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# Performance-guaranteed Robust PID Controller Design for Systems with Unstable Zero

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**Abstract:** *The paper deals with the development of a new robust PID controller design method that guarantees designer-specified maximum overshoot and settling time for non-minimum phase processes with unstable zero. The PID controller design provides guaranteed gain margin $G_M$. The parameter of the tuning rules is a suitably chosen point of the plant frequency response obtained by a sine-wave signal with excitation frequency $\omega_n$. Then, the designed controller moves this point into the phase crossover with the required gain margin $G_M$. The couple $(\omega_n;G_M)$ is specified with respect to closed-loop performance requirements in terms of $\eta_{max}$ (maximum overshoot) and $t_s$ (settling time) according to developed parabolic dependences. The new approach has been verified on a vast batch of benchmark examples; subsequently, the developed algorithm has been extended to robust PID controller design for plants with unstable zero and unstructured uncertainties.*

*Keywords: Gain margin, PID tuning, guaranteed performance, unstable zero*

## 1. INTRODUCTION

The proposed new method is applicable for control of linear single-input-single-output non-minimum phase systems even with unknown mathematical model with unstructured uncertainties. A survey on PID controller tuning can be found in (Åström and Hägglund, 1995), (Åström and Hägglund, 2000), (Blickley, 1990), (Grabbe et al., 1959-61), (Karaboga and Kalinli, 1996), (Kristiansson and Lennartson, 2002), (Morilla and Dormido, 2000), (O'Dwyer, 2000), (Tinham, 1989), (Veselý, 2003), (Visioli, 2006), (Yu, 2006), in the famous paper (Ziegler and Nichols, 1942) and references therein. The control objective is to provide required *nominal maximum overshoot* $\eta_{max}$ *and settling time* $t_s$ of the controlled process variable $y(t)$. The key idea behind guaranteeing specified values $\eta_{max}$ and $t_s$ consists in extending validity of the relations $\eta_{max}=f(G_M)$ and $t_s=f(\omega_n)$ derived for 2[nd] order systems (Reinisch, 1974) for arbitrary plant orders; two-parameter quadratic dependences were obtained for both the maximum overshoot $\eta_{max}=f(G_M,\omega_n)$ and settling time $t_s=f(G_M,\omega_n)$. The resulting plots called *B-parabolas* enable the designer choosing such a couple $(G_M,\omega_n)$ that guarantees fulfillment of specified performance requirements thus allowing consistent and systematic shaping of the closed-loop step response with regard to the controlled plant (Bucz and Kozáková, 2012).

## 2. PID CONTROLLER DESIGN OBJECTIVES FOR PROCESSES WITH UNSTABLE ZERO

It is a well known difficulty to control the class of non-minimum phase systems $G(s)=(1-\alpha s)/(1+Ts)^n$ with unstable zero $z=+1/\alpha$, even for small values of α; moreover, control complexity increases with increasing $\alpha$ (Vítečková et al., 2000). Fig. 1 shows Nyquist plots of the non-minimum phase plant $G(s)$ for $n=3$ and $T=1$, with an unstable zero ($\alpha$=0.1,0.2,0.5,1,2,5 are considered). Fig. 1 reveals, that with increasing $\alpha$ the gain margin of the plant decreases, and the phase crossover moves closer to $(-1,j0)$. Due to significant changes of the gain margin of the plant brought about by the non-minimum phase behavior, it is beneficial to use *gain margin* $G_M$ as a performance measure when designing the PID controller.

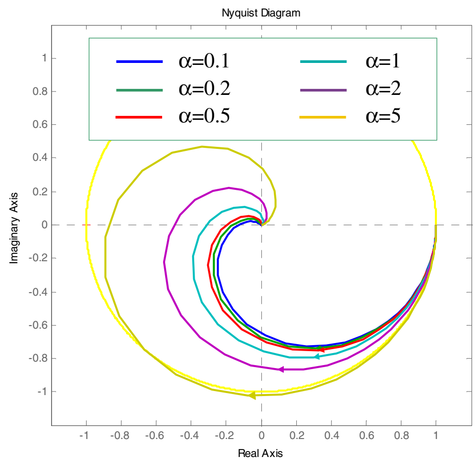


Fig. 1. Nyquist plots of $G(s)=(1\text{-}\alpha s)/(Ts+1)^n$ for $n=3$, $T=1$ and different values of $\alpha$

Consider a multipurpose loop shown in Fig. 2 (the switch in position $SW=1$). Let $G(s)$ be transfer function of an uncertain non-minimum phase plant, and $G_R(s)$ the PID controller.

