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Equivalence of Multi-Formulated Optimal Slip Control for Vehicular Anti-Lock Braking System

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Abstract: This paper presents the equivalence of time-optimal and optimal distance slip-control approach for purposes of performing antilock braking. A dynamic braking model is developed encorporating a slip state to facilitate slip tracking. Optimal distance braking is performed with control constraints on the dynamic slip based braking model. A similar treatment is applied for time optimal braking on the braking model. The key contribution is the demonstration of the equivalence of optimal distance and time optimal braking of a vehicle. A generalised none zero-terminal condition is utilised in the optimal formulations. Simulation results demonstrate the validity of the approach along with the development of a key optimality condition for the equivalence approach.

Keywords: antilock braking systems, optimal control, performance measure

1. INTRODUCTION

In order to brake a wheeled vehicle, a braking torque is applied to the wheel leading to a reduction in the wheel's angular velocity. The reduced linearised wheel's angular velocity in turn leads to the wheel skidding relative to the road/driving surface. A greater braking torque leads to a greater wheel angular velocity reduction and so a greater skidding effect. This skidding is formally called slip and while braking, slip can vary from a minimum of zero to a maximum of unity.

The case for zero slip corresponds to when there is no braking torque is applied so the linearised wheel's angular velocity is the same as the car speed relative to the road surface. The unity or 100% slip case on the opposite extreme corresponds to the case when the wheel speed is zero but the car's speed relative to the road surface is not zero as the car has not come to rest. The 100% slip case is also called *wheel locking* or full-skid. Antilock braking systems (ABS) are utilised in wheeled vehicles with the specific aim of avoiding the locking of the wheel. The effective braking force while braking rises to a maximum as the slip increases from zero after which the effective braking force decreases as the slip value approaches 100% or the locking value. Pacejka models and various works have demonstrated this explained observation such as (Ting et al., 2005), (Tsiotras et al., 2000), (Pedro et al, 2009), (Nyandoro et al, 2011a). Hence slip control is a feature that allows ABS type braking to not only avoid locking but also to have improved performance by attempting brake at the best slip value i.e. the slip value that provides the best braking force for a wheeled vehicle.

The goal of an optimal Anti-lock Braking System (ABS) is to optimally reduce the speed of a vehicle from an initial traveling speed towards rest in either the minimum possible distance, (Nyandoro et al, 2011a; Choi, 2008) or minimum possible time. Optimal control theory has been successfully applied to ABS to prove the need for slip control of ABS for various types of vehicle braking systems such as pneumatic brakes, electro-pneumatic brakes and electrical brakes in works such as (Ting et al., 2005), (Tsiotras et al., 2000). Various disturbances and ABS uncertainties such as road conditions, initial vehicle speeds, braking actuator dynamics in (Lin, 2003; Choi et al, 2006), wheel bearing friction, suspension effects in (Ting et al., 2005) and wind resistance have also been treated in applying optimal theory and other controllers to ABS in (Baslamisli et al, 2007; Austin et al, 2000; Petersen et al, 2003; Lin et al, 2003), (Tsiotras et al., 2000) with significant degree of success for ABS which is a typical safety critical system as highlighted in (Ting et al., 2005),(Pedro et al, 2009).

A major challenge in designing the best controller for ABS is the analytical evaluation of best performance. As yet most works such as (Tsiotras et al., 2000) have utilised minimum braking distance as the measure of best performance. Yet other measures do exist such as minimum braking time of ABS. Since ABS only avoids wheel locking, slip control of ABS allows for a better braking model allowing for optimal slip control. Yet again most works such as (Ting et al., 2005; Nyandoro et al, 2011a) have utilised braking models with slip being an implicit parameter and not explicit slip control. With different measures of best performance comes the need to then demonstrate that different evaluations of best performance lead to the same unique best performance. The goal of this paper is thus to generate a slip control model and demonstrate that unique optimal performance is achieved even via different performance evaluations. An optimality condition required to ensure the uniqueness of optimal slip control is formulated.

