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Decentralized Controller Design by Continuous Pole Placement for Commensurate-Time-Delay Systems

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Abstract: Decentralized controller design problem for linear time-invariant retarded commensurate-time-delay systems is considered. The continuous pole placement algorithm, which has recently been introduced for static state vector feedback controller design for retarded time-delay systems, is first extended to design dynamic output feedback controllers. This algorithm is then used in the proposed decentralized controller design algorithm. An example is also presented to demonstrate the proposed approach.

Keywords: Decentralized control; time-delay systems; dynamic output feedback; stabilization; pole placement.

1. INTRODUCTION

Many systems may involve time-delays in their dynamics, inputs, and/or outputs. The controller design for such systems, which are called *time-delay systems*, is more difficult than for finite-dimensional systems in general (e.g., see Loiseau et al. (2009)).

A decentralized control structure may be necessary, or at least preferable, for many large-scale systems (Šiljak (1991)). Although the subject of decentralized controller design for finite-dimensional systems has been discussed in the literature for the past four decades, the same problem for time-delay systems has found place in the literature rather recently (e.g., see Bakule (2008) and references therein).

In the present work, we propose a decentralized controller synthesis procedure for linear time-invariant (LTI) retarded commensurate-time-delay systems. This procedure is based on the *decentralized pole assignment algorithm* of Davison and Chang (1990), which was proposed for finite-dimensional systems. In this algorithm a centralized controller is designed for each control agent sequentially. Therefore, the algorithm requires a centralized controller design procedure, which is to be used by each control agent. For this purpose, we use the *continuous pole placement algorithm* of Michiels et al. (2002). This algorithm, however, was proposed for static state vector feedback. Therefore, after stating our problem formally in Section 2, in Section 3 we extend the continuous pole placement algorithm to design dynamic output feedback controllers. This algorithm is then used in the decentralized controller design algorithm proposed in Section 4.

Throughout the paper, $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ denote the sets of, respectively, complex numbers, real numbers, and non-negative integers. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ denotes the real part of s. For $\mu \in \mathbb{R}$, $\mathbb{C}_\mu^- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < \mu\}$. $\mathbb{R}[\cdot]$ denotes the ring of polynomials in $\cdot$ with real coefficients. For $k, l \in \mathbb{N}$, $\mathcal{F}^k$ and $\mathcal{F}^{k\times l}$ respectively denote the spaces of k-dimensional vectors and $k \times l$-dimensional matrices with elements in $\mathcal{F}$, where $\mathcal{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{R}[\cdot]$. i denotes the imaginary unit. I_k and $0_{k\times l}$ respectively denote the $k \times k$-dimensional identity and the $k \times l$-dimensional zero matrices. When the dimensions are apparent, we use I and 0 to denote respectively the identity and the zero matrices. For a matrix or vector M, M^T and M^* respectively denote the transpose and the complex-conjugate transpose of M. For a (vector) function $x(\cdot)$, $\dot{x}(\cdot)$ denotes the derivative of $x(\cdot)$. Finally, $\|\cdot\|$, $\det(\cdot)$, and $\mathrm{rank}(\cdot)$ respectively denote the 2-norm, the determinant, and the rank of $(\cdot)$ and $\mathrm{bdiag}(\cdots)$ denotes a block diagonal matrix with $(\cdots)$ on the main diagonal.

2. PROBLEM STATEMENT

Consider a decentralized LTI retarded commensurate-time-delay system Σ with ν control agents,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{\sigma} \left(A_i x(t-h_i) + \sum_{j=1}^{\nu} B_{j,i} u_j(t-h_i) \right) \\ y_j(t) &= \sum_{i=0}^{\sigma} C_{j,i} x(t-h_i) \ , \quad j = 1, \ldots, \nu \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time t and $u_j(t) \in \mathbb{R}^{p_j}$ and $y_j(t) \in \mathbb{R}^{q_j}$ are, respectively, the input and the output vectors at time t, accessible by the j^{th} control agent ($j = 1, \ldots, \nu$). The matrices A_i, $B_{j,i}$ and $C_{j,i}$ ($i = 0, \ldots, \sigma,\ j = 1, \ldots, \nu$) are constant real matrices, $h_0 := 0$ (thus, $i = 0$ in (1) corresponds to the delay-free part), and $h_i := ih$, $i = 1, \ldots, \sigma$, are the time-delays, which are assumed to be commensurate with a common divisor $h > 0$.

