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Solving Algebraic Riccati Equations via Proper Orthogonal Decomposition ⋆

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Abstract: In this paper we present a method to solve algebraic Riccati equations by employing a projection method based on Proper Orthogonal Decomposition. The method only requires simulations of linear systems to compute the solution of a Lyapunov equation. The leading singular vectors are then used to construct a projector which is employed to produce a reduced order system. We compare this approach to an extended Krylov subspace method and a standard Gramian based method.

1. INTRODUCTION

Riccati equations play an important role in optimal control and filtering. Moreover, solutions of the algebraic Riccati equation can be important for certain model reduction algorithms, such as LQG balanced truncation, Antoulas, 2005, Sec. 7.5. Moreover, solving Algebraic Riccati Equations (ARE) is a computationally challenging task. Over the past 45 years many methods and techniques were developed to efficiently solve nonlinear matrix equations of Riccati type. The methods include invariant subspace methods Guo and Laub, 1979; Bunse-Gerstner and Mehrmann, 1986, spectral projection methods Byers, 1987, rational and global Krylov subspace methods Saad and Gv, 1990; Heyouni and Jbilou, 2009; Simoncini et al., 2013 as well as the Newton-Kleinman method Kleinman, 1968; Burns et al., 2008; Feitzinger et al., 2009; Benner and Saak, 2010; Singler and Batten, 2012. A good survey for solving large Riccati and Lyapunov equations can be found in Benner and Saak, 2013 and Bini et al., 2012.

Based on simple intuition, high order models often yield high order optimal controllers, which are impractical for real time implementation in a physical device. The capability of the physical device dictates the implementable model order, which in many cases is low. Moreover, when considering the finite time horizon LQG control problem, one faces additional challenges. Linear differential equations involving the solution of an ARE have to be solved in real time. Integrating the full order model is hence unfeasible. Therefore, reduced order models, including a reduced order approximate solution of ARE are needed.

Proper Orthogonal Decomposition (POD), also known as KL-expansion and Principal Component Analysis, is widely used in the engineering and mathematical community for simulation and control, see Berkooz et al., 1993; Kunisch and Volkwein, 2002; Hay et al., 2009; Volkwein, 2013 and the references therein. In this paper, we suggest the use of POD to devise a projection method to compute approximate solutions to ARE efficiently and accurately. In particular, a POD method is employed to approximate the left singular vectors of the observability Gramian and construct a projection. This is then used to obtain matrices of low dimensions for which the algebraic Riccati equation can be solved easily.

For ease of presentation, we assume that a finite dimensional dynamical system $\Sigma = (A, B, C)$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$
$$y(t) = Cx(t), \tag{2}$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$ and $C \in \mathbb{R}^{p\times N}$ and $\dot{x}$ denotes the usual time derivative of x. The standard inner product in $\mathbb{R}^N$ is used.

Consider the algebraic Riccati equation

$$A^TP + PA - PBB^TP + C^TC = 0 \quad \in \mathbb{R}^{N\times N} \tag{3}$$

arising, for instance, in optimal design of a linear quadratic regulator, Kwakernaak and Sivan, 1972. Developing efficient solution strategies for this nonlinear algebraic equation is still an ongoing research effort. In this work, the focus is on projection methods to solve equation (3).

Many algorithms to solve Riccati equations exploit the intrinsic connection to the Lyapunov equation

$$A^TX + XA + C^TC = 0 \quad \in \mathbb{R}^{N\times N}. \tag{4}$$

One should note that (4) is a linear matrix equation obtained from (3) by ignoring the nonlinear term. The connection between the Riccati and Lyapunov equation can also be observed numerically. The recent work of Simoncini et al., 2013 investigates the relationship for the convergence speed of the Lyapunov solutions and Riccati solutions. The authors have investigated the case where both (3) and (4) are projected onto the same extended Krylov subspace and found error bounds relating both residuals.

