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Finite-Time Output Energy Measure for Polynomial Systems With Applications in Observability Analysis

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Abstract: This paper presents a set-based output energy measure for constrained polynomial systems with parameter uncertainties. Output energy is measured in terms of the L_2-norm on a finite-time interval while the initial conditions and parameters are allowed to take values from a set. By specifying a bound on the output norm, the measure allows further to determine the set of initial conditions and parameters which lead to satisfaction of this bound. Furthermore, this set characterizes whether an uncertain system can be estimated by a norm-observer and, therefore, can be applied for observability analysis. The derivation of the set is based on recasting a nonlinear program with embedded differential equations into an infinite-dimensional linear program. This is achieved by reformulating the system dynamics in terms of occupation measures. The chosen relaxation approach of the linear program generically guarantees that the obtained outer-approximation converges, for increasing relaxation order, to the true set of initial conditions and parameters satisfying the specified bound on the output norm.

1. INTRODUCTION

State estimation is in practice as well as theory typically the first step to control or supervise a plant. A particular instance of the state estimation problem is to relate the norm of the output to the norm of the states. This builds the foundation for controller design methods based on the output-to-state stability concept, see e. g. (Astolfi and Praly, 2006; Sontag and Wang, 1997; Hespanha *et al.*, 2005) and references therein. Other applications of output norms are observability Gramians in observability analysis. Such Gramians can be used to quantify observability of linear systems e. g. as needed in model reduction. Several extensions of Gramians have been proposed in the literature, i. a. (Ionescu and Scherpen, 2009; Lall *et al.*, 2002; Hahn *et al.*, 2003; Streif *et al.*, 2006) for nonlinear systems and (Petersen, 2002; Sastry and Desoer, 1982; Wang and Michel, 1994; Sojoudi *et al.*, 2009) for uncertain linear systems.

This work investigates an output energy measure using the L_2-norm of the output similar to (Gray and Mesko, 1999), however, for a finite-time horizon and uncertain polynomial systems. By posing a bound on this measure it is then possible to determine the set of initial conditions and parameters that lead to an output energy smaller than the posed bound. This is equivalent to a norm-observer. To derive this set of initial conditions and parameters, we formulate a polynomial program with embedded differential equations. Instead of directly employing dynamic optimization, we employ the concept of occupation measures to derive an infinite-dimensional linear program, see e. g. (Lasserre *et al.*, 2008; Lasserre, 2010). This program can then be addressed by a hierarchy of LMI relaxations, which can be solved efficiently. This approach generically guarantees that the estimated initial condition/parameter set converges for increasing relaxation order to the true set of initial conditions for which the observability measure is bounded. Further, it is shown that this measure can be employed in observability analysis in the sense that in a special case a norm-observer does not provide any information on the initial states.

This contribution is structured as follows. We formally state the problem of output energy quantification for polynomial systems in Sec. 2. The nonconvex and nonlinear optimization problem with embedded differential equations is stated and reformulated using occupation measures in Sec. 3. Sec. 4 illustrates how the connection between output energy and initial conditions can be employed for observability analysis. The approach is illustrated considering two examples in Sec. 5 and discussed in Sec. 6. Note that the presented approach also allows the quantification of parameter identifiability due to a simple reformulation of the nonlinear dynamics.

Notation: Sets and function spaces are denoted by calligraphic letters, e. g. $\mathcal{X}$ and $\mathscr{P}$. In particular, the space of continuous functions in variable x is denoted by $\mathscr{C}(x)$ and the space of continuously differentiable functions by $\mathscr{C}^1(x)$. $\mathbb{R}[x]$ denotes the ring of polynomials in variables $(x_1, \ldots, x_{n_x})$ with coefficients from $\mathbb{R}$. $\Sigma[x] \subset \mathbb{R}[x]$ denotes the convex cone of polynomials that are sum of squares (SOS) of polynomials and $\Sigma_r[x] \subset \Sigma[x]$ its subcone of SOS polynomials with degree at most $2r$. The symbol T denotes transposition of matrices and vectors. We denote the dimension of a variable x with $n_x \in \mathbb{N}$. State and output trajectories starting at the initial condition $x_0 \in \mathbb{R}$ are denoted by $x(t|x_0)$ and $y(t|x_0)$, respectively. The finite-time output measure on the time interval $[0,T]$ is denoted by $\boldsymbol{M}_T \in \mathbb{R}$.