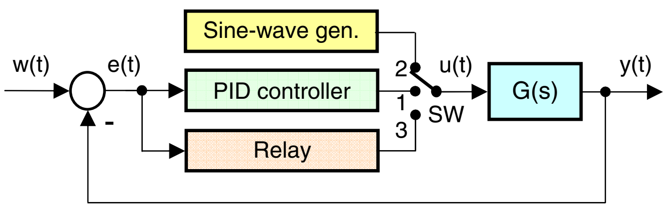


Fig. 2. Multipurpose loop for the designed sine-wave method

The corresponding closed-loop characteristic equation $c(s)=1+L(s)=1+G(s)G_R(s)=0$ expresses the closed-loop stability can easily be broken down into the magnitude and phase conditions

$$\left|G(j\omega_p^*)\right|\left|G_R(j\omega_p^*)\right|=1/G_M\,,\quad arg G(\omega_p^*)+arg G_R(\omega_p^*)=-\pi\,, \quad (1)$$

where $G_M$ is required gain margin, $L(j\omega)$ is the open-loop transfer function, and $\omega_p^*$ is the open-loop phase crossover frequency. Denote $\varphi=argG(\omega_p^*)$, $\Theta=argG_R(\omega_p^*)$, and consider the ideal PID controller in the form

$$G_R(s)=K\left[1+\frac{1}{T_i s}+T_d s\right], \qquad (2)$$

where $K$ is the proportional gain, and $T_i$, $T_d$ are integral and derivative time constants, respectively. After comparing the two forms of the PID controller frequency transfer functions

$$G_R(j\omega_p^*)=K+jK\left[T_d\omega_p^*-\frac{1}{T_i\omega_p^*}\right], \qquad (3)$$

$$G_R(j\omega_p^*)=\left|G_R(j\omega_p^*)\right|\left[cos\Theta+j\,sin\Theta\right], \qquad (4)$$

PID coefficients can be obtained from the complex equation at $\omega=\omega_p^*$

$$K+jK\left[T_d\omega_p^*-\frac{1}{T_i\omega_p^*}\right]=\frac{cos\Theta}{G_M\left|G(j\omega_p^*)\right|}+j\frac{sin\Theta}{G_M\left|G(j\omega_p^*)\right|} \qquad (5)$$

using the substitution $|G_R(j\omega_p^*)|=1/[G_M|G(j\omega_p^*)|]$ resulting from (1a). The complex equation (5) is then solved as a set of two real equations

$$K=\frac{cos\Theta}{G_M\left|G(j\omega_p^*)\right|}\,,\quad K\left[T_d\omega_p^*-\frac{1}{T_i\omega_p^*}\right]=\frac{sin\Theta}{G_M\left|G(j\omega_p^*)\right|}\,, \qquad (6)$$

where (6a) is a general rule for calculating the controller gain $K$; substituting (6a) into (6b), a quadratic equation in $T_d$ is obtained

$$T_d^2\left(\omega_p^*\right)^2-T_d\omega_p^* tg\Theta-\frac{1}{\beta}=0\,,\ \text{where}\ \beta=\frac{T_i}{T_d}\,. \qquad (7)$$

Expression for calculating $T_d$ is the positive solution of (7)

$$T_d=\frac{tg\Theta}{2\omega_p^*}+\frac{1}{\omega_p^*}\sqrt{\frac{tg^2\Theta}{4}+\frac{1}{\beta}}\,. \qquad (8)$$

Hence, (6a), (7b) and (8) are the resulting PID tuning rules, where the angle $\Theta$ is obtained from the phase condition (1b)

$$\Theta=-180°-arg\,G(\omega_p^*)=-180°-\varphi\,. \qquad (9)$$

## 3. PLANT IDENTIFICATION BY A SINUSOIDAL EXCITATION INPUT

Consider again Fig. 2; if $SW$=2, a sinusoidal excitation signal $u(t)=U_n sin(\omega_n t)$ with magnitude $U_n$ and frequency $\omega_n$ is injected into the plant $G(s)$. The plant output $y(t)=Y_n sin(\omega_n t+\varphi)$ is also sinusoidal with magnitude $Y_n$, where $\varphi$ is the phase lag between $y(t)$ and $u(t)$. After reading the values $Y_n$ and $\varphi$ from the recorded values of $u(t)$ and $y(t)$, a particular point of the plant frequency response corresponding to the excitation frequency $\omega_n$

$$G(j\omega_n)=\left|G(j\omega_n)\right|e^{j\,arg\,G(\omega_n)}=\left[Y_n/U_n\right]e^{j\varphi(\omega_n)} \qquad (10)$$

can be plotted in the complex plane.

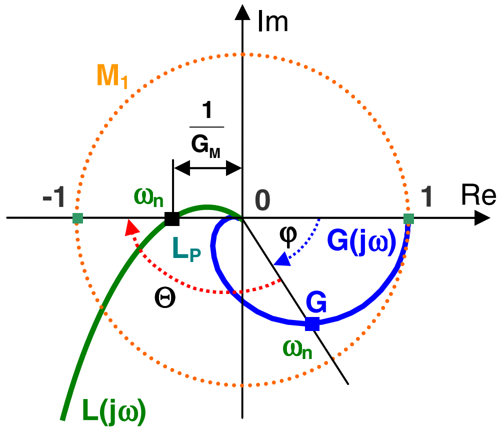


Fig. 3. Graphical representation of the PID tuning principle

Excitation frequency $\omega_n$ is taken from the interval

$$\omega_n\in\left\langle 0.5\omega_c, 1.25\omega_c\right\rangle\,, \qquad (11)$$

where the *plant critical frequency* $\omega_c$ can be obtained by the well-known relay experiment (Åström and Hägglund, 1995), i.e. for $SW=3$.