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In order to obtain a more compact representation of (1), let us introduce the delay operator, τ, by h, i.e., $\tau f(t) = f(t-h)$ for any function f of time t. Then, define the matrix operators whose elements are in $\mathbb{R}[\tau]$:

$$A(\tau) := \sum_{i=0}^{\sigma} A_i \tau^i \ , \quad B_j(\tau) := \sum_{i=0}^{\sigma} B_{j,i}\tau^i \ , \tag{2}$$

and

$$C_j(\tau) := \sum_{i=0}^{\sigma} C_{j,i}\tau^i \ , \tag{3}$$

for $j = 1, \ldots \nu$. Also define

$$B(\tau) := [\, B_1(\tau) \ B_2(\tau) \ \cdots \ B_\nu(\tau) \,] \tag{4}$$

and

$$C(\tau) := \left[\, C_1^T(\tau) \ C_2^T(\tau) \ \cdots \ C_\nu^T(\tau) \,\right]^T \ . \tag{5}$$

Then, the system Σ can be compactly represented as

$$\begin{aligned} \dot{x}(t) &= A(\tau)x(t) + B(\tau)u(t) \\ y(t) &= C(\tau)x(t) \end{aligned} \tag{6}$$

where $u(t) := \left[\, u_1^T(t) \cdots u_\nu^T(t) \,\right]^T \in \mathbb{R}^p$ and $y(t) := \left[\, y_1^T(t) \cdots y_\nu^T(t) \,\right]^T \in \mathbb{R}^q$, where $p := \sum_{j=1}^{\nu} p_j$ and $q := \sum_{j=1}^{\nu} q_j$.

Definition 1. For any given $\mu \in \mathbb{R}$, the set of *μ-modes* of the system Σ, described by (1) or, equivalently, by (6), is defined as

$$\Omega_\mu(\Sigma) := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \mu \text{ and } \phi_\Sigma(s) = 0\} \tag{7}$$

where $\phi_\Sigma(s) := \det\left(sI - \bar{A}(s)\right)$ is the *characteristic function* of the system Σ, where $\bar{A}(s)$ is obtained from the operator matrix $A(\tau)$, defined in (2), by replacing the operator τ by the function e^{-hs}, i.e., $\bar{A}(s) := \sum_{i=0}^{\sigma} A_i e^{-sh_i}$.

Definition 2. For any given $\mu \in \mathbb{R}$, the system Σ is said to be *μ-stable* if $\Omega_\mu(\Sigma) = \emptyset$. Furthermore, a controller K is said to *μ-stabilize* the system Σ, if the closed-loop system obtained by applying the controller K to system Σ is μ-stable.

Definition 3. $\lambda \in \mathbb{C}$ is said to be a *μ-centralized fixed mode (μ-CFM)* of Σ if $\mathrm{Re}(\lambda) \geq \mu$ and

$$\det\left(\lambda I - \bar{A}(\lambda) - \bar{B}(\lambda)K\bar{C}(\lambda)\right) = 0$$

for all $K \in \mathbb{R}^{p\times q}$, where, similar to $\bar{A}(s)$, $\bar{B}(s)$ and $\bar{C}(s)$ are respectively obtained from the operator matrices $B(\tau)$ and $C(\tau)$, respectively defined in (4) and (5), by replacing the operator τ by the function e^{-hs}.

Definition 4. $\lambda \in \mathbb{C}$ is said to be a *μ-decentralized fixed mode (μ-DFM)* of Σ if $\mathrm{Re}(\lambda) \geq \mu$ and

$$\det\left(\lambda I - \bar{A}(\lambda) - \sum_{j=1}^{\nu} \bar{B}_j(\lambda)K_j\bar{C}_j(\lambda)\right) = 0$$

for all $K_j \in \mathbb{R}^{p_j\times q_j}$, $j = 1, \ldots, \nu$, where $\bar{B}_j(s)$ and $\bar{C}_j(s)$ are respectively obtained from the operator matrices $B_j(\tau)$ and $C_j(\tau)$, respectively defined in (2) and (3), by replacing the operator τ by the function e^{-hs}.

Note that, λ is a μ-CFM or a μ-DFM of Σ, only if it is a μ-mode of Σ. For any finite $\mu \in \mathbb{R}$, μ-CFMs and μ-DFMs of a retarded time-delay system, like Σ, can be determined by finite amount of computation (e.g., see Erol and İftar (2013)).

The objective in this work is to introduce a controller synthesis technique to design decentralized controllers, where only feedback from y_j to u_j is allowed for $j = 1, \ldots, \nu$, so that the closed-loop system is μ-stable, for some given real μ (normally $\mu \leq 0$). For this purpose, we propose to adapt the *decentralized pole assignment algorithm* of Davison and Chang (1990) to the present case. In this algorithm a centralized controller synthesis is used for each control agent sequentially. Thus, to use this algorithm, we first need to adopt a centralized controller synthesis algorithm. For this, we propose to use the *continuous pole placement algorithm*, introduced by Michiels et al. (2002). This algorithm was originally presented for static state vector feedback controllers. However, in our case, the whole state vector is not generally available to any control agent. Furthermore, using static feedback almost never produce useful results to control a decentralized time-delay system. Therefore, in the next section, we first extend the continuous pole placement algorithm of Michiels et al. (2002) to the case of centralized dynamic output feedback controllers.