2. PROBLEM SETUP AND DEFINITION

2.1 Polynomial System and Uncertainty Description

We consider nonlinear systems of the form

$$\begin{aligned} \dot{\tilde{x}}(t) &= f\big(t, \tilde{x}(t), p, u(t)\big), \\ \tilde{y}(t) &= h\big(t, \tilde{x}(t), p, u(t)\big), \end{aligned} \tag{1}$$

with states $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$, time-invariant parameters $p \in \mathbb{R}^{n_p}$, outputs $\tilde{y} \in \mathbb{R}^{n_{\tilde{y}}}$, and inputs $u \in \mathbb{R}^{n_u}$. We assume f, h to be polynomials, i. e. $f, h \in \mathbb{R}[t, \tilde{x}, p, u]$.

To simplify the notation, we combine the parameter and the state vector to $x = [\tilde{x}^\mathsf{T}, p]^\mathsf{T} \in \mathbb{R}^{n_x}$. Thus, (1) becomes

$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \\ y(t) &= h\big(t, x(t), u(t)\big), \end{aligned} \tag{2}$$

where the time derivatives of states corresponding to parameters are set to zero. For the subsequent analysis, we assume given inputs $u(t) = u_s(t)$, where $u_s \in \mathbb{R}[t]$.

This work assumes constraints on the states and initial conditions (including the parameters) as given, e. g., from physical insight or a-priori knowledge of admissible values of uncertain parameters, or controller constraints. Note that in principle, the bounds can be rather large for states not representing parameters, while they might be tight for some well known parameters. Such uncertainties are also commonly referred to as unknown-but-bounded or error-in-variables uncertainties. We assume that these constraints are given as compact sets defined by polynomial inequalities $g_i(\cdot) \geq 0, g_i \in \mathbb{R}[x]$, i. e.

$$\mathcal{X} := \Big\{x :\ g_{x,i}(x) \geq 0,\ \forall\, i = 1, \ldots, m_x\Big\} \subset \mathbb{R}^{n_x}, \tag{3a}$$

$$\mathcal{X}_0 := \Big\{x :\ g_{x_0,i}(x) \geq 0,\ \forall\, i = 1, \ldots, m_{x_0}\Big\} \subseteq \mathcal{X}. \tag{3b}$$

That is, $x(0) \in \mathcal{X}_0$ and $x(t) \in \mathcal{X}, \forall\, t \in (0,T]$. It is important to note that the presented approach implicitly considers only trajectories that are staying within $\mathcal{X}$ over the time interval $[0,T]$. However, these constraints on the trajectory can be relaxed, i. e. allowing for trajectories leaving $\mathcal{X}$, but this results in a more involved interpretation of the derived measures and some of the convergence results hold only under additional assumptions, e. g. convexity of f, g, and h (Jeyakumar *et al.*, 2013b) or an extended Archimedean quadratic module (Jeyakumar *et al.*, 2013a).

2.2 Output Energy

We quantify the energy visible at the output $y(t|x_0)$ starting from a specific initial condition using the L_2-norm. The goal is now to link an initial condition to the output energy, in the sense that the norm of the output is used to derive bounds on the initial condition similar to the concept of norm-observability, see e. g. (Hespanha *et al.*, 2005). This work considers the output energy over a finite-time interval as defined next.

Definition 1. (Finite-Time Output Energy Measure). The output energy measure $\boldsymbol{M}_T(x_0)$ of an initial condition $x_0 \in \mathbb{R}^{n_x}$ depends on the resulting output trajectory $y(t|x_0) \in \mathbb{R}^{n_y}$ of (2) on the finite-time interval $[0,T]$, $T > 0$ by

$$\boldsymbol{M}_T(x_0) := \int_0^T \|y(t|x_0)\|^2\, dt. \qquad \blacksquare$$

In (Gray and Mesko, 1996) an equivalent infinite-time measure is is used for observability analysis for linear systems and systems for which the so-called zero-state assumption holds. Note that Def. 1 only allows to quantify the output energy of a single initial condition.

If there are no uncertainties, the quantification of the energy could be determined by deriving the initial condition x_0 that maximizes the L_2-norm.

Problem 1. (Initial Condition with Highest/Lowest Output Energy). Determine the initial condition $x_0 \in \mathcal{X}_0$, including parameter values, for which $\boldsymbol{M}_T(\cdot)$ is minimized or maximized to give $\boldsymbol{M}_T^{min}$ or $\boldsymbol{M}_T^{max}$, respectively. ■

In case of uncertainties, however, $\boldsymbol{M}_T^{min}$ and $\boldsymbol{M}_T^{max}$ do not provide much insight, as typically uniqueness of the optima is lost resulting in entire sets of initial conditions that lead to the same output energy. In addition, a system that is unobservable in the classical sense might still have a high output energy.