The rest of this paper is structured as follows, first a model for ABS is obtained followed by slip control motivation, and the major contribution of formulating a stable linearisation approach for the ABS model. Simulation results for a linearised controller are provided to demonstrate the effectiveness of the linearising technique.

2. MODELING

2.1 Physical Model

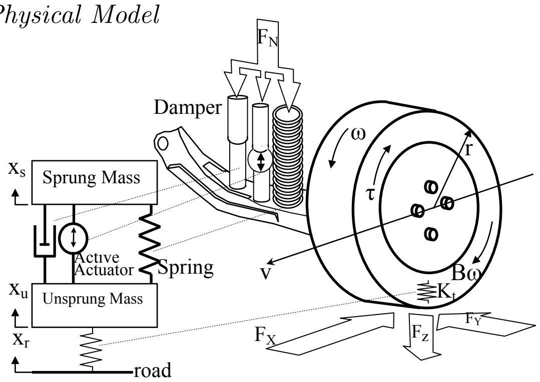

Fig. 1. Quarter car braking model

The general quarter-car model used in (Ting et al., 2005; Pedro et al, 2009; Nyandoro et al, 2011a,b) is also utilised in this paper. Fig. 1 illustrates a quarter-car wheel and braking system. At any point in time, t, between the instant of braking commencement $t = t_0$ and final braking time $t = t_f$ the car has a forward longitudinal velocity, $v(t)$, and the wheel has an angular velocity, $\omega(t)$. A braking force is applied to generate a braking torque $\tau(t)$ acting on the wheel. Typically it is assumed that the weight is approximately equally distributed on the four wheels of the vehicle and that each of the four wheels of the car contribute approximately equally to the car's total braking force. Further it is assumed in this quarter-car model that cornering forces, road roughness and related forces are negligible.

2.2 Mathematical Model

The car while traveling has brakes applied at an initial time $t = t_0 = 0$ and comes to a stop at a final time $t = t_f$. As the brakes are applied the car's longitudinal velocity $v(t)$ is initially $v(t_0)$ and at $t = t_f$ the car's velocity will have come to zero i.e. $v(t_f) = 0$. Newton's laws applied to the quarter-car wheel and braking system shown in Fig.1 gives the governing equations of motion. The vehicle's translational dynamics are:

$$M\dot{v} = -\mu(\gamma)F_z - C_x v^2 \tag{1}$$

where M is the quarter-car's total mass, μ is the longitudinal friction coefficient, F_z is the normal force acting on the vehicle wheel, and C_x is the vehicle aerodynamic friction coefficient. For slip control as explained later μ is a function of the slip ratio γ as detailed later in Fig.2. The vehicle wheel's rotational dynamics are:

$$I\dot{\omega} = \mu(\gamma)F_z r - B\omega - \tau_b \tag{2}$$

where I is the moment of inertia of the wheel, r is the wheel radius, B is the wheel bearing friction coefficient, and τ_b is the input braking torque. An electro-mechanical set of brakes is used to apply a braking torque, τ_b, on the disk/drum brakes. The weight component of the quarter-car, F_z, is given by:

$$F_z = Mg \tag{3}$$

where g is the acceleration due to gravity. By definition the braking slip ratio γ is:

$$\gamma = \frac{v - r\omega}{v} \tag{4}$$

The typical relationship between μ and γ is given in Fig.2, and is modeled by the approximate equation, see (Ting et al., 2005):

$$\mu(\gamma) = 2\mu_0 \frac{\gamma_0 \gamma}{\gamma_0^2 + \gamma^2} \tag{5}$$

The peak μ value μ_0 occurs at a γ value γ_0 and from Fig.2 for a dry asphalt road surface $\mu_0 = 0.8$ and $\gamma_0 = 0.18$ respectively.