Using the PID controller with the coefficients $\{K;T_i=\beta T_d;T_d\}$, the identified point $G(j\omega_n)$ with coordinates (10) can be moved into the phase crossover $L_P\equiv L(j\omega_p^*)$ on the negative real half-axis, where the required gain margin $G_M$ is guaranteed (Fig. 3), if the following identity between the excitation and phase crossover frequencies $\omega_n$ and $\omega_p^*$, respectively, is fulfilled

$$\omega_p^*=\omega_n\,. \qquad (12)$$

Considering (11), the following relations result

$$\left|G(j\omega_p^*)\right|=\left|G(j\omega_n)\right|,\ arg\,G(\omega_p^*)=arg\,G(\omega_n)=\varphi\,, \qquad (13)$$

$$\Theta=-180°-arg\,G(\omega_n) \qquad (14)$$

and the phase crossover coordinates are $L_P=[|L(j\omega_n)|,argL(\omega_n)]=[1/G_M,-180°]$. Substituting (13a) into (6a) and (12) into (8), the PID controller coefficients guaranteeing the required gain margin $G_M$ are obtained using the sine-wave type tuning rules expressed in the following form

$$K=\frac{cos\Theta}{G_M\left|G(j\omega_n)\right|}\,,\quad T_d=\frac{tg\Theta}{2\omega_n}+\frac{1}{\omega_n}\sqrt{\frac{tg^2\Theta}{4}+\frac{1}{\beta}}\,, \qquad (15)$$

$$\beta=4\,,\quad T_i=\beta T_d\,,\quad \Theta=-180°-\varphi\,. \qquad (16)$$

## 4. CLOSED-LOOP PERFORMANCE UNDER THE DESIGNED PID CONTROLLER

This section answers the following question: how to transform the maximum overshoot $\eta_{max}$ and settling time $t_s$ as required by the designer into the couple of frequency-domain parameters

$(\omega_n, G_M)$ needed for identification and PID controller tuning? Consider typical gain margins $G_M$ given by the set

$$\{G_{Mj}\}=\{3dB,5dB,7dB,9dB,11dB,13dB,15dB,17dB\}, \quad (17)$$

$j=1\ldots8$; let us split (11) into 5 equal sections of the size $\Delta\omega_n=0{,}15\,\omega_c$ and generate the set of excitation frequencies

$$\{\omega_{nk}\}=\{0.5\omega_c,0.65\omega_c,0.8\omega_c,0.95\omega_c,1.1\omega_c,1.25\omega_c\}, \quad (18)$$

$k=1\ldots6$; its elements divided by the plant critical frequency $\omega_c$ determine excitation levels $\sigma_k=\omega_{nk}/\omega_c$ given by the set

$$\{\sigma_k\}=\{0.5,0.65,0.8,0.95,1.1,1.25\}, \quad (19)$$

$k=1\ldots6$. Fig. 4 shows the closed-loop step response shaping for different $G_M$ and $\omega_n$ using the PID controller design for the plant (20b) with parameters $T_2=0.75$, $\alpha_2=1.3$, and required gain margins $G_M=5dB$, $9dB$, $11dB$ and $13dB$ at different excitation levels $\sigma_1=\omega_{n1}/\omega_c=0.5$, $\sigma_3=\omega_{n3}/\omega_c=0.8$ and $\sigma_5=\omega_{n5}/\omega_c=1.1$.

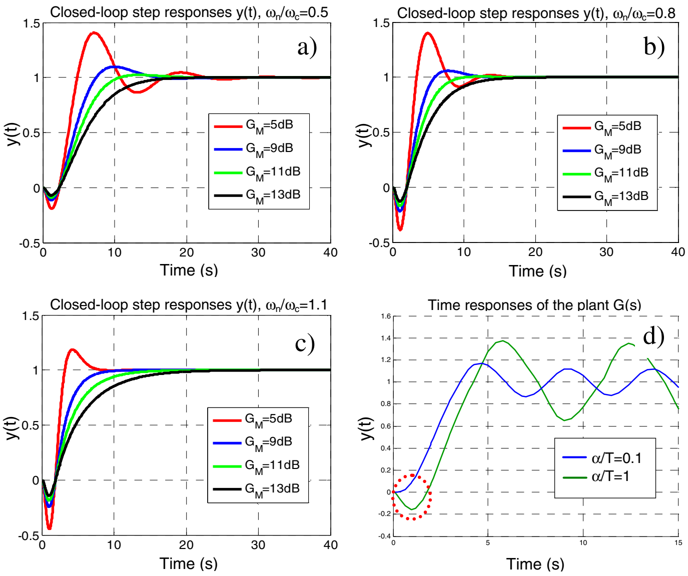


Fig. 4. *a)-c)* Closed-loop step responses of $G_2(s)$ with $T_2=0.75$, $\alpha_2=1.3$ for various $G_M$ and $\omega_n$; *d)* Time responses of $G_2(s)$ for $\alpha/T=1$ and $\alpha/T=0.1$ during the relay test