3. CENTRALIZED DYNAMIC OUTPUT FEEDBACK CONTROLLER DESIGN BY CONTINUOUS POLE PLACEMENT

As mentioned at the end of the previous section, the purpose of the present section is to extend the continuous pole placement algorithm of Michiels et al. (2002) to the case of centralized dynamic output feedback controllers. Throughout this section, we will consider a centralized retarded commensurate-time-delay system of the form (6), where the whole output, $y(t)$, is available for feedback to the whole input $u(t)$. The controllers we will consider are of the following form:

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gy(t) \\ u(t) &= Hz(t) + Ky(t) \end{aligned} \tag{8}$$

where $z(t) \in \mathbb{R}^m$ is the state vector of the controller at time t, where $m \in \mathbb{N}$ is the controller dimension, and $F \in \mathbb{R}^{m\times m}$, $G \in \mathbb{R}^{m\times q}$, $H \in \mathbb{R}^{p\times m}$, and $K \in \mathbb{R}^{p\times q}$. Note that when $m = 0$, such a controller reduces to a centralized static output feedback controller.

It was shown by Kamen et al. (1985) that a system of the form (6) can be μ-stabilized by a controller of the form (8) (with a sufficiently large m) if and only if the system does not have any μ-CFMs. Since our final aim is to use the centralized synthesis approach of the present section in a decentralized framework, even though the overall system can be μ-stabilized (i.e., it does not have any μ-DFMs - see Erol and İftar (2013)), the system from a particular input channel to the corresponding output channel may have μ-CFMs. To mitigate the problem caused by the μ-CFMs, we will first obtain the *controllable and observable part* of the given system, and then apply the stabilization algorithm to this part only. To identify the controllable and observable part, we first need to present the following definition and lemma from Lee et al. (1982).

Definition 5. The system (6), equivalently the pair $(A(\cdot), B(\cdot))$, is said to be *controllable* if the matrix

$$\left[B(\tau) \quad A(\tau)B(\tau) \quad \ldots \quad (A(\tau))^{n-1}B(\tau)\right]$$

is full rank over $\mathbb{R}[\tau]$. Also, the system (6), equivalently the pair $(C(\cdot), A(\cdot))$, is said to be *observable* if the matrix

$$\left[C^T(\tau) \;\; A^T(\tau)C^T(\tau) \;\; \ldots \;\; \left(A^T(\tau)\right)^{n-1} C^T(\tau) \right]$$

is full rank over $\mathbb{R}[\tau]$. Furthermore, the triple $(C(\cdot), A(\cdot), B(\cdot))$ is said to be *controllable and observable* if the pair $(A(\cdot), B(\cdot))$ is controllable and the pair $(C(\cdot), A(\cdot))$ is observable.

Lemma 1. Consider the system (6). There exist a unimodular transformation matrix $T(\tau) \in \mathbb{R}[\tau]^{n\times n}$ such that the transformed system has the following canonical form

$$\begin{bmatrix} \dot{x}_{co}(t) \\ \dot{x}_{c\bar{o}}(t) \\ \dot{x}_{\bar{c}}(t) \end{bmatrix} = \begin{bmatrix} A_{co}(\tau) & 0 & A_{13}(\tau) \\ A_{21}(\tau) & A_{c\bar{o}}(\tau) & A_{23}(\tau) \\ 0 & 0 & A_{\bar{c}}(\tau) \end{bmatrix} \begin{bmatrix} x_{co}(t) \\ x_{c\bar{o}}(t) \\ x_{\bar{c}}(t) \end{bmatrix} + \begin{bmatrix} B_{co}(\tau) \\ B_{c\bar{o}}(\tau) \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \left[\; C_{co}(\tau) \;\; 0 \;\; C_{\bar{c}}(\tau) \;\right] \begin{bmatrix} x_{co}(t) \\ x_{c\bar{o}}(t) \\ x_{\bar{c}}(t) \end{bmatrix}$$

where the triple $\left(C_{co}(\cdot), A_{co}(\cdot), B_{co}(\cdot)\right)$ is controllable and observable. Furthermore, the system (6) is zero-state equivalent to the system

$$\begin{aligned} \dot{x}_{co}(t) &= A_{co}(\tau)x_{co}(t) + B_{co}(\tau)u(t) \\ y(t) &= C_{co}(\tau)x_{co}(t) \end{aligned} \tag{9}$$

which is both controllable and observable.

Proof. See Lee et al. (1982). □

We will refer to the system (9) as the *controllable and observable part* of the system (6). The modes of the rest of the system, i.e., the roots of $\det\left(sI - \bar{A}_{c\bar{o}}(s)\right) \det\left(sI - \bar{A}_{\bar{c}}(s)\right) = 0$, will be called *structural fixed modes* (SFMs) of the system, where $\bar{A}_{c\bar{o}}(s)$ and $\bar{A}_{\bar{c}}(s)$ are respectively obtained from the operator matrices $A_{c\bar{o}}(\tau)$ and $A_{\bar{c}}(\tau)$ by replacing the operator τ by the function e^{-hs}. Note that, any SFM is a CFM of the original system (6). Unlike, finite-dimensional systems, however, the controllable and observable part of the system may still have fixed modes, which are also CFMs of the original system (6) (Erol (2014)). Here, these modes will be called *unstructural fixed modes* (UFMs).