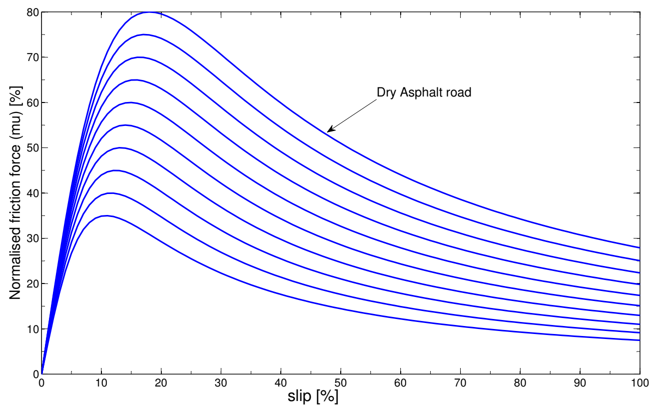

Fig. 2. Typical tire longitudinal friction μ-γ curves

The effects of suspension dynamics via the manipulation of F_z is dealt in our previous work (Nyandoro et al, 2011a).

The road roughness x_r and cornering force F_Y in Fig.1 are assumed to be negligible.
Different road surfaces are modeled by different γ_0 and corresponding μ_0 values as they are unique for each road surface. Since the peak friction coefficient μ_0 is obtained when γ has the value γ_0, i.e. $\mu_0 = \mu(\gamma_0)$ hence the goal of slip control is to generate a braking torque τ_b to maintain the braking slip value always close or equal to its optimal value γ_0.

2.3 Performance Criteria

Control theory generally attempts to generate a control input to meet certain desired plant parameter trajectories at times constrained by time and/or input constraints. However optimal control is unique among control techniques in identifying a performance measure and specifically in utilising the performance measure to generate a control input that maximises/minimises this performance criteria. Constraints such as input, output, among other system parameter constraints can be incorporated into the optimal control formulation. So optimal control theory has these advantages while uniquely optimising a particular performance measure thus providing best performance.

However certain practical performance attributes provide potentially conflicting optimal control formulations. In particular for ABS optimal/best braking could be validly measured via minimum braking distance. The goal of optimal control would be to formulate the evaluation of an input braking torque that would minimise the braking distance of the system in Fig.1. However an equal claim to best braking would be to minimise the braking time thus leading to an optimal time control problem. This paper thus seeks to find the equivalence of both these criteria namely optimal distance and optimal time ABS braking. This equivalence is achieved by attempting to formulate optimal torque for both the performance criteria and if the two respective optimal torques are the same then the ABS performances from the equivalent torques would be the same.

While (Tsiotras et al., 2000) demonstrates optimal distance ABS braking the formulation utilised claims optimal slip evaluation. However a dynamic ABS model is utilised with slip the key parameter being an implicit parameter in the model. Further a number of assumptions are made in the optimal distance formulation of (Tsiotras et al., 2000). A key assumption is the requirement for zero terminal conditions which practically is not necessary to be met. In this paper an explicit dynamic slip model is formulated and then utilised in formulating both optimal distance and also the time optimal control problems for ABS. Further the zero-terminal condition restriction is removed thus allowing for non-zero terminal condition of states and a more practically applicable result as optimal ABS braking can be applied without enforcing braking to rest. The equivalence of both optimal controllers is to be demonstrated and the condition of this equivalence.