Consider the following benchmark plants

$$G_1(s)=\frac{-\alpha_1 s+1}{(T_1 s+1)^{n_1}},\; G_2(s)=\frac{-\alpha_2 s+1}{(s+1)(T_2 s+1)(T_2^2 s+1)(T_2^3 s+1)}. \quad (20)$$

The proposed method has been applied for each element of the Cartesian product $\omega_{nk}\times G_{Mj}$ of the sets (18) and (17) for $j=1\ldots8$ and $k=1\ldots6$. Significant differences between dynamics of individual control loops under designed PID controllers can be observed for the benchmark systems (20).

The *settling time* $t_s$ can be expressed by the relation

$$t_s=\frac{\gamma\pi}{\omega_n}, \quad (21)$$

where $\gamma$ is the *curve factor of the step response*. To examine settling times of closed-loops for various plant dynamics, it is advantageous to define the *relative settling time* $\tau_s=t_s\omega_c$. Substituting $\omega_n=\sigma\omega_c$ we obtain relation for the relative settling time

$$t_s\omega_c=\frac{\pi}{\sigma}\gamma \Rightarrow \tau_s=\frac{\pi}{\sigma}\gamma, \quad (22)$$

where $t_s$ is related to the *plant critical frequency* $\omega_c$. Due to introducing $\omega_c$, the l.h.s. of (22a) is constant for the given plant and independent of $\omega_n$. The dependence (22b) obtained empirically for different excitation frequencies $\omega_{nk}$ is depicted in Fig. 5b and Fig. 6b, respectively; it is evident that with increased phase margin $G_M$ at every excitation level $\sigma$ the relative settling time $\tau_s$ first decreases and after achieving its minimum $\tau_{s\_min}$, it increases again. Consider the benchmark plants $G_1(s)$ and $G_2(s)$ with following parameters: $G_{1.1}(s)$: $(T_1,n_1,\alpha_1)=(0.75,8,0.2)$; $G_{1.2}(s)$: $(1,3,0.1)$; $G_{1.3}(s)$: $(0.5,5,1)$; $G_2(s)$: $T_2=0.5$, $\alpha_2=1.3$. Couples of examined plants $[G_2(s), G_{1.3}(s)]$ and $[G_{1.2}(s), G_{1.1}(s)]$ differ principally by the *ratio* $\alpha/T$, which for the 1st couple is $[\alpha_2/T_2=2.6,\ \alpha_{1.3}/T_{1.3}=2]$ and for the 2nd couple $[\alpha_{1.2}/T_{1.2}=0.1,\ \alpha_{1.1}/T_{1.1}=0.27]$. Hence, the ratio of the *parameter* $\alpha$ and the *(dominant) time constant* $T$ of the plant is significant for the closed-loop performance assessment under the PID controller designed for a plant with unstable zero. Based on the previous analysis of design results of a series of benchmark examples, unknown plants with unstable zero can be classified according to the ratio $\alpha/T$ in following two groups:

1. plants with the ratio $\alpha/T<0{,}3$;
2. plants with the ratio $\alpha/T>0{,}3$.

According to this classification, empirical dependences $\eta_{max}=f(G_M)$, $\tau_s=f(G_M)$ for non-minimum phase systems with an unstable zero were constructed for different open-loop gain margins $G_M$ and excitation levels $\sigma$, and are depicted in Fig. 5a (for $\alpha/T>0.3$), and Fig. 6a (for $\alpha/T<0.3$). The network of dependences shows that increasing gain margin $G_M$ brings about decreasing of $\eta_{max}$.