For stable A, it can be shown (Antoulas, 2005, Prop.4.27) that the exact solution to the Lyapunov equation is given by the *Observability Gramian*

$$X = \int_0^\infty e^{tA^T} C^T C e^{tA} dt. \tag{5}$$

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Proper Orthogonal Decomposition can be used to obtain an approximation of (5). This is the starting point of our new algorithm to solve ARE.

2. PROJECTION METHODS

Projection methods were demonstrated to be efficient methods to compute solutions of Riccati and Lyapunov equations Jaimoukha and Kasenally, 1994; Jbilou, 2006; Jbilou and Riquet, 2006; Simoncini, 2007; Heyouni and Jbilou, 2009; Simoncini et al., 2013. To this end, let us assume the availability of a projection matrix

$$V_r = [v_1,\ v_2, \ldots, v_r] \in \mathbb{R}^{N\times r}, \qquad V_r^T V_r = I_r, \quad (6)$$

where $r \ll N$ is the reduced dimension and I_r is the identity matrix in $\mathbb{R}^{r\times r}$. Once a suitable projector is found, a reduced order model $\Sigma_r = (A_r, B_r, C_r)$ can be constructed via

$$A_r = V_r^T A V_r, \quad (7)$$

$$B_r = V_r^T B, \quad (8)$$

$$C_r = C V_r. \quad (9)$$

One can compute the reduced order solution P_r of the projected Riccati equation

$$A_r^T P_r + P_r A_r - P_r B_r B_r^T P_r + C_r^T C_r = 0 \quad \in \mathbb{R}^{r\times r} \quad (10)$$

with any of the methods described earlier. Having solved the low order ARE, equation (10), define an approximate solution to (3) through

$$P_r^{(N)} := V_r P_r V_r^T \approx P. \quad (11)$$

In this work, we are interested in the convergence behavior $P_r^{(N)} \to P$ as $r \to N$. Numerical results in the literature employ the norm of the residual

$$\mathcal{R}(\tilde{P}) := A^T\tilde{P} + \tilde{P}A - \tilde{P}BB^T\tilde{P} + C^TC \quad \in \mathbb{R}^{N\times N} \quad (12)$$

as a standard measure for numerical accuracy of solutions to ARE. Equation (10) can also be thought of as enforcing the Galerkin orthogonality condition on the residual, $V_r^T\mathcal{R}(P_r^{(N)})V_r = 0$. For in general a full order solution to ARE is not available, the residual (matrix) provides a good measure for the accuracy of solutions. For practical purposes, we compute the Frobenius norm of the residual. The Frobenius norm of a matrix A is defined as $||A||_F = \sqrt{tr(A^TA)}$. We shall investigate the performance of several algorithms and track convergence for the residual norm. As will be demonstrated, some of the projection based methods we compared do not obey monotone decreasing errors. This can be of concern, since the extra 'work' of increasing the projection basis size does not result in a smaller residual. Also, note that the residual in $\mathbb{R}^{N\times N}$ is expensive (or impossible) to evaluate. For the extended Krylov subspace method described below, an efficient way to compute the residual norm is available. Deriving a cheap evaluation of the norm of the residual independent of the method will be part of a future publication.

When dealing with projection methods, one needs to ensure that stability is preserved under projection. In some applications, stability is enforced through post processing: Whenever the reduced order model has unstable poles, a restarted Arnoldi or Lanczos algorithm is used to remove those. However, we have the following

Lemma 1. (Stability Preservation under Projection, Casta-Selga et al., 2012): Let $(\Sigma) = (A, B, C)$ be a continuous time system with $\mathfrak{Re}(\lambda(A + A^T)) \leq 0$, where $\lambda(A)$ denotes any eigenvalue of A. The projected reduced order model $(\Sigma_r) = (A_r, B_r, C_r)$ is stable if $V_r \in \mathbb{R}^{N\times r}$ has full column rank.

Note that as a special case, if A is normal (symmetric, skew-symmetric or orthogonal matrices form a subset of normal matrices), then reduced order models obtained via projection as presented in Lemma 1 are stable.