3. OPTIMAL CONTROL FORMULATION

3.1 Optimal Control Criteria

Given a plant with dynamics

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, u, t\right) \tag{6}$$

and a scalar performance index $J(x)$:

$$J\left(\mathbf{x}\right) = \int_0^{t_f} L\left(\mathbf{x}, u, t\right) \tag{7}$$

and Hamiltonian H:

$$H\left(\mathbf{x}, u, \lambda, t\right) = L\left(\mathbf{x}, u, t\right) + \lambda^T \mathbf{f}\left(\mathbf{x}, u, t\right) \tag{8}$$

let

$$J_1 = \int_0^{t_f} \left(H - \lambda^T \dot{\mathbf{x}}\right) dt = \int_0^{t_f} \left[L + \lambda^T \left(\mathbf{f} - \dot{\mathbf{x}}\right)\right] dt \tag{9}$$

from Leibniz's rule

$$\begin{aligned} dJ_1 = &\left(H - \lambda^T \dot{\mathbf{x}}\right) dt|_{t_f} - \left(H - \lambda^T \dot{\mathbf{x}}\right) dt|_0 \\ &+ \int_0^{t_f} \left[\frac{\partial H}{\partial \mathbf{x}}^T \delta\mathbf{x} + \frac{\partial H}{\partial u}^T \delta u - \lambda^T \delta\dot{\mathbf{x}} + \left(\frac{\partial H}{\partial \lambda} - \dot{\mathbf{x}}\right)^T \delta\lambda\right] dt \end{aligned} \tag{10}$$

$$d\mathbf{x}(t_f) = \delta\mathbf{x}(t_f) + \dot{\mathbf{x}}(t_f) dt_f \tag{11}$$

Hence

$$\begin{aligned} dJ_1 = &-\lambda^T d\mathbf{x}|_{t_f} + H dt|_{t_f} + \lambda^T d\mathbf{x}|_0 - H dt|_0 \\ &+ \int_0^{t_f} \left[\frac{\partial H}{\partial \mathbf{x}}^T \delta\mathbf{x} + \frac{\partial H}{\partial u}^T \delta u - \lambda^T \delta\dot{\mathbf{x}} + \left(\frac{\partial H}{\partial \lambda} - \dot{\mathbf{x}}\right)^T \delta\lambda\right] dt \end{aligned} \tag{12}$$

3.2 Optimal Control Evaluation

Hence for optimal control we have plant dynamics (6), performance index (7) and Hamiltonian (8) thus (12) gives co-state dynamics

$$-\dot{\lambda} = \frac{\partial H}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}}^T \lambda + \frac{\partial L}{\partial \mathbf{x}} \tag{13}$$

with the initial boundary condition $\mathbf{x}(0)$ specified and

$$\lambda^T|_{t_f} d\mathbf{x}(t_f) + H|_{t_f} dt_f = 0 \tag{14}$$

The stationarity condition is

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial f}{\partial u}^T \lambda + \frac{\partial L}{\partial u} \tag{15}$$

from which the optimal input u_{opt} is obtained. However numerous cases exist where (15) can not provide u_{opt} e.g. when a singularity exists and hence evaluation of u_{opt} is via

$$u_{opt} = argmin \quad H\left(\mathbf{x}, u, \lambda, t\right) \tag{16}$$

From the total derivative for H

$$\dot{H} = \frac{\partial H}{\partial \mathbf{x}}^T \dot{\mathbf{x}} + \frac{\partial H}{\partial u}^T \dot{u} + \dot{\lambda}^T \mathbf{f} \tag{17}$$

so $\dot{H} = 0$ on the optimal trajectory the Hamiltonian is constant provided the Hamiltonian is not an explicit function of time. The foregoing formulations follow from general optimal control theory from (Lewis, 2012) or other works such as (Bryson, 1975).

4. OPTIMAL SLIP CONTROL FORMULATION

4.1 Slip Control Model Formulation

From the quarter car model physical equations (1)-(2) the following simplified state equations are used for the ABS quarter car model

$$\dot{v} = -\frac{\mu F_z}{M} = f_v \tag{18a}$$

$$\dot{\omega} = \frac{\mu F_z R}{I_w} - \frac{\tau}{I_w} \tag{18b}$$

v parameters	ω parameters	γ parameters
$v(0) = 120km/hr$	$\omega(0) = R * 120000/3600$	$\gamma(0) = 0$
$v(t_f) = 0$	$\omega(t_f) = 0$	$\gamma(t_f) = \gamma_0$

Table 1. Boundary parameters

This set of state model equations with state vector $[v \;\; \omega]^T$ does not include the air drag force and the wheel bearing resistance torque for simplicity of analysis. This can be partly justified by the practical order of magnitude of the air drag forces compared to the frictional braking forces and also relative magnitude of bearing friction torque relative to the braking torque magnitude.