During the continuous pole placement algorithm, when a controlled mode approaches to a fixed mode, the sensitivity of the controlled mode with respect to changes in the controller parameters become considerably small and this causes very large changes in the controller parameters even for very small desired displacements for the controlled modes (Michiels et al. (2002)). This is one of the major reasons for the failure of the continuous pole placement algorithm. By using the transformation given in Lemma 1, SFMs can be separated from the system. Thus, this problem will be avoided for SFMs by using only the controllable and observable part in the stabilization algorithm instead of the whole system. However, it should be noted that, even if such a decomposition is done, UFMs, if any, may still cause problem. Presence of a real μ-CFM to the left of any real controlled mode may result in the failure of the stabilization algorithm. Because, when only the real parts of modes are controlled, approach of a real mode to this μ-CFM would cause sensitivity matrices with considerably small norms which result in very large changes in the controller parameters. Also, a similar situation occurs for any real transmission zero located between μ and any real controlled mode (Erol (2014)). In the rest of the paper, we will assume that the controllable and observable part (9) does not have any μ-CFMs (i.e., the system (6) does not have any UFMs with real part greater than or equal to μ) or any real transmission zeros greater than or equal to μ.

Before presenting our algorithm, let us first define

$$\bar{A}^e_{co}(s) := \begin{bmatrix} \bar{A}_{co}(s) & 0 \\ 0 & 0_{m\times m} \end{bmatrix}, \quad \bar{B}^e_{co}(s) := \begin{bmatrix} \bar{B}_{co}(s) & 0 \\ 0 & I_m \end{bmatrix},$$

and $\bar{C}^e_{co}(s) := \begin{bmatrix} \bar{C}_{co}(s) & 0 \\ 0 & I_m \end{bmatrix}$, where $\bar{A}_{co}(s)$, $\bar{B}_{co}(s)$, and $\bar{C}_{co}(s)$ are respectively obtained from the operator matrices $A_{co}(\tau)$, $B_{co}(\tau)$, and $C_{co}(\tau)$, appearing in (9), by replacing the operator τ by the function e^{-hs}. Also define $K^e := \begin{bmatrix} K & H \\ G & F \end{bmatrix} \in \mathbb{R}^{(m+p)\times(m+q)}$, where F, G, H, and K are the matrices of the controller defined by (8), which are structured in a canonical form (e.g., see Chapter 6 of Chen (1984)) with

$$\hat{m} := m(p+q) + pq \tag{10}$$

free parameters. Let $\tilde{K}^e \in \mathbb{R}^{\hat{m}}$ be the vector of the free parameters, $\tilde{k}^e_1, \ldots, \tilde{k}^e_{\hat{m}}$, of the controller.

The characteristic function of the closed-loop system (obtained by applying the controller (8) to the system (9)) is then obtained as

$$\phi_{\Sigma,K}(s) = \det\left[sI - \bar{A}^e_{co}(s) - \bar{B}^e_{co}(s)K^e\bar{C}^e_{co}(s)\right]. \tag{11}$$

Therefore, finding a controller (8) which μ-stabilizes (9) is equivalent to finding a $\tilde{K}^e \in \mathbb{R}^{\hat{m}}$ such that all roots of $\phi_{\Sigma,K}(s) = 0$ have real parts less than μ. In the sequel we will refer to the roots of $\phi_{\Sigma,K}(s) = 0$ with $\mathrm{Re}(s) \geq \mu$ as the *μ-roots* of (11). Now, let $s_i \in \mathbb{C}$ be a mode of the closed-loop system, i.e., $\phi_{\Sigma,K}(s_i) = 0$. Then

$$\begin{aligned} \left(s_iI - \bar{A}^e_{co}(s_i) - \bar{B}^e_{co}(s_i)K^e\bar{C}^e_{co}(s_i)\right) v_i &= 0 \\ N(v_i) &= 1 \end{aligned} \tag{12}$$

where $v_i \in \mathbb{C}^{n_{co}+m}$ is a non-zero vector, where n_{co} is the dimension of x_{co} in (9), and $N(\cdot)$ is a normalizing function, for example, one can choose $N(v) = v^*v$. Differentiating (12) with respect to a component $\tilde{k}^e_\psi$ $\left(\psi = 1, \ldots, \hat{m}\right)$ of $\tilde{K}^e$, we obtain a linear system of equations as follows

