2} - \frac{k_4}{2} - \frac{\sigma}{2}\right)\|\theta_1\|^2 \\ & + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\theta_0^2 + \frac{1}{2k_4}\|d\|^2 - \epsilon\tilde{u}^T\dot{\hat{u}}\end{aligned}$$

Upon substitution of $\dot{\hat{u}} = -\frac{1}{\tau_I}\hat{\theta}_1$, the following inequality results:

$$\begin{aligned}\dot{\mathcal{W}} \leq& - k_a\|\tilde{\eta}\|^2 - \left(k_b - \frac{k_g}{2k_3}\right)\|\tilde{\theta}\|^2 - \alpha\|x - \pi(\hat{u})\|^2 \\ & - \left(k_g - \frac{k_g k_3}{2} - \frac{k_4}{2} - \frac{\sigma}{2}\right)\|\theta_1\|^2 \\ & + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\theta_0^2 + \frac{1}{2k_4}\|d\|^2 + \frac{\epsilon}{\tau_I}\tilde{u}^T\hat{\theta}_1\end{aligned}$$

The term $\tilde{u}^T\theta_1$ can be rewritten as:

$$\begin{aligned}\tilde{u}^T\theta_1 =& \frac{\partial h(\pi(\hat{u}))}{\partial x} g(\pi(\hat{u}))\tilde{u} \\ & + \left(\frac{\partial h(x)}{\partial x} g(x) - \frac{\partial h(\pi(\hat{u}))}{\partial x} g(\pi(\hat{u}))\right)\tilde{u}.\end{aligned}$$

By local convexity of the steady-state objective function, it follows that:

$$\tilde{u}^T\theta_1 \leq -\alpha_u\|\tilde{u}\|^2 - \left(\frac{\partial h(x)}{\partial x} g(x) - \frac{\partial h(\pi(\hat{u}))}{\partial x} g(\pi(\hat{u}))\right)\tilde{u}.$$

Given that $h(x)$ and $g(x)$ are smooth, it follows that there exists Lipschitz constants L_g and L_h such that:

$$\tilde{u}^T\theta_1 \leq -\alpha_u\|\tilde{u}\|^2 + L_g L_h\|\tilde{u}\|\|x - \pi(\hat{u})\|.$$

As a result, one can write the following inequality for $\mathcal{W}$:

$$\begin{aligned}\dot{\mathcal{W}} \leq& - k_a\|\tilde{\eta}\|^2 - \left(k_b - \frac{k_g}{2k_3}\right)\|\tilde{\theta}\|^2 - \alpha\|x - \pi(\hat{u})\|^2 \\ & - \left(k_g - \frac{k_g k_3}{2} - \frac{k_4}{2} - \frac{\sigma}{2}\right)\|\theta_1\|^2 \\ & + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\theta_0^2 + \frac{1}{2k_4}\|d\|^2 \\ & - \alpha_u\left(\frac{1}{\tau_I}\right)\|\tilde{u}\|^2 + \left|\frac{1}{\tau_I}\right| L_g L_h\|\tilde{u}\|\|x - \pi(\hat{u})\| \\ & - \frac{1}{\tau_I}\tilde{u}^T\tilde{\theta}_1.\end{aligned}$$

Finally, we bound the last term for some positive constant k_5

$$\begin{aligned}\dot{\mathcal{W}} \leq& - k_a\|\tilde{\eta}\|^2 - \left(k_b - \frac{k_g}{2k_3} - \frac{k_5}{2}\right)\|\tilde{\theta}\|^2 - \alpha\|x - \pi(\hat{u})\|^2 \\ & - \left(k_g - \frac{k_g k_3}{2} - \frac{k_4}{2} - \frac{\sigma}{2}\right)\|\theta_1\|^2 \\ & + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\theta_0^2 + \frac{1}{2k_4}\|d\|^2 \\ & - \alpha_u\left(\frac{1}{\tau_I} - \frac{1}{2\tau_I^2 k_5}\right)\|\tilde{u}\|^2 + \left|\frac{1}{\tau_I}\right| L_g L_h\|\tilde{u}\|\|x - \pi(\hat{u})\|.\end{aligned}$$