Transformation of the state model (18) to a slip model gives the following ABS slip based model (25) where v state is maintained while the slip state replaces the ω state to give a new state vector $\mathbf{x} = [v \;\; \gamma]^T$.

Since by definition

$$\gamma = \frac{v - R\omega}{v} \tag{19}$$

and one version of the *Magic Formula* relates the coefficient of road friction μ with the slip γ (Austin et al, 2000):

$$\mu = \frac{2\mu_0\gamma_0\gamma}{\gamma_0^2 + \gamma^2} \tag{20}$$

the following useful relations are obtained.

$$\omega = \frac{v}{R}(1-\gamma) \tag{21}$$

$$\frac{\partial\gamma}{\partial\omega} = -\frac{R}{v} \tag{22}$$

$$\frac{\partial\gamma}{\partial v} = \frac{R\omega}{v^2} = \frac{(1-\gamma)}{v} \tag{23}$$

$$\frac{\partial\mu}{\partial\gamma} = \frac{2\mu_0\gamma_0(\gamma_0^2 - \gamma^2)}{\gamma_0^2 + \gamma^2} \tag{24}$$

The state transformation $\mathbf{x}_{old} = [v \;\; \omega]^T$ to $\mathbf{x} = [v \;\; \gamma]^T$ is thus performed to give the transformed ABS state equation $\dot{\mathbf{x}} = [\dot{v} \;\; \dot{\gamma}]^T = \mathbf{f}(\mathbf{x}, u, t) = [f_v \;\; f_\gamma]^T$:

$$\dot{v} = -\frac{\mu F_z}{M} = f_v \tag{25a}$$

$$\dot{\gamma} = \frac{\partial\gamma}{\partial v}\dot{v} + \frac{\partial\gamma}{\partial\omega}\dot{\omega} = -\frac{R\omega}{v^2}\left(\frac{\mu F_z}{M} - \frac{R}{v}\frac{\mu F_Z R}{I_w} - \frac{\tau}{I_w}\right)$$
$$= \frac{-R\omega\mu F_z}{mv^2} - \frac{\mu F_z R^2}{vI_w} + \frac{r\tau}{vI_w} = f_\gamma \tag{25b}$$

4.2 Optimal Distance Torque Formulation

With the ABS model state equation (25) the objective becomes that of minimising the braking distance J_{opt_v}:

$$J_{opt\_v} = \int_0^{t_f} v(\tau)d\tau \tag{26}$$

subject to the initial and final conditions in Table 1

where $\gamma_0 = 0.18$ optimises μ on a μ-slip curve. So the final-state constraint is

$$\gamma - \gamma_0 = 0 \tag{27a}$$

So the optimal controller is derived by definition from the Hamiltonian, state equation, co-state/adjoint equation, stationarity condition, and finally the boundary condition as formulated above.

The Hamiltonian for the state equation (25) and performance index (26) is

$$H(v, \gamma, u, t) = v + \lambda_v f_v + \lambda_\gamma f_\lambda \tag{28}$$

where the co-state vector is $\lambda = [\lambda_v \;\; \lambda_\gamma]^T$ with the co-state state equation defined by

$$\dot{\lambda} = [\dot{\lambda_v} \;\; \dot{\lambda_\gamma}]^T = -\frac{\partial H}{\partial \mathbf{x}} = -\frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}\lambda - \frac{\partial v}{\partial \mathbf{x}}, \quad t \le t_f \tag{29}$$