$$\begin{aligned} &\left(s_iI - \bar{A}^e_{co}(s_i) - \bar{B}^e_{co}(s_i)K^e\bar{C}^e_{co}(s_i)\right)\frac{\partial v_i}{\partial \tilde{k}^e_\psi} \\ &+\left(I - \frac{\partial \bar{A}^e_{co}(s_i)}{\partial s_i} - \frac{\partial \bar{B}^e_{co}(s_i)}{\partial s_i}K^e\bar{C}^e_{co}(s_i) - \bar{B}^e_{co}(s_i)K^e\frac{\partial \bar{C}^e_{co}(s_i)}{\partial s_i}\right)v_i\frac{\partial s_i}{\partial \tilde{k}^e_\psi} \\ &-\left(\bar{B}^e_{co}(s_i)\frac{\partial K^e}{\partial \tilde{k}^e_\psi}\bar{C}^e_{co}(s_i)\right)v_i = 0 \end{aligned}$$

and

$$\left(\frac{\partial N(v_i)}{\partial v_i}\right)\frac{\partial v_i}{\partial \tilde{k}^e_\psi}=0 \ ,$$

which represent $n_{co}+m+1$ equations in $n_{co}+m+1$ unknowns, where the unknowns are $\frac{\partial s_i}{\partial \tilde{k}^e_\psi}$ and the $n_{co}+m$ components of $\frac{\partial v_i}{\partial \tilde{k}^e_\psi}$.

Assume that $k \le \hat{m}$ modes, say $s_1,\dots,s_k$, are desired to be shifted towards the μ-stable region $\mathbb{C}^-_\mu$. These modes will be referred to as the *controlled modes*. Now define the *sensitivity matrix* Θ_k as follows

$$\Theta_k := [\theta_{i,\psi}] \in \mathbb{C}^{k\times\hat{m}} \ \text{ where } \ \theta_{i,\psi} := \frac{\partial s_i}{\partial \tilde{k}^e_\psi} \ . \tag{13}$$

Let $\Delta\tilde{S}^d_k := \left[\Delta s^d_1 \ \dots \ \Delta s^d_k\right]^T \in \mathbb{C}^k$ be the desired displacement of the k controlled modes. Assuming that $\Delta\tilde{S}^d_k$ is in the range space of Θ_k, the corresponding change $\Delta\tilde{K}^e$ for $\tilde{K}^e$ can be computed from

$$\Theta_k\Delta\tilde{K}^e = \Delta\tilde{S}^d_k \ . \tag{14}$$

We note that $\Delta\tilde{S}^d_k$ must be chosen such that all elements of $\Delta\tilde{K}^e$ are real. This is achieved by choosing $\Delta\tilde{S}^d_k$ such that complex-conjugate modes remain as complex-conjugate or both become real and no real mode becomes a complex mode unless another real mode becomes its complex-conjugate. As in Michiels et al. (2002), when rank$(\Theta_k)=k$, a solution to (14), with minimal $\|\Delta\tilde{K}^e\|$, is given by

$$\Delta\tilde{K}^e = \Theta^\dagger_k\Delta\tilde{S}^d_k \ , \tag{15}$$

where $\Theta^\dagger_k$ is the Moore-Penrose inverse of Θ_k (Penrose and Todd (1956)).

Now, we can present the following.

Algorithm 1.

1) Initialize the controller dimension as $m=0$.
2) Initialize $\tilde{K}^e = 0_{\hat{m}\times 1}$, where $\hat{m}$ is as in (10).
3) Compute (e.g., by the method of Wu and Michiels (2012)) the $(\mu-\varepsilon)$-roots of (11) for some $\varepsilon>0$. If there are no μ-roots, stop: μ-stability is achieved with the current $\tilde{K}^e$. Otherwise, let η be the real part of the rightmost root and k be the number of roots with real part greater than or equal to $\eta-\varepsilon$ (note that $k\ge 1$). If $k>\hat{m}$, increase m so that $k\le\hat{m}$ and go to step 2. Otherwise, define the rightmost k roots as the *controlled modes* and continue with step 4.
4) Compute the sensitivity matrix, Θ_k, defined in (13). Let $\rho := \text{rank}\,(\Theta_k)$.
5) If $\rho=k$, choose the desired displacement of the k controlled modes, $\Delta\tilde{S}^d_k$, such that all k controlled modes move towards $\mathbb{C}^-_\mu$. Compute $\Delta\tilde{K}^e$ by (15) and go to step 7.
6) If $\rho<k$, check if a $\Delta\tilde{S}^d_k$ in the range space of Θ_k can be chosen so that all k controlled modes move towards $\mathbb{C}^-_\mu$. If so, using this $\Delta\tilde{S}^d_k$, compute a suitable $\Delta\tilde{K}^e$ which satisfies (14) and go to step 7. Otherwise, increase the controller dimension m by one and go to step 2.
7) Update $\tilde{K}^e$ as $\tilde{K}^e+\Delta\tilde{K}^e$ and go to step 3.