For this reason, consider the subsequent problem more suitable for the considered problem setup.

Problem 2. (Initial Conditions with Bounded Output Energy Measure). For a given $\boldsymbol{m} \in \mathbb{R}$. Determine the set of initial conditions $\mathcal{X}_0^*$ for which every initial condition $x_0 \in \mathcal{X}_0^*$ satisfies $\boldsymbol{M}_T(x_0) \leq \boldsymbol{m}$. ■

In the following, a computational approach is proposed to determine the set $\mathcal{X}_0^*$ for a given bound $\boldsymbol{m}$.

3. OPTIMIZATION-BASED FINITE-TIME OUTPUT ENERGY MEASURE

This section addresses Prob. 1 and Prob. 2 and derives an optimization-based method to determine the set of initial conditions with bounded output energy measure employing occupation measures.

Prob. 1 can be directly translated into a nonlinear minimization problem to obtain the lower bound:

$$\begin{aligned} \boldsymbol{M}_T^{min} := \min_{x_0}\ & \int_0^T \|y(t|x_0)\|^2\, dt \\ \text{s.t. } & \dot{x}(t) = f(t, x(t), u_s(t)), \\ & y(t) = h(t, x(t), u_s(t)), \\ & x_0 \in \mathcal{X}_0, \\ & x(t) \in \mathcal{X}, \quad \forall\, t \in (0,T]\,. \end{aligned} \tag{4}$$

Note that this problem can be solved by the methods presented in (Henrion *et al.*, 2008). As stated before, Prob. 1 is inapplicable in case of uncertainties. We therefore modify (4) to address Prob. 2 as follows.

$$\begin{aligned} \mathcal{X}_0^* := \text{find }\ & x_0 \\ \text{s.t. } & \dot{x}(t) = f(t, x(t), u_s(t)), \\ & y(t) = h(t, x(t), u_s(t)), \\ & \int_0^T \|y(t|x_0)\|^2\, dt \leq \boldsymbol{m}, \\ & x_0 \in \mathcal{X}_0, \\ & x(t) \in \mathcal{X}, \quad \forall\, t \in (0,T]\,. \end{aligned} \tag{5}$$

With (5), we aim to find initial conditions/parameter values x_0 which lead to a L_2-norm of the output that is bounded by $\mathbf{m}$, thus solving Prob. 2. Clearly, addressing (5) is difficult due to the present nonconvexity. Also the embedded differential equation has to be taken care of. The purpose of the subsequent sections is to derive a convex optimization problem that directly takes the dynamics into account and considers entire sets of parameters and initial conditions, thus solving Prob. 2.

The crucial idea we employ in this work is to reformulate the nonlinear optimization problem with embedded differential equation in terms of occupation measures (see e. g. (Lasserre, 2010; Savorgnan *et al.*, 2009; Henrion *et al.*, 2008)). The occupation measures contain information about the initial condition and parameter values of the system, as well as the nonlinear dynamics (i. e. trajectories) of the system. The main advantage of this reformulation is the resulting linear relationship between the occupation measures. This gives rise to a convex problem, albeit infinite-dimensional, which can be solved efficiently by a LMI relaxation hierarchy as shown in the following.

3.1 Reformulation of the System Dynamics

We give a short overview of the steps necessary to reformulate the above nonlinear optimization problem in terms of an infinite dimensional linear program, for more details we refer to (Henrion *et al.*, 2008; Savorgnan *et al.*, 2009; Lasserre, 2010). The employed procedure is based on a reformulation of the dynamics in terms of Borel measures. The main idea here is to derive constraints such a measure has to fulfill and that provide a direct connection to the nonlinear dynamics. For this purpose, we introduce the following notation. We denote the space of finite Borel measures supported on the set $\mathcal{A}$ with $\mathscr{B}(\mathcal{A})$. From duality it follows that each element of this space corresponds to a linear bounded functional in the dual space $\mathscr{C}(\mathcal{A})'$. We call measures μ which are nonnegative and fulfill $\mu(\mathcal{A}) = 1$, probability measures and the corresponding space of probability measures is denoted by $\mathscr{P}(\mathcal{A})$. Note that although the following derivations are using probability measures the end product is deterministic.