$$= \begin{bmatrix}\dot{\lambda_v} \\ \dot{\lambda_\gamma}\end{bmatrix} = \begin{bmatrix} -1 - \lambda_1\frac{\partial f_v}{\partial v} - \lambda_2\frac{\partial f_\gamma}{\partial v} \\ -\lambda_1\frac{\partial f_v}{\partial \lambda} - \lambda_2\frac{\partial f_\gamma}{\partial \lambda}\end{bmatrix}, \quad 0 \le t \le t_f \tag{30}$$

and

$$\dot{\lambda}_v = -1 + \frac{\partial\mu}{\partial\gamma}\frac{(1-\gamma)}{v}\left(\frac{\lambda_v F_z}{M} + \frac{\lambda_\gamma F_z(1-\gamma)}{Mv} + \frac{\lambda_\gamma F_z R^2}{I_w v}\right)$$
$$-2\frac{\lambda_\gamma F_z \mu(1-\gamma)}{Mv^2} - \frac{\lambda_\gamma \mu F_z R^2}{I_w v^2} + \frac{\lambda_\gamma R}{I_w v^2}\tau \tag{31}$$

$$\dot{\lambda}_\gamma = \frac{\partial\mu}{\partial\gamma}\left(\frac{\lambda_v F_z}{M} + \frac{\lambda_\gamma F_z(1-\gamma)}{Mv} + \frac{\lambda_\gamma F_z R^2}{I_w v}\right)$$
$$-2\frac{\lambda_\gamma F_z \mu}{Mv} - \frac{\lambda_\gamma \mu F_z R^2}{I_w v(1-\gamma)} + \frac{\lambda_\gamma R}{I_w v(1-\gamma)}\tau \tag{32}$$

The final time is not specified but initial system states v and ω are non zero but both have zero terminal values. The transformed slip state γ on the other hand is specified to have a initial zero value and terminal non-zero value $\gamma(t_f) = \gamma_0$. The boundary condition is

$$\left(\psi_{\mathbf{x}}^T\lambda_e - \lambda\right)^T|_{t_f} d\mathbf{x}(t_f) + \left(\psi_t^T\lambda_e + H\right)|_{t_f} dt_f = 0 \tag{33}$$

where λ_e is a constant co-state. Since at $t = t_f$ the variables v, w and γ have specified fixed values $d\mathbf{x}(t_f) \equiv 0$. However dt_f is not necessarily zero since the final time is not fixed. Yet

$$\psi_t^T|_{t_f} = \gamma(t = t_f) - \gamma_0|_{t_f} = 0$$

So with an un-fixed final time, since $\psi_t^T|_{t_f} = 0$, so necessarily $H|_{t_f} = 0$. $H(t)$ is not an explicit function of time hence $H(t) = 0$ for all braking time i.e $0 \le t \le t_f$ along the optimal trajectory.

The optimal stationarity condition gives a singularity

$$\frac{\partial H}{\partial \tau} = 0 \tag{34}$$

Typically braking torque is physically limited by two extreme values such that

$$0 \le \tau \le \tau_{max} \tag{35}$$

So

$$\tau_{opt} = argminH\left(\mathbf{x}, \lambda, \tau\right) \quad (36)$$

From (15) and (16) obtain a switching function H_γ:

$$H_\gamma = -\frac{\lambda_\gamma}{I_w} \quad (37)$$

with optimal braking torque as

$$\tau_{optimal} = \begin{cases} \tau_{max} & for\ H_\gamma > 0 \\ 0 & for\ H_\gamma < 0 \\ \tau_{sing} & for\ H_\gamma \equiv 0 \end{cases} \quad (38)$$