As the empirical dependences in Fig. 5 and Fig. 6 were approximated by quadratic regression curves they are called *B-parabolas* (Bucz and Kozáková, 2012). B-parabolas are a useful design tool to carry out the transformation $\Re:(\eta_{max},t_s)\rightarrow(\omega_n,G_M)$ that enables to choose appropriate values of gain margin $G_M$ and excitation frequency $\omega_n$, respectively, to guarantee the performance specified by the designer in terms of maximum overshoot $\eta_{max}$ and settling time $t_s$ (Bucz and Kozáková, 2012). Note that pairs of B-parabolas at the same level (Fig. 5a, Fig. 5b) or (Fig. 6a, Fig. 6b) are to be used. When a *real plant* with an unstable zero is to be controlled, the *ratio* $\alpha/T$ cannot be specified exactly due to unavailability of the plant model. To decide to which category a given plant belongs ($\alpha/T>0.3$ or $\alpha/T<0.3$) it is sufficient to analyze the rise portion of the output variable during the relay test for finding $\omega_c$. If $y(t)$ has an *S-form with a tiny undershoot,* the plant is included in the category $\alpha/T<0.3$ and B-parabolas from Fig. 6 are to be used. If a considerable undershoot of $y(t)$ occurs having a *"square root sign" form* (Fig. 4d in the *red dashed ellipse*), the plant belongs to the category $\alpha/T>0.3$ and its performance will be assessed using B-parabolas in Fig. 5.

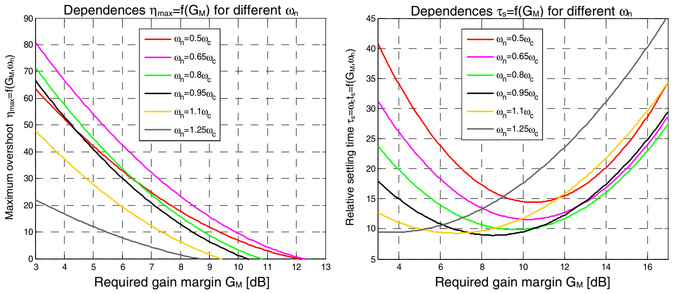


Fig. 5. B-parabolas: *a)* $\eta_{max}=f(G_M)$; *b)* $\tau_s=\omega_c t_s=f(G_M)$ for identification levels $\omega_{nk}/\omega_c$, *k=1,2,3,4,5,6* valid for non-minimum phase systems with the ratio $\alpha/T>0.3$

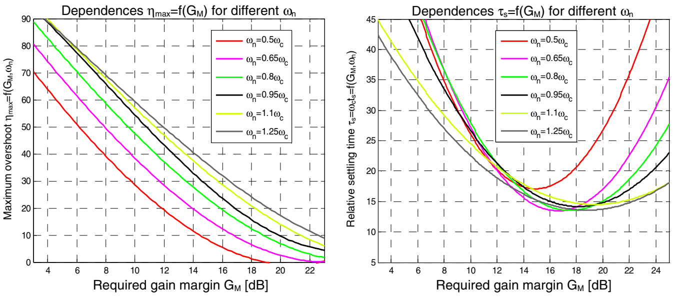


Fig. 6. B-parabolas: *a)* $\eta_{max}=f(G_M)$; *b)* $\tau_s=\omega_c t_s=f(G_M)$ for identification levels $\omega_{nk}/\omega_c$, *k=1,2,3,4,5,6* valid for non-minimum phase systems with the ratio $\alpha/T<0.3$

## 5. ROBUST SINE-WAVE TYPE PID CONTROLLER DESIGN

The main idea of the uncertain plant identification consists in repeating the sine-wave type excitation for individual uncertainty changes using the excitation signal frequency $\omega_n$ yielding a set of identified points $G_i$ of the uncertain plant frequency responses

$$G_i(j\omega_n)=\left|G_i(j\omega_n)\right|e^{j\arg G_i(\omega_n)}=a_i+jb_i,\ \text{i=1,2...N.} \tag{23}$$

Plant parameter changes are reflected in magnitude and phase changes $|G_i(j\omega_n)|$ and $\arg G_i(\omega_n)$, where $i=1...N$; $N=2^p$ is the number of identification experiments and $p$ is the number of varying technological quantities of the plant. The nominal plant model $G_0(j\omega_n)$ at $\omega_n$ is obtained as mean values of real and imaginary parts of $G_i(j\omega_n)$, respectively

$$G_0(j\omega_n)=a_0+jb_0=\frac{1}{N}\sum_{i=1}^{N}a_i+j\frac{1}{N}\sum_{i=1}^{N}b_i\,,\ \text{i=1,2...N,} \tag{24}$$

where $|G_0(j\omega_n)|=(a_0^2+b_0^2)^{0,5}$, $\varphi_0(\omega_n)=\arg G_0(\omega_n)=arctg(b_0/a_0)$. The points $G_i$ representing unstructured uncertainties of the plant can be enclosed in the circle $M_G$ centered in $G_0(j\omega_n)$ with the radius $R_G\equiv R_G(\omega_n)$ obtained as a maximum distance between the *i-th* identified point $G_i(j\omega_n)$ and the nominal point $G_0(j\omega_n)$

$$R_G=\max_i\left\{\sqrt{(a_i-a_0)^2+(b_i-b_0)^2}\right\},\ \text{i=1,2...N.} \tag{25}$$

The dispersion circle $M_G$ centered in the nominal point $G_0$ with the radius $R_G$ encircles all identified points $G_i$ of the uncertain plant (Fig. 7).