We define the following measure

$$\mu(\mathcal{A}\times\mathcal{B}) := \int_{\mathcal{T}}\int_{\mathcal{X}} I_{\mathcal{A}\times\mathcal{B}}(t, x(t|x_0))\mu_0(dx_0)\,dt \qquad (6)$$

for all subsets $\mathcal{A}\times\mathcal{B}$ in the Borel σ-algebra of subsets of $\mathcal{T}\times\mathcal{X}$, where $\mathcal{T} := [0, T]$. Here, $I_{\mathcal{A}}(x)$ is the indicator function of the set $\mathcal{A}$, which is equal to one if $x \in \mathcal{A}$, and zero otherwise. The probability measure $\mu_0 \in \mathscr{P}(\mathcal{X})$ describes the distribution of the random initial condition x_0. Note that μ_0 is not assumed to be known.

We refer to $\mu \in \mathscr{P}(\mathcal{T}\times\mathcal{X})$ as an occupation measure, whereas this term is motivated by the observation that the value $\int_{\mathcal{T}}\mu(dt,\mathcal{B}) = \mu(\mathcal{T}\times\mathcal{B})$ is equal to the total time a trajectory spends in the set $\mathcal{B}\subset\mathcal{X}$. In addition, note that μ encodes the system trajectories, in the sense that for a scalar valued smooth function $v \in \mathscr{C}^\infty(\mathcal{T}\times\mathcal{X};\mathbb{R})$ and μ_0 being the Dirac measure at x_0, i. e. $\mu_0 = \delta_{x_0}$, integration of v w.r.t. μ amounts to time integration along the system trajectory starting at x_0:

$$\int_{\mathcal{T}}\int_{\mathcal{X}} v(t,x)\mu(dt,dx) = \int_{\mathcal{T}} v\big(t, x(t|x_0)\big)\,dt.$$

With these notations, for all sufficiently regular test functions $v \in \mathscr{C}^1(\mathcal{T}\times\mathcal{X};\mathbb{R})$, it holds that

$$\begin{aligned}\int_{\mathcal{X}} v(T,x)\mu_T(dx) - \int_{\mathcal{X}} v(0,x)\mu_0(dx) = \\ \int_{\mathcal{T}}\int_{\mathcal{X}} \frac{d}{dt}\, v\big(t, x(t|x_0)\big)\mu_0(dx_0),\end{aligned} \qquad (7)$$

which corresponds to the evolution of all trajectories along v starting from an initial condition x_0 as specified by the distribution μ_0. The right-hand-side of the above equation can be rewritten as

$$\begin{aligned}&\int_{\mathcal{T}}\int_{\mathcal{X}}\Big(\frac{\partial}{\partial t}v\big(t,x(t|x_0)\big) + \\ &\qquad \operatorname{grad} v\big(t,x(t|x_0)\big)\cdot f\big(t,x(t|x_0)\big)\Big)\,\mu_0(dx_0)\,dt \\ &= \int_{\mathcal{T}}\int_{\mathcal{X}}\Big(\frac{\partial}{\partial t}v(t,x) + \operatorname{grad} v(t,x)\cdot f(t,x)\Big)\,\mu(dt,dx).\end{aligned}$$

To simplify notation, we introduce the Liouville operator $\mathcal{L} : \mathscr{C}^1(\mathcal{T}\times\mathcal{X}) \to \mathscr{C}(\mathcal{T}\times\mathcal{X})$ as $\mathcal{L}v := \frac{\partial v}{\partial t} + \operatorname{grad} v\cdot f$ and its adjoint $\mathcal{L}' : \mathscr{C}(\mathcal{T}\times\mathcal{X})' \to \mathscr{C}^1(\mathcal{T}\times\mathcal{X})'$ such that for the bilinear form $\langle\mathcal{L}v,\mu\rangle = \langle v,\mathcal{L}'\mu\rangle$ holds for all $v \in \mathscr{C}^1(\mathcal{T}\times\mathcal{X})$, i. e. $\mathcal{L}'\mu := -\frac{\partial\mu}{\partial t} - \operatorname{div}(\mu f)$.

With these notations, we write (7) concisely as

$$\langle\mathcal{L}v,\mu\rangle = \langle v,\delta_T\mu_T\rangle - \langle v,\delta_0\mu_0\rangle \qquad (8)$$

for all $v \in \mathscr{C}^1(\mathcal{T}\times\mathcal{X})$, where δ_0 and δ_T refers to $t = 0$ and $t = T$, respectively. Equivalently, we can write

$$\mathcal{L}'\mu = \delta_T\mu_T - \delta_0\mu_0.$$

3.2 Reformulation of the Output Norm

To link the L_2-norm of the output to the introduced measures, consider the mapping H that maps the output space to $[0,\infty]$ and

$$H(\mu(\mathcal{A})) = \mu(h^{-1}(\mathcal{A})) \qquad (9)$$

holds, where $h^{-1}(\mathcal{A}) := \{(t,x) \in \mathcal{T}\times\mathcal{X} : h(t,x) \in \mathcal{A}\}$. Note that H is commonly called a pushforward operator in measure theory as it transports a measure from a measurable space to another according to function h.