4. PROPOSED DECENTRALIZED CONTROLLER DESIGN APPROACH

As stated in Section 2, our objective is to design decentralized controllers for the system Σ, described by (1), so that the closed-loop system is μ-stable. The controllers we will consider for this purpose are of the form

$$\begin{aligned}\dot{z}_j(t) &= F_jz_j(t)+G_jy_j(t)\\ u_j(t) &= H_jz_j(t)+K_jy_j(t)\end{aligned} \ , \tag{16}$$

for $j=1,\dots,\nu$, where $z_j(t)\in\mathbb{R}^{m_j}$ is the state vector of the j^{th} controller at time t, where $m_j\in\mathbb{N}$ is the dimension of the j^{th} controller, and $F_j\in\mathbb{R}^{m_j\times m_j}$, $G_j\in\mathbb{R}^{m_j\times q_j}$, $H_j\in\mathbb{R}^{p_j\times m_j}$ and $K_j\in\mathbb{R}^{p_j\times q_j}$. Note that when $m_j=0$, for any j, the controller for the j^{th} control agent reduces to a static output feedback controller. It was proven by Momeni et al. (2010) that a decentralized retarded time-delay system Σ of the form (1) can be μ-stabilized by decentralized controllers of the form (16) if and only if Σ has no μ-DFMs.

Now, suppose that decentralized controllers of the form (16) has been designed for the first r control agents, where $r<\nu$. Let $m^r := \sum_{j=1}^r m_j$, $p^r := \sum_{j=1}^r p_j$, and $q^r := \sum_{j=1}^r q_j$. Define

$$\hat{B}_r(\tau) := [\,B_1(\tau)\ B_2(\tau)\ \dots\ B_r(\tau)\,] \ ,$$

$$\hat{C}_r(\tau) := \left[\,C_1^T(\tau)\ C_2^T(\tau)\ \dots\ C_r^T(\tau)\,\right]^T \ ,$$

$$\hat{A}^e_r(\tau) := \begin{bmatrix} A(\tau) & 0\\ 0 & 0_{m^r\times m^r}\end{bmatrix} \ , \ \hat{B}^e_r(\tau) := \begin{bmatrix}\hat{B}_r(\tau) & 0\\ 0 & I_{m^r}\end{bmatrix} \ ,$$

$$\hat{C}^e_r(\tau) := \begin{bmatrix}\hat{C}_r(\tau) & 0\\ 0 & I_{m^r}\end{bmatrix} \ ,$$

and $\hat{K}^e_r := \begin{bmatrix}\hat{K}_r & \hat{H}_r\\ \hat{G}_r & \hat{F}_r\end{bmatrix} \in \mathbb{R}^{(m^r+p^r)\times(m^r+q^r)}$, where $\hat{F}_r :=$ bdiag$\,(F_1,\dots,F_r)$, $\hat{G}_r :=$ bdiag$\,(G_1,\dots,G_r)$, $\hat{H}_r :=$ bdiag$\,(H_1,\dots,H_r)$, and $\hat{K}_r :=$ bdiag$\,(K_1,\dots,K_r)$. Also define $\xi_r(t) := \left[\,x^T(t)\ z_1^T(t)\ \cdots\ z_r^T(t)\,\right]^T \in \mathbb{R}^{n+m^r}$.

Then, for the $(r+1)^{\text{th}}$ control agent, the resultant system, with input u_{k+1} and output y_{k+1}, is described by

$$\begin{aligned}\dot{\xi}_r(t) &= \left(\hat{A}^e_r(\tau)+\hat{B}^e_r(\tau)\hat{K}^e_r\hat{C}^e_r(\tau)\right)\xi_r(t)\\ &\quad + \begin{bmatrix}B_{r+1}(\tau)\\ 0_{m^r\times p_{r+1}}\end{bmatrix}u_{r+1}(t)\\ y_{r+1}(t) &= \left[C_{r+1}(t)\quad 0_{q_{r+1}\times m^r}\right]\xi_r(t)\end{aligned} \tag{17}$$

We will denote the above system by Σ_r. Note that Σ_0, as described above, is same as the system Σ with only u_1 as its input and y_1 as its output. Also note that, Σ_ν denotes the overall closed-loop system with all the control loops closed. Since Σ_r, for $r=0,\dots,\nu-1$, is a centralized control system, it can be decomposed as in Lemma 1 and its controllable and observable part can be obtained. We will denote the controllable and observable part of Σ_r by Σ^{co}_r. Now, we propose the following algorithm to design decentralized controllers for the system Σ in order to μ-stabilize it.

Algorithm 2.

1) Fix upper limits, $\bar{m}_1,\dots,\bar{m}_\nu$, on the dimensions of the decentralized controllers.