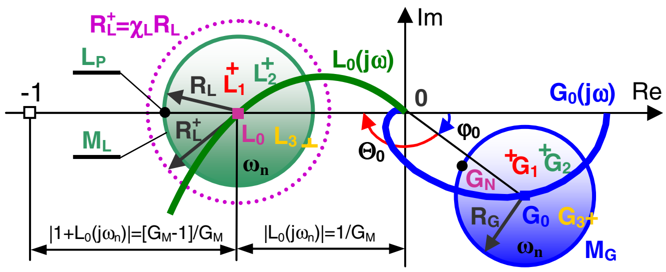


Fig. 7. Dispersion circles $M_G$ and $M_L$

The proposed control law generated by the robust controller $G_{Rrob}(s)$ designed for the nominal point $G_0(j\omega_n)$ actually carries out the transformation $\wp:\{R_G\rightarrow R_L:R_L=|G_{Rrob}|R_G\}$ of the set of identified points $G_i(j\omega_n)$ encircled by $M_G$ with the radius $R_G$ into the set of points $L_i(j\omega_n)$ delimited by $M_L$, and also calculates the radius $R_L\equiv R_L(\omega_n)$ of the dispersion circle $M_L$ corresponding to the points $L_i(j\omega_n)$ of the Nyquist plot so as to guarantee fulfillment of the robust stability condition. The robust PID controller is designed using the sine-wave method described in sections 2 and 3; the input data for the nominal model $G_0(j\omega_n)$ are its coordinates: $\{|G_0(j\omega_n)|;\ \varphi_0=\arg G_0(\omega_n)\}$. Substituting them into (15) and (16) the following expressions for calculating robust PID controller parameters are obtained

$$K_{rob}=\frac{\cos\Theta_0}{G_M\left|G_0(j\omega_n)\right|}\,,\ T_{d\,rob}=\frac{tg\Theta_0}{2\omega_n}+\frac{1}{\omega_n}\sqrt{\frac{tg^2\Theta_0}{4}+\frac{1}{\beta}}\,, \tag{26}$$

$$T_{irob}=\beta T_{irob},\ \Theta_0=-180^\circ-\varphi_0,\ \beta=4. \tag{27}$$

It can be seen that the gain margin $G_M$ appearing in (26a) is at the same time a robust PID controller tuning parameter required for guaranteeing robust stability.

***Theorem 1 (Sufficient condition of robust stability under a PID controller)***

Consider an uncertain continuous-time stable dynamic system described by unstructured uncertainty. The closed-loop system $T(s)$ under the controller $G_R(s)$ is robustly stable if the nominal closed-loop system ($G_0(s)$ under a PID controller $G_R(s)$) is stable, and

$$G_M>1+\frac{\chi_L R_G(\omega_n)}{\left|G_0(j\omega_n)\right|}\,, \tag{28}$$

where $G_M$ is the required gain margin, $\omega_n$ is the excitation frequency, $\chi_L$ is the safety factor, $R_G(\omega_n)$ is the radius of the dispersion circle of the Nyquist plots of the plant at $\omega_n$, and $G_0(j\omega_n)$ is a point on the Nyquist plot of the nominal plant at $\omega_n$.

***Proof***

The proof can easily be performed according to Fig. 7. If the nominal open-loop $L_0(s)=G_0(s)G_R(s)$ is stable, then according to the Nyquist stability criterion the closed-loop with the uncertain plant will be stable if the distance between $L_0$ and the point *(-1,j0)*, i.e. $|1+L_0(j\omega_n)|$ is greater than the radius $R_L(\omega_n)$ of the circle $M_L$ centered in $L_0$, i.e.

$$R_L(j\omega_n)<\left|1+L_0(j\omega_n)\right|, \tag{29}$$

where $\omega_n$ is the sine-wave generator frequency. The distance $|1+L_0(j\omega_n)|$ is a complementary distance $|0,L_0|=|L_0|$ to the unit value. Thus

$$\left|L_0(j\omega_n)\right|+\left|1+L_0(j\omega_n)\right|=1,\ \left|1+L_0(j\omega_n)\right|=1-\left|L_0(j\omega_n)\right|. \quad (30)$$

From the principles of the proposed PID controller tuning method results that the robust controller shifts the nominal point of the plant frequency response $G_0(\omega_n)$ to a point $L_0$ on the negative real half-axis of the complex plane. Thus, the magnitude $|L_0(j\omega_n)|=|G_0(j\omega_n)||G_R(j\omega_n)|=1/G_M$ yielding the ratio $|G_R(j\omega_n)|=1/[G_M|G_0(j\omega_n)|]$ between the radii $R_G$ and $R_L=|G_R|R_G$ of the circles $M_G$ and $M_L$, respectively. The radius $R_L$ of the dispersion circle $M_L$ is calculated as

$$R_L = R_G\frac{1}{G_M\left|G_0(j\omega_n)\right|}. \quad (31)$$

Substituting (30b) and (31) into the general robust stability condition (29) and considering the *safety factor* $\chi_L$, the following inequality holds

$$\frac{G_M-1}{G_M} > \frac{\chi_L R_G}{G_M\left|G_0(j\omega_n)\right|}, \quad (32)$$

which after some manipulations is identical to the proven condition (28). Let $\chi_L=1.2$. According to the robust stability condition the chosen value $G_M$ is substituted into (26a) and afterwards the robust PID controller parameters are obtained from (26) and (27). A setup of the proposed method is extensively illustrated on the following example.