To be able to represent the previous statement also in terms of continuous functions as needed in the following, consider the canonical basis of monomials up to degree r:

$$m_r(x) := (1, x_1, \ldots, x_{n_x}, x_1^2, x_1x_2, \ldots, x_1^{r-1}x_2, \ldots, x_{n_x}^r)^\mathsf{T}$$

and the Riesz functionals

$$\begin{aligned}z_0 &:= \int_{\mathcal{X}} m_r(x)\mu_0(dx),\ \ z_T := \int_{\mathcal{X}} m_r(x)\mu_T(dx),\\ z &:= \int_{\mathcal{T}\times\mathcal{X}} m_r(x)\mu(dt,dx),\ \bar{z} := \int_{\mathcal{T}\times\mathcal{X}} m_r(x)\circ m_r(x)\mu(dt,dx),\end{aligned}$$

where $\circ$ is the Hadamard product. Then we can simply define a vector $c \in \mathbb{R}^{n_{m_r}}$ such that $h(t,x,u_s) = c^\mathsf{T} m_r(x)$ and $\int_{\mathcal{T}} h(t,x,u_s) = \int_{\mathcal{T}\times\mathcal{X}} c^\mathsf{T}\bar{z}\mu(dt,dx)$.

3.3 Infinite-Dimensional Linear Program and Relaxation

By applying the results presented in the previous sections to (5), we obtain the infinite-dimensional linear program in the positive cone of the space of finite signed Borel measures:

$$\begin{aligned}
\sup_{\mu_0} \ & \langle 1, \mu_0 \rangle \\
\text{s.t.} \ & \delta_T \mu_T - \mathcal{L}' \mu = \delta_0 \mu_0, \\
& \boldsymbol{m} - H(\mu) \geq 0, \\
& \mu_0 + \hat{\mu}_0 = \lambda, \\
& \mu_0, \mu_T, \mu, \hat{\mu}_0 \geq 0, \\
& \mathrm{supp}\mu_0, \mathrm{supp}\mu_T, \mathrm{supp}\hat{\mu}_0 \subset \mathcal{X}, \\
& \mathrm{supp}\mu \subset [0,T] \times \mathcal{X}, \\
& \mu_0, \mu_T \in \mathscr{P}(\mathcal{X}), \mu \in \mathscr{P}([0,T] \times \mathcal{X}).
\end{aligned} \tag{10}$$

To derive a meaningful problem, we additionally impose the bound $\mu_0 + \hat{\mu}_0 = \lambda$, where $\hat{\mu}_0$ is a slack variable (or complementary measure). This bound ensures that the derived measure is dominated by the Lebesque measure λ. In practice that is not a restriction as one still searches for the infimum in the infinite dimensional measure space; however, it is advantageous from a computational perspective since Lebesque moments of sets are readily available, see also (Henrion and Korda, 2013).

The optimization (10) describes basically the hypervolume of $\mathcal{X}_0^*$. To derive an exact, explicit description of $\mathcal{X}_0^*$ we represent (10) as its infinite dimensional dual problem over the space of continuous functions in terms of nonnegative polynomials as

$$\begin{aligned}
\inf_{w} \ & \langle w(x), \lambda \rangle \\
\text{s.t.} \ & w(x) - v(0,x,u_s) \geq 1, \forall\, x \in \mathcal{X}, \\
& -\mathcal{L}v(t,x,u_s) - h^2(t,x,u_s) + \boldsymbol{m} \geq 0, \\
& \quad \forall\, x \in [0,T] \times \mathcal{X}, \\
& v(T,x,u_s), w(x) \geq 0, \forall\, x \in \mathcal{X}, \\
& w(x) \in \mathcal{C}(\mathcal{X}), v \in \mathcal{C}^1([0,T] \times \mathcal{X}.
\end{aligned} \tag{11}$$

As both LPs are infinite-dimensional they cannot be solved directly. We employ here Lasserre's hierarchy (e. g. (Lasserre, 2010)) to derive a solution.