It seems reasonable that the accuracy and convergence behavior of projection methods depends on the richness of the approximation space as well as the structure and dimension of the projector V_r. This paper compares promising methods to generate a projection matrix V_r and presents a new approach based on POD. This builds upon recent work of Singler, 2011 for computing the solution of infinite dimensional Lyapunov equations. First, an overview of existing work is provided.

Early work on projection methods for solving Lyapunov equations is given in Saad and Gv, 1990. The authors used the standard Krylov subspace $\mathcal{K}_l(A, b) = span\{b, Ab, A^2b, \ldots, A^{l-1}b\}$ to approximate the matrix exponential times a vector by a polynomial in A times a vector as $e^Ab \approx p_{l-1}(A)b$. In the case of Lyapunov equations, exponentials of A^T occur in the kernel of the integral representation of the observability Gramian (5). The idea is that a good approximation of the integrand should be sufficient for convergence of the Gramian. This is not trivial, but Grasedyck, 2004 showed that under rather mild assumptions a quadrature with sinc quadrature points and appropriate weights yields good low rank approximations of X. More early results on projected Lyapunov equations are provided in Jaimoukha and Kasenally, 1994. The authors therein sought low rank approximations of X and compared the GMRES method and the block Arnoldi method. Moreover, error bounds for both methods were derived.

In Druskin and Knizhnerman, 1998, it was shown that the enriched Krylov space

$$\mathbb{K}_{2l}(A, b) = \mathcal{K}_l(A, b) + \mathcal{K}_l(A^{-1}, b) \quad (13)$$

yields better approximations for a product of a matrix function in A and a vector b. This property has then been exploited in the design of an iterative method for the solution of the Lyapunov equation, see Simoncini, 2007. The proposed "Extended Krylov Subspace Method (EKSM)" is then compared to the Cholesky factorized-ADI method. The projection based EKSM was found to be competitive and in some examples cheaper than the Cholesky factorized-ADI method.

In this paper, a new POD based projection method to compute approximate solutions of ARE is presented. We first compute the left singular vectors of an approximate observability Gramian via the algorithm proposed in Willcox and Peraire, 2002; Singler, 2011. Next, a reduced order model is obtained via projection of (A, B, C) with those left singular vectors. The dimension of the necessary singular value decomposition is limited by $\min(N, pS)$, with S being the number of snapshots collected during the simulation of a linear system.

2.1 Extended Krylov Subspace Method (EKSM)

Heyouni and Jbilou, 2009 used the extended Krylov subspace $\mathbb{K}_r = \mathcal{K}_{r/2}(A^T, C^T) + \mathcal{K}_{r/2}(A^{-T}, C^T)$ to obtain V_r, where $A^{-T} = (A^{-1})^T$.

The procedure is summarized in Algorithm 1. Note, that this algorithm is built for multi-input multi-output systems (MIMO) where $m, p > 1$.

Algorithm 1 : Extended Block Arnoldi (EBA) Algorithm (Heyouni and Jbilou (2009))

Input: $A^T \in \mathbb{R}^{N\times N}$, $C^T \in \mathbb{R}^{N\times p}$ and an integer r.
Output: $V_r \in \mathbb{R}^{N\times r}$, an orthogonal projection matrix.

```
Compute the QR-decomposition of [C^T, A^{-T}C^T], i.e.
  [C^T, A^{-T}C^T] = V_1 Λ; Set 𝒱_0 = { }.
for j = 1, 2, ..., r do
  Set V_j^(1): first p colums of V_j and V_j^(2): second p
    columns of V_j.
  𝒱_j = [𝒱_{j-1}, V_j]; V̂_{j+1} = [A^T V_j^(1), A^{-T} V_j^(2)].
  Orthogonalize V̂_{j+1} with respect to 𝒱_j to get V_{j+1},
    i.e.
  for i = 1, 2, ..., j do
    H_{i,j} = V_i^T V̂_{j+1} ;
    V̂_{j+1} = V̂_{j+1} - V_i H_{i,j};
  end for
  Compute the QR-decomposition of V̂_{j+1}, i.e. V̂_{j+1} =
    V_{j+1} H_{j+1,j}.
end for
```