The optimal control operates with either minimum 0 or maximum torque. However during braking for any $finite-time$ a subset of the braking time $0 \le t \le t_f$ over which $H_\gamma(t) = 0$ yields the singular control τ_{sing} in the time interval of singular torque $H_\gamma(t) = 0$. So

$$\dot{H}_\gamma = 0 = \dot{\lambda}_\gamma = \frac{\partial\mu}{\partial\gamma}\left(\frac{\lambda_v F_z}{M} + \frac{\lambda_\gamma F_z(1-\gamma)}{Mv} + \frac{\lambda_\gamma F_z R^2}{I_w v}\right) - 2\frac{\lambda_\gamma F_z \mu}{Mv} - \frac{\lambda_\gamma \mu F_z R^2}{I_w v(1-\gamma)} + \frac{\lambda_\gamma R}{I_w v(1-\gamma)}\tau \quad (39)$$

$$= \lambda_v \frac{F_z}{M}\frac{\partial\mu}{\partial\gamma} \quad (40)$$

where (40) is obtained from (39) and in turn from (37) along the singular arc $\lambda_\gamma = 0$. Since $\lambda_\gamma = 0$ along the singular arc, $\lambda_v \ne 0$ in (40) for non-zero co-states on optimal trajectory. Hence $\frac{\partial\mu}{\partial\gamma} = 0$ in (40). Taking again a derivative of $\dot{H}_\gamma$.

$$\ddot{H}_\gamma = 0 = \frac{F_z}{M}\frac{\partial\mu}{\partial\gamma}\dot{\lambda}_v + \frac{F_z}{M}\lambda_v \frac{d}{dt}\frac{\partial\mu}{\partial\gamma} \quad (41)$$

$$= \frac{F_z}{M}\lambda_v \frac{d}{dt}\left(\frac{\partial\mu}{\partial\gamma}\right) \quad (42)$$

Again (42) is obtained from (41) because $\frac{\partial\mu}{\partial\gamma} = 0$ and since $\lambda_v \ne 0$ from (42)

$$\frac{d}{dt}\frac{\partial\mu}{\partial\gamma} = 0 = \frac{\partial^2\mu}{\partial\gamma^2}\dot{\gamma} = \frac{\partial^2\mu}{\partial\gamma^2}\left(\frac{-R\omega\mu F_z}{mv^2} - \frac{\mu F_z R^2}{vI_w} + \frac{r\tau_{sing}}{vI_w}\right) \quad (43)$$

giving along the singular arc

$$\frac{R\omega\mu|_{sing}F_z}{mv^2} + \frac{\mu|_{sing}F_z R^2}{vI_w} - \frac{R\tau_{sing}}{vI_w} = 0 \quad (44)$$

and

$$\tau_{sing} = \frac{I_w\omega\mu|_{sing}F_z}{mv} + \mu|_{sing}F_z R \quad (45)$$

$$= \frac{I_w(1-\gamma_0)\mu_0 F_z}{mR} + \mu_0 F_z R \quad (46)$$

The derivation has been based on the deductions that along the singular arc, $\lambda_v \ne 0$, and $\frac{\partial\mu}{\partial\gamma} = 0$.

Parameters	Parameters	Parameters
$I_w = 1.6\ kgm^2$	$M = 400\ kg$	$r = 0.3\ m$
$\gamma_0 = 0.20$	$\mu_0 = 0.9$	t_f = unspecified
$v(0) = 120\ km/hr$	$\gamma(0) = 10\ rad\ s^{1}$	$v(t_f) = 0\ km/hr$
$\gamma(t_f) = 10\ rad\ s^{1}$	$\tau_{min} = 0\ Nm$	$\tau_{max} = 2950\ Nm$

Table 2. Quarter-car model simulation parameters

4.3 Time Optimal Torque Formulation

The time optimal control perfromance index is

$$J_{opt\_t} = \int_0^{t_f} d\tau = t_f \quad (47)$$

giving a Hamiltonian

$$H(v,\gamma,u,t) = 1 + \lambda_v f_v + \lambda_\gamma f_\lambda \quad (48)$$

With only the difference in formulations being the $L_{opt\_v} = v$ while $L_{opt\_t} = 1$ the formulation for the optimal control gives the same switching function H_γ and singular τ_{sing} and same optimal control. The only other difference besides the Hamiltonian differenes being the co-state equation $\dot{\lambda}_v$ thus affecting how fast the switching function activates the singular control. Hence the singular control is shown to be the same. Additionally if the braking can be such that singular torque is utilised for the greater part of the braking time then the braking for both controllers will be practically equivalent.