2) Let $r = 0$.
3) If Σ_r^{co} is μ-stable, let $m_{r+1} = 0$ and choose a random non-zero $K_{r+1} \in \mathbb{R}^{p_{r+1} \times q_{r+1}}$ such that the closed-loop system obtained by applying the static output feedback $u_{r+1}(t) = K_{r+1} y_{r+1}(t)$ to Σ_r^{co} is μ-stable (by the continuity of the modes with respect to the feedback gains, there exists such a K_{r+1} - see Momeni and Aghdam (2008)) and go to step 5. Otherwise, continue with step 4.
4) Apply Algorithm 1 to Σ_r^{co} to design a controller of the form (16) with $j = r+1$ of dimension not greater than $\bar{m}_{r+1}$ to μ-stabilize it. If such a controller can not be designed, use the last controller with dimension $\bar{m}_{r+1}$ which moves as many controlled modes as possible towards $\mathbb{C}_\mu^-$.
5) If $r = \nu - 1$ go to step 6. Otherwise, set $r = r + 1$ and go to step 3.
6) If the overall closed-loop system Σ_ν is μ-stable, stop: the desired decentralized controller has been obtained. Otherwise, increase the upper limits, $\bar{m}_1, \ldots,$ $\bar{m}_\nu$, and go to step 2.

The above algorithm is an extension of the decentralized pole assignment algorithm of Davison and Chang (1990) to the time-delay case, where the continuous pole placement algorithm of Michiels et al. (2002), as extended in Section 3, is used to design a centralized controller for each control agent at each step. The reason for choosing upper limits in step 1 is to avoid using unnecessarily high-dimensional controllers for the lower indexed control agents. The reason for applying a static output feedback controller in step 3, whenever Σ_r^{co} is μ-stable, is to make sure that any μ-mode of Σ, which is not a μ-DFM, is a mode of Σ_s^{co}, for some $s > r$ (so that it can be eventually moved towards $\mathbb{C}_\mu^-$). As indicated by Davison and Chang (1990), if such a feedback loop is not closed, some μ-modes may not appear as the modes of Σ_r^{co} for any r, even if they are not μ-DFMs.

Remark. A reviewer for the present paper has indicated that the results of Ravi et al. (1995) might be useful in determining the upper limits, $\bar{m}_1, \ldots, \bar{m}_\nu$, in step 1. These results, however, are valid only for finite-dimensional systems and it is not apparent how to extend them to the time-delay case. The same reviewer also questioned whether, rather than using the approach of Davison and Chang (1990), it would be possible to use an approach, such as the one by Stanković et al. (2007), which designs all the controllers simultaneously. This is of course possible, but is beyond the scope of the present work.

5. EXAMPLE

Let us consider a LTI retarded time-delay system described as in (1) with $\nu = 2$, $\sigma = 1$, $h = h_1 = 1$,

$$A_0 = \begin{bmatrix} 7 & 9 & 7 & 9 \\ 0 & -1 & 4 & -2 \\ -11 & -6 & -7 & -11 \\ -22 & -12 & -4 & -27 \end{bmatrix}, \quad B_{1,0} = \begin{bmatrix} -4 \\ -3 \\ 2 \\ 4 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -4 & 6 & -8 & -1 \\ 0 & 4 & 0 & 0 \\ 5 & -3 & 9 & 1 \\ 10 & -6 & 6 & 8 \end{bmatrix}, \quad B_{1,1} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix},$$

$$B_{2,0} = [\,3\ 0\ -3\ -5\,]^T, \quad B_{2,1} = [\,1\ 0\ -1\ -1\,]^T,$$

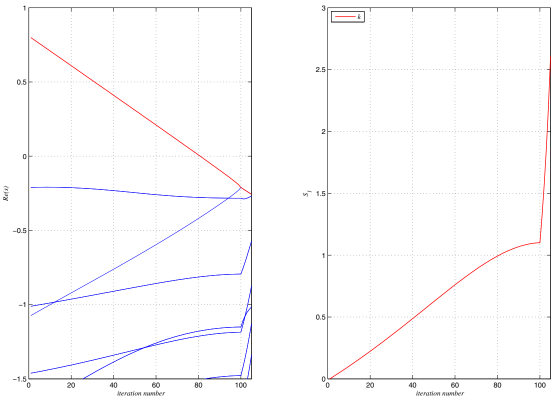

Fig. 1. Real parts of the rightmost modes (left) and the controller parameter (right), as a function of the iterations, for the first control agent.

$$C_{1,0} = [\,0\ 1\ 4\ -2\,], \quad C_{1,1} = [\,1\ -1\ 1\ 0\,],$$

$$C_{2,0} = [\,1\ -1\ 2\ -0.5\,], \quad C_{2,1} = [\,1\ 0\ 0\ 1\,].$$

By using the programs of Wu and Michiels (2012), for $\epsilon = 1$, we obtain the $-\epsilon$-modes of the system as $\Omega_{-1}(\Sigma) = \{0.7990, 0.1523, -0.1904 \pm 5.4367\text{i}, -0.2104 \pm 4.8730\text{i}, -0.7049 \pm 11.3571\text{i}\}$. Since the system has two modes, $s_1 = 0.7990$ and $s_2 = 0.1523$, with non-negative real parts, the system is not μ-stable for $\mu = 0$. Furthermore, s_1 is a CFM for control agent 2 and s_2 is a CFM for control agent 1. Hence, the system is not stabilizable by any one of the control agents alone. However, neither s_1, nor s_2 is a DFM, hence it is possible to μ-stabilize the system by decentralized feedback. To obtain a stabilizing decentralized controller, we apply Algorithm 2.