## 6. VERIFICATION OF THE PROPOSED ROBUST PID CONTOLLER DESIGN METHOD

Consider the following uncertain plant $G_3(s)$ with an unstable zero

$$G_3(s)=\frac{K_3(-\alpha_3 s+1)}{(T_3 s+1)^3}, \quad (33)$$

$$G_{30}(s)=\frac{K_{30}(-\alpha_{30}s+1)}{(T_{30}s+1)^3}=\frac{0{,}8(-7.5s+1)}{(27.5s+1)^3} \quad (34)$$

with parameters $K_3$, $T_3$ and $\alpha_3$ varying within $\pm 15\%$ around the *nominal values;* $G_{30}(s)$ is the nominal model. For the above plant, a robust PID controller is to be designed to guarantee a *maximum overshoot* $\eta_{max0}=5\%$ and a *maximum relative settling time* $\tau_{s0}=12$ for the *nominal model* (33), and *stability* of the *family of plants* $G_3(s)$ (32) (robust stability).

**1.** The measured *critical frequency* of the *nominal model* is $\omega_c=0.0488\ s^{-1}$. From requirements *on the nominal closed-loop performance* results $t_s=\tau_{s0}/\omega_c=12/0.0488= =245.9\ s$.

**2.** To achieve the *expected nominal performance* $(\eta_{max0},\tau_{s0})= =(5\%,12)$, the gain margin and excitation frequency are chosen $(G_M,\omega_n)=(18dB,0.65\omega_c)$ using the *„pink“ B-parabolas* in Fig. 6 as according to (34) $\alpha_{30}/T_{30}=7.5/27.5=0.27<0.3$. Uncertainties of the plant are included in three parameters: $K_3$, $T_3$ and $\alpha_3$, the number of identification experiments is therefore $N=2^3=8$.

**3.** Using the sine-wave method, eight points of Nyquist plots of the uncertain plant were identified at $\omega_n=0{,}65\omega_c=0{,}65.0{,}04880= =0{,}03172\ s^{-1}$: $G_{31}(j\omega_n)...G_{38}(j\omega_n)$ (depicted by *blue „x“* in Fig.11). The nominal point $G_{30}(j\omega_n)$, which position was calculated from the coordinates of identified points $G_{3i}(j\omega_n)$, $i=1...8$, is located on the Nyquist plot of the nominal model $G_{30}(j\omega_n)$ *(blue curve)* thus proving correctness of the identification. Radius of the dispersion circle $M_G$ drawn from the nominal point $G_{30}(j\omega_n)$ is $R_G=0.164$.

**4.** As $G_M=18dB$ and the r.h.s. of (27) $G_{0\_RS}=3.52\ dB$, the robust stability condition (26) $G_M>G_{0\_RS}$ is satisfied. The designed robust PID controller moves the nominal point $G_{30}(j\omega_n)$ on the negative half-axis into $L_{30}(j\omega_n)= =G_{30}(j\omega_n)G_{R\_rob}(j\omega_n)=0.12e^{-j180°}$, through which passes the Nyquist plot of the nominal open-loop $L_{30}(j\omega_n)$ (Fig. 10 in green), where the gain margin $G_M=18\ dB$ is guaranteed. The nominal closed-loop step response *(Fig. 11a, green curve)* proves achieving the required nominal performance $\eta_{max0\_obtained}=4.55\%$, $\tau_{s0\_obtained}=\omega_c t_{s0\_obtained}=0{,}0488.243=11{,}86$.

**5.** The dispersion circle $M_L$ *(in green)* radius $R_L=0.0573$ encompasses all points $L_{3i}(j\omega_n)=G_{3i}(j\omega_n)G_{R\_rob}(j\omega_n)$ for $i=1…8$. The PID controller has moved the worst point $G_{3N}(j\omega_n)$ of the plant (*blue symbol „+“* in Fig. 8) into $L_{3N}(j\omega_n)=0.16e^{-j197°}$, according to it the estimated worst gain margin is $G_{MN}=14.9\ dB$.

**6.** The smallest gain margin with the worst point $G_{3N}(j\omega_n)$ of the plant (*blue symbol „+“* in Fig. 8) is specified by the intersection of the red Nyquist plot with the negative real axis, where the open-loop gain margin is $G^{+}_{MN}=13.1\ dB;$ here $\eta_{maxN}=25\%$ and the relative settling time $\tau_{sN}=16$ are expected (according to *„pink“ curves* in Fig. 6 at $\omega_n=0.65\omega_c$). Achieved performance $\eta_{maxN\_obtained}=13.5\%$, $t_{sN\_obtained}=301\ s$ (*red step response* in Fig. 9b) prove this fact.