As a result, it follows that there exists a strictly positive τ_I such that the last inequality can be written as:

$$\begin{aligned}\dot{\mathcal{W}} \leq& - k_a\|\tilde{\eta}\|^2 - \left(k_b - \frac{k_g}{2k_3} - \frac{k_5}{2}\right)\|\tilde{\theta}\|^2 \\ & - \left(k_g - \frac{k_g k_3}{2} - \frac{k_4}{2} - \frac{\sigma}{2}\right)\|\theta_1\|^2 \\ & + k_c\|\dot{\theta}\|^2 + \frac{\sigma}{2}\theta_0^2 + \frac{1}{2k_4}\|d\|^2 \\ & - \alpha\beta_1(\tau_I^*)\|x - \pi(\hat{u})\|^2 - \alpha_u\beta_2(\tau_I^*)\|\tilde{u}\|^2.\end{aligned}$$

from corresponding positive constants $\beta_1(\tau_I^*)$ and $\beta_2(\tau_I^*)$. As a result,iIt follows that, for every $\tau_I > \tau_I^*$, $\tilde{\eta}$, $\tilde{\theta}$, $\tilde{u}$ and θ_1 converge to an $\mathcal{O}(D/\tau_I)$ neighbourhood of the origin. As u approaches a neighbourhood of u^*, the state x enters a neighbourhood of the steady-state optimum $\pi(u^*)$. This is achieved by using estimation gains K and k_T that are larger than the optimization gain k_g to ensure that all constants multiply the corresponding norms ($\|\tilde{\eta}\|^2$, $\|\tilde{\theta}\|^2$, $\|\theta_1\|^2$, $\|\tilde{u}\|^2$ and $\|x - \pi(\hat{u})\|^2$) are negative. ■

5. SIMULATION EXAMPLE

In this section, we consider the following dynamical system:

$$\begin{aligned}\dot{x}_1 =& - x_2 + u \\ \dot{x}_2 =& x_1 - x_2\end{aligned}$$

The cost function to be minimized is given by: $y = x_1^4 + 6(x_1 - 1)$. The optimum occurs at $u^* = -1.145$, $x_1^* = -1.145$, $x_2^* = -1.145$ where $y^* = -11.15$. The tuning parameters are chosen as: $k_T = 100$, $K = 100I$, $k_g = 0.1$ and $\tau_I = 1$. The dither signal is $d(t) = \sin(1000t)$. The simulation results are shown in Figures 1 and 2. Figure 1 shows the objective function and the corresponding input trajectories for the ESC. Figure 2 shows the state trajectories. The results demonstrate the effectiveness of the PI-ESC to locate the unknown optimum.

6. CONCLUSION

In this paper, an alternative proportional-integral ESC technique was proposed. The technique is based on the time-varying parameter estimation that allows one to exploit a simple parameterization of the unknown dynamics of the cost function. The ESC algorithm is shown to provide local asymptotic convergence of the closed-loop system to the unknown optimum. In contrast to existing ESC techniques, no time-scale separation is required to achieve the steady-state optimum. Future work will be focussed on the generalization of the technique to a large class of systems. In particular, the proposed ESC will be considered for the solution of steady-state optimization problems in unstable systems

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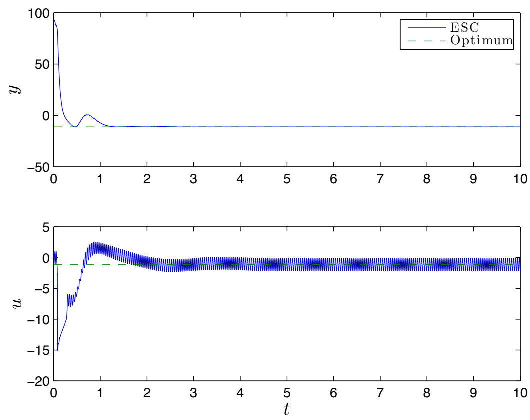

Fig. 1. Plot of the cost function and the corresponding control as a function of time.

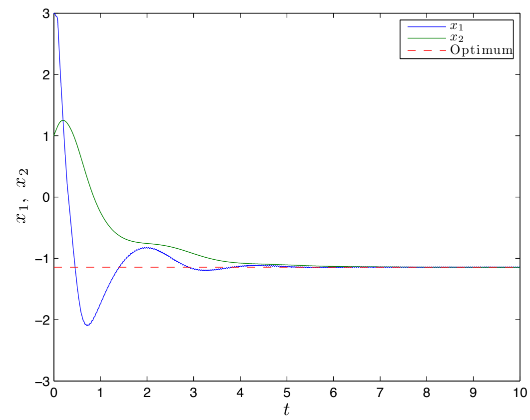

Fig. 2. State trajectories for the PI-ESC closed-loop process.