At every iteration step of Algorithm 1, a new column is added to V_r. The inversion is implemented by precomputing a LU-decomposition of A and then solving triangular systems. In Heyouni and Jbilou, 2009, the above algorithm is embedded into a procedure to solve ARE as in Algorithm 2 below, yielding an hierarchical method. The authors constructed a cheap evaluation of $r_m = ||\mathcal{R}(P_r^{(N)})||$, which serves as a stopping criterion for the algorithm. To achieve this, the Arnoldi recurrence turned out to be crucial. With this step, only matrices of size r are required to compute the stopping criterion.

2.2 Gramian Based Projection

An approximate solution of the Lyapunov equation is computed with `lyap` in `Matlab` as

$$X^{(N)} = \texttt{lyap}(A^T, C^T C). \tag{14}$$

`Matlab` uses the SLICOT SB03MD routine for this problem. At first, the Schur decomposition of A^T is computed and then the new system solved by forward substitution. The algorithm is backward stable and requires $O(N^3)$ operations, therefore becoming unfeasible for large N.

We then compute the singular value decomposition of the approximate controllability Gramian

$$V\Sigma W^T = X^{(N)}, \tag{15}$$

where the columns of $V = [v_1 \ v_2 \ \dots v_N]$ span the rangespace of $X^{(N)}$. Truncation of V after r vectors yields the projection matrix as $V_r = [v_1 \ v_2 \ \dots v_r]$.

For medium sized problems, this method performed very well in numerical examples and we thus included it for research purposes. However, for large systems, this direct computation of the solution to the Lyapunov equation became unfeasible. Krylov-based projection techniques such as EKSM described earlier as well as the POD method described below are used in this case.

2.3 POD Projection Method

Proper Orthogonal Decomposition is a powerful model reduction technique based on measurements or simulations of a system. In Willcox and Peraire, 2002, a snapshot based approach was used to approximate solutions of Lyapunov equations. In particular, the authors suggested to use snapshots of simulations of the dual equations to compute the observability Gramian (5). The dual equations of the underlying optimal control problem are

$$\dot{x}_i(t) = A^T x_i(t), \tag{16}$$

$$x_i(0) = c_i^T, \tag{17}$$

for all $i = 1, \dots, p$ and $t \in [0, \bar{T}]$. Here, $\bar{T}$ specifies a final time to stop simulations. In Singler, 2011 this approach was extended to infinite dimensional systems and error bounds for the convergence of a low rank solution of the Lyapunov equation to the infinite dimensional Gramian were obtained. Note, that the transpose observation matrix is $C^T = [c_1^T \ c_2^T \dots c_p^T]$, so one has to simulate the system p times to get fully independent data. Therefore, systems with few outputs p are more favorable for this method. This is a priori not a disadvantage. Krylov subspace methods for $p > 1$ also require block multiplications and block inversions which become increasingly expensive as p grows.

By the theory of ordinary differential equations, the solution to (16) - (17) at $t = t_j$ is $x_i(t_j) = e^{t_j A^T} c_i^T$ for $i = 1, \dots, p$. The method of Proper Orthogonal Decomposition requires data of the system. Thus, simulate (16)-(17) and take S snapshots $x_i(t_j)$ at equally spaced time intervals in $[0, \bar{T}]$. Abusing notation, $x_i(t_j)$ is used for both the exact analytical solution as well as for the finite dimensional approximation, stemming from discretization. By the method of snapshots, for a given initial condition c_i^T we assemble the snapshots in

$$Y_i = [x_i(0) \ x_i(t_1) \dots x_i(\bar{T})] \in \mathbb{R}^{N\times S}. \tag{18}$$