By replacing the measures in (10) with the canonical basis $m_r(x)$ we obtain the standard primal moment relaxation

$$\begin{aligned}
\sup_{z_0} \ & (z_0)_1 \\
\text{s.t.} \ & A_T z_T - A z = A_0 z_0, \\
& -c^\mathsf{T} \bar{z} + \boldsymbol{m} \geq 0, \\
& M_r(z_T) \succeq 0, L_{r-d}(g_{x,i} z_T) \succeq 0, \\
& \quad \forall\, i = 1, \ldots, m_x, \\
& M_r(z) \succeq 0, L_{r-\max(d, deg(t^2-t))}(g_{x,i}, z) \succeq 0, \\
& \quad \forall\, i = 1, \ldots, m_x,
\end{aligned} \tag{12}$$

where M_r and L_{r-d} denote the moment matrix and the localizing matrix of degree r (d is the degree of the respective constraint g_{x_i}), respectively. The vector c is defined as before and the matrices A_T, A_0, A are derived by comparing the coefficients of (8) and the Riesz functionals with the system dynamics.

Deriving the standard dual of (12) and then deriving a sum-of-squares strengthening of this dual leads to

$$\begin{aligned}
\inf_{w_{c,r}} \ & w_{c,r}^\mathsf{T} l \\
\text{s.t.} \ & w_r(x) = v(0,x) + 1 + r_0(x) + \\
& \quad + \sum\nolimits_{i=1}^{m_x} r_{0,i}(x) g_{x,i}(x), \\
& -\mathcal{L}v(t,x) - h^2(t,x,u_s) + \boldsymbol{m} = p(t,x) \\
& \quad + q_0(t,x) t(T-t) + \sum\nolimits_{i=1}^{m_x} q_i(t,x) g_{x,i}(x), \\
& w_r(x) = p_0(x) + \sum\nolimits_{i=1}^{m_x} q_{0,i}(x) g_{x,i}(x), \\
& v(1,x) = p_1(x) + \sum\nolimits_{i=1}^{m_{x_k}} q_{1,i}(x) g_{x,i}(x),
\end{aligned} \tag{13}$$

where l is the vector of Lebesgue moments over $\mathcal{X}$ indexed in the same basis in which the polynomial $w_r(x)$ with coefficients $w_{c,r}$ is expressed. The minimum is over polynomials $v(t,x)$ and $w_r(x)$, and polynomial sum-of-squares $p(t,x)$, $q_0(t,x)$, $q_i(t,x)$, $p_0(x) \in \Sigma_r[t,x]$, $q_{0,i}(x)$, $p_1(x)$, $r_0(x), r_{0,i}(x) \in \Sigma_r[x], \forall\, i = 1,\ldots,m_x$ and $q_{1,i}(x), \forall\, i = 1,\ldots,m_{x_k}$ of appropriate degrees. The constraints that polynomials are sum-of-squares can be written explicitly as LMI constraints, and the objective is linear in the coefficients of the polynomial $w_r(x)$. Therefore, Prob. 2 can be formulated as an semi-definite program. Furthermore, the set $\mathcal{W}_r := \{x : w_r(x) \geq 1\}$ is an outer-approximation of $\mathcal{X}_0^*$, i. e. $\mathcal{X}_0^* \subseteq \mathcal{W}_r$, and the Lebesque measure of $\mathcal{W}_r$ converges to the Lebesque measure of $\mathcal{X}_0^*$ for $r \to \infty$, see (Henrion and Korda, 2013, Thm. 5).

In the next section, it is illustrated how $\mathcal{X}_0^*$ can be applied in observability analysis.

4. APPLICATION TO OBSERVABILITY ANALYSIS

This section illustrates that the set $\mathcal{X}_0^*$ can contain more information on the initial conditions than the original set $\mathcal{X}_0$. In particular, if the set $\mathcal{X}_0^*$ is not contained in the interior of $\mathcal{X}_0$ then $\mathcal{X}_0^*$ does not provide any additional information on the location of possible initial conditions. Therefore, to strengthen the expressiveness of $\mathcal{X}_0^*$ the following additional requirement is defined.

Definition 2. (Set-Observability). A state x_i of system (2) with initial conditions in the set $\mathcal{X}_0$ is said to be set-observable if the projection of $\mathcal{X}_0^*$ onto x_i has a smaller Lebesque measure λ than the projection of $\mathcal{X}_0$ onto x_i, i. e. $\lambda(\perp_{x_i} \mathcal{X}_0^*) < \lambda(\perp_{x_i} \mathcal{X}_0)$. System (2) is said to be set-observable if this condition holds for all $i = 1, \ldots, n_x$. ∎

The idea behind set-observability is to compare the length of the projections of $\mathcal{X}_0$ and $\mathcal{X}_0^*$ onto different state directions. If the former length is larger than the latter then this means that the bound on the output energy can be used to reduce the initially present uncertainties. According to Def. 2 the observability of a system corresponds, therefore, not only to the output energy but also to the shape and size of $\mathcal{X}_0^*$. Prob. 2 then becomes the following

Problem 3. (Set-observable Initial Conditions).
Determine the set $\mathcal{X}_0^*$ and determine which states are set-observable w.r.t. Def. 2 for a given bound $\boldsymbol{m}$. ∎

With the help of (13) we can solve Prob. 3. Moreover, it allows us to analyze the set-observability of system (2) according to Def. 2. We can state the following result:

Theorem 1. (Sufficient Condition for Set-Observability). Given an outer-approximation $\mathcal{W}_r$. Assuming $\mathcal{X}_0^* \neq \emptyset$, then the following statements are equivalent.