4.4 Remarks and Results

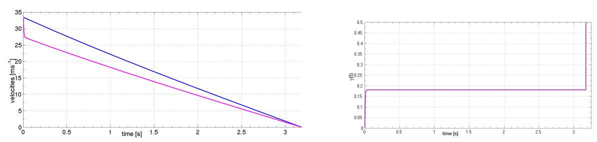

Fig. 3. Optimal v and w (left) Optimal slip (right)

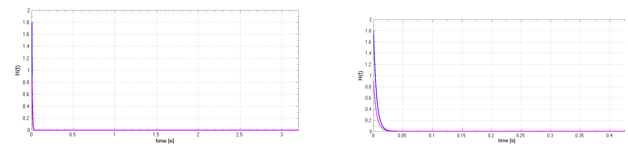

Fig. 4. Optimal H (left) Optimal H time zoom (right)

Lastly the optimal torque control in (45) needs to satisfy the *Legendre − Clebsch* condition (Tsiotras et al., 2000)

$$\frac{\partial}{\partial\tau}\left(\frac{d^{2q}H_\gamma}{dt^{2q}}\right) \ge 0 \quad (49)$$

where q is the order of the singular arc in this case $q = 1$. For the *Legendre − Clebsch* condition on $\ddot{H}_\gamma$ from (41) gives

$$\frac{R}{vI_w} \geq 0 \tag{50}$$

which is true for all braking time, $0 \leq t \leq t_f$, as long as $v \neq 0$, since $v(t)$ is always positive when braking is initiated at $t = 0$ until the vehicle comes to rest at $t = t_f$.

Physical constraints are assumed for τ_{sing} and in particular the assumption is that $0 = \tau_{min} \leq \tau_{sing} \leq \tau_{max}$. The upper limit needs to be observed and typically for most vehicles the braking system can generate enough braking torque to lock the wheels thus guaranteeing that indeed $\tau_{sing} \leq \tau_{max}$.

For the Quarter Car Simulation Model with simulation parameters in Table 2 τ_{sing}=977.41Nm which is less than τ_{max}. The optimal torque is one arc constituted of three sub arcs firstly an initial very small time interval τ_{max} that drives the slip γ towards γ_0, secondly the constant singular torque $\tau sing$ which lasts for practically the total braking time period albeit except for the initial and final impulse time intervals, third and lastly the torque may take another infinitesimally small time interval at the end of the braking process with torque values either τ_{min} or τ_{max}.

Hence for practical purposes the optimal torque can be replaced by two impulses of magnitude τ_{max} at the start and end of the braking process and also the fixed singular torque between the two impulses. H is none zero for a very small time interval at the start of the braking process after which the Hamiltonian satisfies the switching condition for singular control for both time optimal and optimal distance braking. This small time interval is typically less than 0.05s for both optimal controllersHence for practical purposes the equivalence is demonstrated albeit not satisfied for the initial bang time before the switching condition is satisfied.

4.5 Conclusion

The paper presents a formulation for optimal torque control via minimisng the braking distance. Optimal torque for an optimal time control problem is similarly developed. The optimal control in both cases is found to be predominantly the singular torque which appears after an initial time before the switching function condition is satisfied to thus switch to the singular torque. The singular torque for both optimal distance and optimal braking time is the same and this is predominant during the braking time. Simulation results further support formulated equivalence of multiple optimal criteria torque control for slip based ABS braking.

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