Simulations starting with every column of C^T are concatenated in $Y = [Y_1, \ Y_2, \dots, Y_p] \in \mathbb{R}^{N\times pS}$. Since A is assumed to have only eigenvalues with strictly negative real parts, one can approximate

$$X \approx \int_0^{\bar{T}} e^{tA^T} C^T C e^{tA} dt \approx Y\hat{W}Y^T =: \tilde{X}, \tag{19}$$

where $\hat{W}$ is a matrix of weights, here chosen to be the weights for the approximation of the integral with the trapezoidal rule. Note, that the approximate observability Gramian can be Cholesky-factored as

$$\tilde{X} = ZZ^T, \tag{20}$$

where $Z = Y\hat{W}^{1/2} \in \mathbb{R}^{N\times pS}$. Since we seek the left singular vectors of $\tilde{X}$ to construct V_r, there is no need to form the Gramian explicitly. Instead, we follow the approach in Volkwein, 2013, Algorithm 2, Ch.1. If $N > pS$, compute the singular value decomposition $V_1\Sigma_1 W_1^T = Z^T Z$. Let V_r contain the first r columns of V_1 and let Σ_r be the $r \times r$ leading submatrix of Σ_1. Then $V_r = ZV_r\Sigma_r^{-1/2}$. In the case where $N < pS$, we actually compute the singular value decomposition $V_2\Sigma_2W_2^T = ZZ^T = \tilde{X}$ and obtain V_r as the leading r columns of V_2. For medium sized problems, the latter case allows very accurate approximations of the controllability Gramian involving many time snapshots, without increasing the cost of computing the singular value decomposition.

3. NUMERICAL RESULTS

In this section, dynamical systems $\Sigma = (A, B, C)$ as given in (1) are considered. We generate a projection matrix $V_r \in \mathbb{R}^{N\times r}$ through EKSM, Gramian based projection and the POD algorithm as described above. Then, the reduced order models $(\Sigma_r) = (A_r, B_r, C_r)$ are computed via projection and the reduced order Riccati equation (10) is solved with `care` in `Matlab`. This function solves the Hamiltonian eigenvalue problem associated with ARE and is further described in Arnold and Laub, 1984. The approximation of P is then given by $V_rP_r^{(N)}V_r^T$. Assessing the quality of the approximations is done via comparison of the Frobenius norm of the residual (12). This gives a consistent measure for the quality of approximations. A general projection algorithm is used to compare the methods, see Algorithm 2. The approximate $P_r^{(N)}$ does not have to be computed explicitly, but only its Cholesky factor, see the optional steps 7 and 8 of Algorithm 2.

The computational environment was a 2010 MacBook Pro with a 2.66 GHz Intel Core i7 Processor and 4GB RAM. `Matlab` was used as a software in the version of `R2012b`. With 'CPU time' we mean the time needed to compute V_r, measured by `tic,toc`.

Algorithm 2 :General Projected Riccati Algorithm

Input: $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$, $C \in \mathbb{R}^{p\times N}$. A projector $V_r = [v_1, \dots, v_r] \in \mathbb{R}^{N\times r}$ and *dtol*.

Output: $P_l^{(N)}$, solution of ARE.

1: **for** $l = 1, 2, \dots r$ **do**

2: Let $V_l = [v_1, \dots, v_l]$. Compute $A_l = V_l^TAV_l$, $B_l = V_l^TB$, $C_l = CV_l$.

3: Solve

$$A_l^TP_l + P_lA_l - P_lB_lB_l^TP_l + C_l^TC_l = 0. \quad (21)$$

4: Compute full order $P_l^{(N)} = V_lP_lV_l^T$.

5: Compute the residual norm

$$r_l := ||\mathcal{R}(P_l^{(N)})||_F. \quad (22)$$

6: **end for**

7: If desired: Compute the singular value decomposition $P_l = U\Sigma U^T$ where $\Sigma = diag[\sigma_1, \dots, \sigma_l]$ and $\sigma_1 \geq \dots \geq \sigma_l$; Determine k such that $\sigma_{k+1} < dtol < \sigma_k$, set $\Sigma_k = diag[\sigma_1, \dots, \sigma_k]$ and compute $Z_l = V_lU_k\Sigma_k^{1/2}$.