(1) The system is set-observable.
(2) $\exists r \in \mathbb{R} \cup \{\infty\}$ and $\exists \boldsymbol{m} \in \mathbb{R}$ such that $\lambda(\perp_{x_j} \mathcal{X}_0^*) \leq \lambda(\perp_{x_j} \mathcal{W}_r) < \lambda(\perp_{x_j} \mathcal{X}_0)$, $\forall x_j$. ■

Proof: Follows the same lines as the proof of (Henrion and Korda, 2013, Thm. 6). In the case of finite convergence the same argumentation as provided in the following can be employed by replacing ∞ with a sufficiently large constant $R \in \mathbb{R}$. From the convergence of the relaxation it follows that w_r converges to the indicator function $I_{\mathcal{X}_0^*}$ of the set $\mathcal{X}_0^*$. Furthermore, at every relaxation order r we have $\mathcal{X}_0^* \subset \mathcal{W}_r$, i. e. $w_r \geq I_{\mathcal{W}_r} \geq I_{\mathcal{X}_0^*}$. Therefore, we have

$$\lambda(\perp_{x_j} \mathcal{X}_0^*) = \int_{\mathcal{X}} \perp_{x_j} I_{\mathcal{X}_0^*} d\lambda$$
$$= \lim_{r\to\infty} \int_{\mathcal{X}} \perp_{x_j} w_r d\lambda \geq \lim_{r\to\infty} \int_{\mathcal{X}} \perp_{x_j} W_r d\lambda = \lambda(\perp_{x_j} \mathcal{W}_r).$$

As $\mathcal{X}_0^* \subset \mathcal{W}_r$, it follows that $\lambda(\perp_{x_j} \mathcal{X}_0^*) \leq \lambda(\perp_{x_j} \mathcal{W}_r)$. Therefore, $\lambda(\perp_{x_j} \mathcal{X}_0^*) = \lambda(\perp_{x_j} \mathcal{W}_r)$ must hold, which concludes the proof. □

Note that there are certain similarities of the proposed concept of set-observability and norm-observability introduced in (Hespanha *et al.*, 2005, 2002). A system is small-time initial-state norm-observable (SINO), if $\forall \tau$ there exists $\gamma \in \mathcal{K}_\infty$ such that the Euclidean norm of the initial state is upper bounded by the infinity norm of the output, i. e. $|x(0)| \leq \gamma(||y||_{\infty,[0,\tau]})$, see (Hespanha *et al.*, 2005). It is obvious that set-observability and SINO are related in the case that parameters are not unknown-but-bounded. The main difference derives from the fact that this work considers only bounded state space regions, therefore, a strict relationship exists only for $\boldsymbol{m} = 0$. In this case, if a system is not set-observable ($\forall r$) it follows that no γ exists on $\mathcal{X}$.

5. EXAMPLES

Two-Tank Example: We consider the polynomial model of a two-tank as derived in (Labibi *et al.*, 2009). The process consists of two water tanks and one pump modeled by:

$$\begin{aligned} \dot{x}_1 &= 0.073x_1^2 - 1.6x_1 - 0.047x_2^2 + 0.2x_2, \\ \dot{x}_2 &= 0.33x_2^2 - 1.4x_2, \\ y &= h(x_1, x_2), \end{aligned} \tag{14}$$

where $h(x_1, x_2)$ is either x_1 or x_2, state constraints are $\mathcal{X} = \mathcal{X}_0 = [0, 1] \times [0, 1]$ and the end-time T is set to one.

For given bounds $\mathbf{m} \in \{0, 0.01, \ldots, 0.1, 0.2, 0.3\}$ on the output energy measure, the set of initial conditions is derived with (13), cf. Fig. 1. Furthermore, the results show that the system is set-observable for $h(x_1, x_2) = x_1$ and not set-observable for $h(x_1, x_2) = x_2$ according to Thm. 1 as $\mathcal{W}_r$ is up to numerical optimality equal to $\mathcal{X}_0$.