In the first phase, we first obtain the controllable and observable part, Σ_0^{co}, of the system, Σ_0, seen by the first control agent. We note that, although $s_2 = 0.1523$ is a 0-CFM of Σ_0, it is a SFM, and hence does not appear as a mode of Σ_0^{co}. The only unstable mode of Σ_0^{co} is $s_1 = 0.7990$. Next, by applying Algorithm 1 to Σ_0^{co}, the following stabilizing controller, with dimension $m = 0$, is obtained:

$$u_1(t) = 2.6072\, y_1(t)\,. \tag{18}$$

The progress of the algorithm, i.e., the real parts of the rightmost modes and the controller parameter as a function of the iterations, is shown in Fig. 1.

In the second phase, we first close the loop formed by the first control agent by using controller (18) to obtain the system Σ_1 seen by the second control agent. We then obtain the controllable and observable part, Σ_1^{co}, of this system. We note that $s_2 = 0.1523$ is the only unstable mode of both Σ_1 and Σ_1^{co}. For Σ_1^{co}, Algorithm 1 fails to find a stabilizing controller with dimension $m = 0$. Algorithm 1 also fails to find a stabilizing controller, in the controllable canonical form, with dimension $m = 1$ and $m = 2$. However, with $m = 3$, the controller

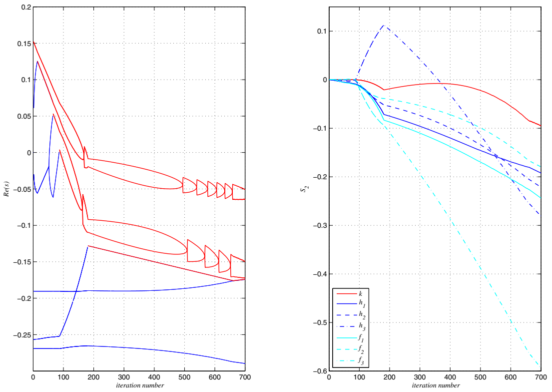

Fig. 2. Real parts of the rightmost modes (left) and the controller parameters (right), as a function of the iterations, for the second control agent.

$$\dot{z}_2(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2440 & -0.1796 & -0.5940 \end{bmatrix} z_2(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y_2(t)$$
$$u_2(t) = [\,-0.1922 \;\; -0.2215 \;\; -0.2812\,]\, z_2(t) - 0.0949\; y_2(t) \tag{19}$$

is found to stabilize Σ_1^{co}. The progress of the algorithm is shown in Fig. 2.

The overall closed-loop system, Σ_2, following the application of the decentralized controllers (18) and (19) to the original system Σ is then obtained. -1-modes of this system are computed as $\Omega_{-1}(\Sigma_2) = \{-0.05080, -0.06406, -0.1495, -0.1724, -0.1741 \pm 0.6199i, -0.1743 \pm 5.4364i, -0.2897 \pm 5.9614i, -0.5837 \pm 11.767i, -0.6897 \pm 11.355i, -0.8814 \pm 17.805i\}$. It is seen that the closed-loop system does not have any modes with non-negative real parts. Hence, the decentralized controllers (18) and (19) stabilize the given system.

6. CONCLUSION

Decentralized controller design problem for LTI retarded commensurate-time-delay systems has been considered. The only reason we considered the case of commensurate time-delays, rather than the more general case of incommensurate time-delays, is that the decomposition presented in Lemma 1 is not possible for the incommensurate case in general. If this decomposition is not used, all the results of the present work directly extends to systems with incommensurate time-delays. As mentioned in Section 3, the current work assumes that the controllable and observable part (9) does not have any μ-CFMs or any real transmission zeros greater than or equal to μ. Alternative approaches can, however, be developed (see Erol (2014)) when this assumption fails. Another possible line of further research is to consider time-delay (or other types of infinite-dimensional) controllers instead of finite-dimensional controllers of the form (8) or (16). Although a centralized time-delay system of the form (6) can be μ-stabilized by a time-delay controller if and only if it can be μ-stabilized by a finite-dimensional controller of the form (8) (Kamen et al. (1985)) and a decentralized time-delay system of the form (1) can be μ-stabilized by decentralized time-delay controllers if and only if it can be μ-stabilized by finite-dimensional controllers of the form (16) (Erol and İftar (2013)), as mentioned by Erol and İftar (2013), use of time-delay controllers may sometimes be advantageous.

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