8: $P_l^{(N)} \approx Z_lZ_l^T$.

When comparing numerical results, we monitor accuracy, convergence and computational effort. The following questions are relevant to us for assessing the quality of the method:

(1) For given r how long does it take to generate the projection matrix V_r ?

(2) What is the convergence behavior for the residual $||\mathcal{R}(P_r^{(N)})||_F$ as r increases ?

3.1 ISS1R Flex Model

This model describes a structural subsystem of the International Space Station (ISS) and is taken from Antoulas, 2005. During the assembly process of the ISS many international partners are involved and robust stability and performance of all stages has to be certified. Many flexible structures, operational modes and control systems result in a very complex dynamical system. For robustness and performance assessment, it is critically important to identify the potential for dynamic interaction between the flexible structure and the control systems. As more components are added to the space station, the original symmetry gets lost, which poses additional simulation challenges. The subsystem 1R is a flex model of the Russian Service Module.

This dynamical system has $m = 3$ inputs, $p = 3$ outputs and consists of $N = 270$ states. The system matrix A is a non-symmetric dense matrix with 63843 nonzero elements and therefore 88% fill-in. The mass matrix is the identity. For $r = 2, 4, \dots, 70$, we computed a reduced order model, solved the low order Riccati equation and computed residual norms arising from those models. The results are plotted in Figure 1 below. The POD based approach employed simulations from $t = 0s$ to $\bar{T} = 15s$ and snapshots were taken every $0.02s$, amounting to a combined 2253 snapshots of the three simulations of the system.

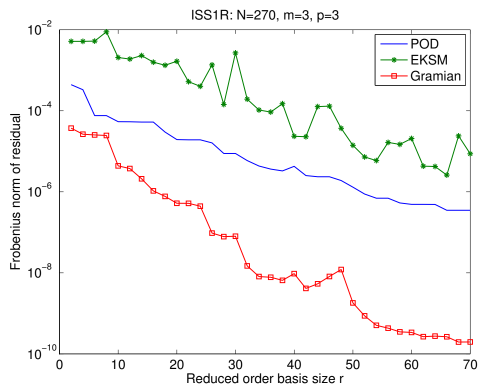

Fig. 1. Residual norms $||\mathcal{R}(P_r^{(N)})||_F$ for ISS1R model. Time to compute projector V_r: POD: 2.24s, EKSM: 0.24s, Gramian: 0.15s.

Some comments are in order. For a small sized system, the Gramian approach with the `lyap` solver as described above performs very well. In other words, projection with the left singular vectors of the observability Gramian is very effective. In this case the Lyapunov solution is almost identical to the Riccati solution. In Figure 2 the singular values of the Lyapunov and Riccati solution using `lyap` and `care` are plotted. For this problem, the singular values of both solutions are extremely close to each other and only seem to spread with numerical error. We will further investigate this for other problems. Note though, that Simoncini et al., 2013 showed that the convergence behavior of both solutions is related.

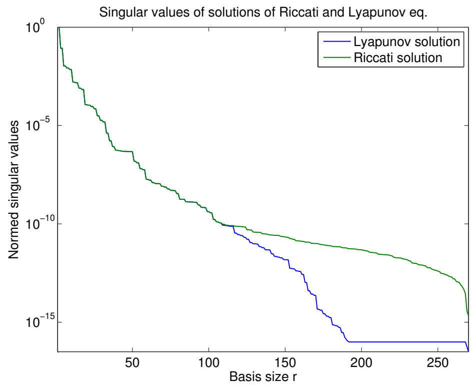

Fig. 2. ISS1R model: Singular values of the observability Gramian $X^{(N)}$ and Riccati solution $P^{(N)}$ using `lyap` and `care`.