Mass-Spring Example: We consider a mass-spring system with a softening spring (for details see (Khalil, 2002)) scaled to the unit box:

$$\begin{aligned} \dot{x}_1 &= (x_2 - 0.5), \\ \dot{x}_2 &= -(p+1)(x_1 - 0.5) - 0.5(x_2 - 0.5) \\ &\quad + 4(p+1)(x_1 - 0.5)^3, \\ y &= x_2. \end{aligned} \tag{15}$$

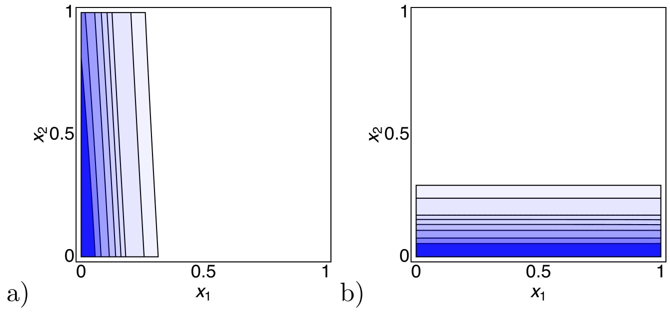

Fig. 1. Two-Tank example: a) set-observable for $y = x_1$, b) set-unobservable for $y = x_2$. Computation time ∼1min with MOSEK 7.0/YALMIP (for each case). Relaxation order $r = 5$.

The state constraints are $\mathcal{X} = \mathcal{X}_0 = [0, 1] \times [0, 1]$, $p \in [0, 1]$ and the end-time T is set to one.

Instead of varying $\mathbf{m}$ as in the previous example we modified (13) such that we can consider a lower and an upper bound on the L_2-norm of the output, namely $\mathbf{m}_1 = 0.6 \leq \int_{\mathcal{T}} ||y(t|x_0)||^2 dt \leq \mathbf{m}_2 = 1.2$. In this case, the output energy measure shows that the states are set-observable and the parameter is not set-observable w.r.t. $\boldsymbol{m}$ as only the projections of $\mathcal{X}_0^*$ onto the axes of x_1, x_2 have a smaller Lebesque measure than the projections of $\mathcal{X}_0$ (see also Thm. 1) as illustrated by Fig. 2.

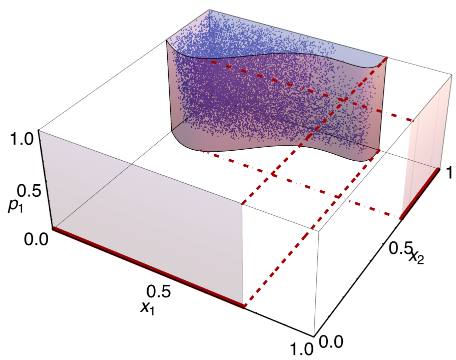

Fig. 2. Mass-spring example: Projections of the computed initial conditions (indicated by dotted lines) show the system is partially set-unobservable. Full red lines correspond to the projections onto the x_1 and x_2 axis, respectively. Computation time ∼30s with MOSEK 7.0/YALMIP. Relaxation order $r = 5$. Dots represent consistent initial conditions obtained by Monte-Carlo sampling.

6. DISCUSSION AND CONCLUSIONS

This work presented a computational approach to determine sets of initial conditions and parameters which satisfy a given bound on the L_2-norm of the output. This was achieved by constructing a polynomial program with embedded differential equations which was recast into an infinite-dimensional linear program with the help of occupation measures. The linear program was solved by a converging hierarchy of LMI problems (Lasserre, 2010). This approach guarantees that the obtained outer-approximation of the initial conditions and parameters converges, for increasing relaxation order, to the true set of initial conditions and parameters satisfying the given bound on the output norm.

Further advantages of the presented approach, besides convergence, are that no sampling is required to find appropriate initial condition/parameter regions. Moreover, as the dynamic optimization is reformulated in terms of occupation measures no integration or numerical approximation of the nonlinear dynamics is needed.

Furthermore, it was shown that the introduced measure can be applied to observability analysis for uncertain systems. However, to derive more rigorous statements and linking the measure to classical observability/identifiability analysis will be subject of future research.

At the moment the presented approach is limited to moderately sized systems as the complexity to solve the LMIs grows polynomially in the number of variables and the relaxation order. However, semidefinite programming is a highly active research field and computational as well as algorithmic improvements are to be expected in the future as suggested by e. g. Permenter and Parrilo (2012); Seiler *et al.* (2013).

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