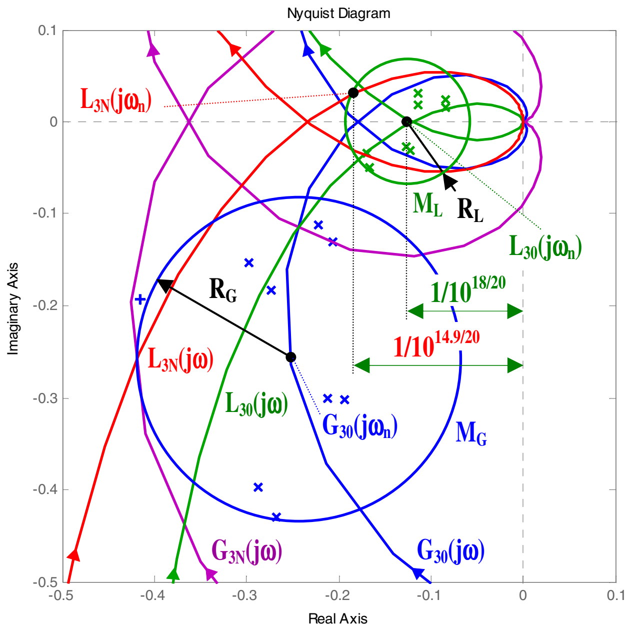


Fig. 8. Nyquist plots of $G_{30}(j\omega)$, $G_{3N}(j\omega)$, $L_{30}(j\omega)$, $L_{3N}(j\omega)$: zoomed, for required performance $\eta_{max0}=5\%$ and $\tau_{s0}=12$

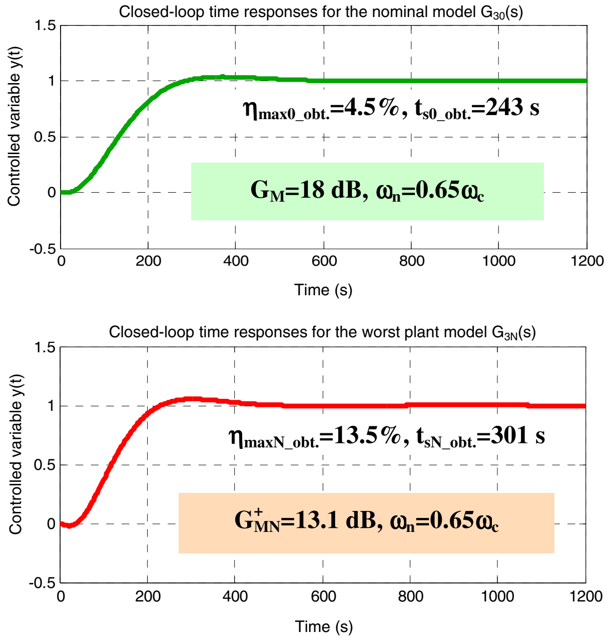


Fig. 9. Closed-loop step responses with the uncertain plant $G_3(s)$ and required values $\eta_{max0}=5\%$ and $\tau_{s0}=12$

***Modified version of sine-wave method***

To avoid using the sine-wave generator, identification of the plant frequency transfer function point can be carried out as follows. Include some filter with the transfer function $F(s)$ in closed-loop which does not violate the closed-loop. Using the classical Ziegler-Nichols experiment (Ziegler and Nichols, 1942) we obtain both the controller ultimate proportional gain $K_c$ and ultimate frequency $\omega_c$. With ultimate parameters the following equation holds

$$K_c F(j\omega_c) G(j\omega_c) = -1 + j0 \tag{35}$$

or, the complex-plane coordinates of $G(j\omega_c)$ are

$$G(j\omega_c) = -\frac{1}{K_c F(j\omega_c)} = m + jn \tag{36}$$

Considering the crossover frequency to be set as $\omega_c=\omega_p$, a PID controller guaranteeing the prescribed gain margin using (15) and (16) can be designed.

***If the plant is unstable,*** the additional transfer function $F(s)$ with P controller can stabilize the closed-loop system; hence *the proposed modification of the sine-wave method or the modified Ziegler-Nichols method can be used for PID controller tuning even for unstable systems.*

## 7. CONCLUSIONS

The proposed robust PID controller design method is applicable for closed-loop output variable response shaping, using various combinations of excitation signal values $\omega_n$ and required gain margins $G_M$. Important contribution of the paper is construction of empirical plots converting time-domain requirements specified by a process technologist (nominal maximum overshoot and settling time) into frequency-domain performance specification in terms of nominal gain margin and phase crossover frequency.

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