The EKSM is rather quick in computing the projection V_r, yet shows higher errors in the residual. As noted above, the POD method essentially reduces to a quadrature for the controllability Gramian. As such, the number and location of the time samples t_j is important. We observed that increasing the final time for simulations gives better approximations of the controllability Gramian and even more so increasing the number of snapshots while maintaining $\bar{T}$.

Moreover, we note that the convergence of the POD approach is rather monotone, where especially the EKSM shows oscillations. For numerical purposes a monotone decreasing error is of interest, since it guarantees that extra work for computing an increased size reduced order model is not wasted.

3.2 ISS12A Flex Model

This is a model of another structural component of the ISS. It describes an advanced stage of the system, including $N = 1412$ states and $m = 3$ inputs and $p = 3$ outputs. The number of non-zero entries in A is 2118, so A is sparse. Moreover, the mass matrix is the identity. For the POD based algorithm, simulations of the linear system for all three initial vectors c_i^T were performed from $t = 0s$ to $\bar{T} = 20s$. This amounts to an overall collection of $S = 3003$ snapshots. The results for the convergence of the residual norm are given in Figure 3.

A plot of the singular values for the solution of the Riccati equation and the Lyapunov equation is given in Figure 4 below. For larger problems it will of course not be possible to compute the Riccati solution via `care`. For comparison purposes, `Matlab` took 65.03s to compute the Riccati solution. Again, the singular values of both solutions are very similar and only separate closer to machine precision. Consequently, the Gramian approach performs very well.

The Proper Orthogonal Decomposition based approach takes $4.9s$ to compute the projector and provides a significantly richer subspace than EKSM. For the POD approach, $N < pS$ so we computed the singular value decomposition of the 1412x1412 Gramian, as described above.

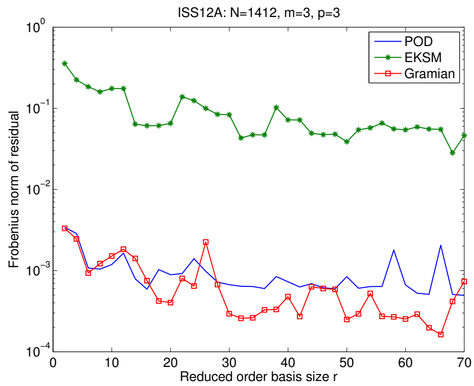

Fig. 3. Residual norms $||\mathcal{R}(P_r^{(N)})||_F$ for ISS12A model. Time to compute projector V_r: POD: 4.9s, EKSM: 0.38s, Gramian: 19.1s

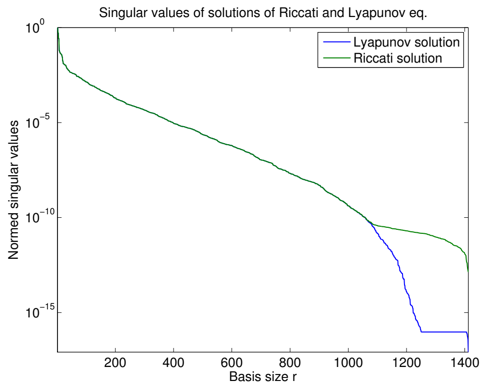

Fig. 4. ISS12A model: Singular values of the observability Gramian $X^{(N)}$ and Riccati solution $P^{(N)}$ using `lyap` and `care`.

4. CONCLUSION

We presented a new approach to compute solutions of algebraic Riccati equations via projection. The method is based on Proper Orthogonal Decomposition to compute an approximation of the solution to the Lyapunov equation via the algorithm in Willcox and Peraire, 2002; Singler, 2011. Its dominant left singular vectors are used for projection and are shown to be sufficiently rich for the projection framework to solve algebraic Riccati equations. We compared this new method to the Extended Krylov Subspace Method in Heyouni and Jbilou, 2009 and a Gramian based approach and demonstrated that the POD based approach performs very well. Larger systems with a mass matrix have been considered as well and performance of the POD method was very satisfactory. Those results, together with a cheap expression for computing the norm of a Riccati residual will be part of a future publication.

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