Parametric Model Order Reduction Using Pseudoinverses for the Matrix Interpolation of Differently Sized Reduced Models

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Abstract: This paper deals with a framework of model order reduction for high-order parametric, linear systems. A set of low-order nonparametric systems with different reduced orders is computed for sample points. Then, two approaches are presented applying pseudoinverses for the introduction of generalized coordinates. Finally, a reduced system for a new parameter value is obtained by interpolating differently sized system matrices. The paper extends current methods to the general case where the local systems have different reduced orders.

Keywords: model reduction; large-scale systems; parameter-dependent systems; modeling.

1. INTRODUCTION

Large-scale systems of ordinary differential equations can arise when dynamical systems are to be modeled accurately. However, the application of high-dimensional systems to optimization, simulation or control fails owing to the high numerical effort. Therefore, the high-order systems have to be approximated by low-order systems by using methods of model order reduction, see e.g. Antoulas (2005). If the models additionally depend on parameters, for example geometry parameters, parametric model order reduction (pMOR) is applied to reduce the order of the large-scale system. Then, the reduced model still possesses the parameter dependencies so that, if a low-order system is to be computed for a new parameter value, the original system does not have to be reduced again.

Besides other methods for pMOR, interpolation based methods were suggested in the literature. They interpolate between a discrete set of low-order systems which are obtained by reducing the original system at different parameter vectors. The reduced systems can have different reduced orders which can e.g. arise if low-order models of similar accuracy for each sampling point are to be calculated. This can be the case for Truncated Balanced Realization with the same error bound for every discrete system. Another occasion is when systems shall be more accurate for some parameter values than for others. The interpolation of the transfer functions of the locally reduced systems was proposed in Baur, and Benner (2009). This procedure allows to interpolate local systems which have different reduced orders. Then, the order of the interpolated system is the sum of the orders of the local systems. Another possibility is the interpolation

of the system matrices of locally reduced models which was proposed in Panzer et al. (2010); Degroote et al. (2010); Amsallem, and Farhat (2011); Geuss et al. (2013). However, the interpolation is so far only possible if the system matrices have the same size. Hence, the method currently has the constraint that the local systems share the same reduced order. We extend the current methods of pMOR by matrix interpolation to the general case where the local systems have different reduced orders. In Section 2 preliminaries are presented. The method of pMOR by matrix interpolation with differently sized local systems is proposed in Section 3. Numerical results are shown in Section 4, followed by conclusions in Section 5.

2. PRELIMINARIES

2.1 Linear time-invariant dynamical systems

In this paper, we consider the linear time-invariant (LTI), parameter-dependent system $G(\mathbf{p})$ in descriptor form:

$$G(\mathbf{p}) : \begin{cases} \mathbf{E}(\mathbf{p})\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p})\mathbf{x}(t) + \mathbf{B}(\mathbf{p})\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}(\mathbf{p})\mathbf{x}(t), \end{cases} \quad (1)$$

where $\mathbf{E}(\mathbf{p}) \in \mathbb{R}^{n \times n}$, $\mathbf{A}(\mathbf{p}) \in \mathbb{R}^{n \times n}$, $\mathbf{B}(\mathbf{p}) \in \mathbb{R}^{n \times r}$, and $\mathbf{C}(\mathbf{p}) \in \mathbb{R}^{m \times n}$ are the system matrices which depend on the vector of parameters $\mathbf{p} \in \mathcal{D}$ with parameter domain $\mathcal{D} \subset \mathbb{R}^d$. Vectors $\mathbf{u}(t) \in \mathbb{R}^r$, $\mathbf{y}(t) \in \mathbb{R}^m$, and $\mathbf{x}(t) \in \mathbb{R}^n$ denote inputs, outputs, and states of the system at time t. In this paper, system $G(\mathbf{p})$ will be computed for k values of parameter vector $\mathbf{p}_i$ with $i = 1, \ldots, k$, where $G_i := G(\mathbf{p}_i)$.

2.2 Projection-based model order reduction

Consider a high-dimensional system G_i. It shall be approximated by a low-dimensional system of order $q_i \ll n$ using Petrov-Galerkin projection. The projection matrices

$\mathbf{V}_i := \mathbf{V}(\mathbf{p}_i) \in \mathbb{R}^{n \times q_i}$ and $\mathbf{W}_i := \mathbf{W}(\mathbf{p}_i) \in \mathbb{R}^{n \times q_i}$ are suitably chosen. They are referred to as right and left reduced order bases (ROBs) and span the right subspace $\mathcal{V}_i := \mathcal{V}(\mathbf{p}_i)$ and left subspace $\mathcal{W}_i := \mathcal{W}(\mathbf{p}_i)$. This leads with $\mathbf{x}_i(t) \approx \mathbf{V}_i \mathbf{x}_{r,i}(t)$ to the reduced order model:

$$G_{r,i} : \begin{cases} \mathbf{E}_{r,i}\dot{\mathbf{x}}_{r,i}(t) = \mathbf{A}_{r,i}\mathbf{x}_{r,i}(t) + \mathbf{B}_{r,i}\mathbf{u}(t), \\ \mathbf{y}_{r,i}(t) = \mathbf{C}_{r,i}\mathbf{x}_{r,i}(t), \end{cases} \quad (2)$$

where $\mathbf{x}_{r,i}(t) \in \mathbb{R}^{q_i}$ is the reduced state vector and

$$\begin{aligned} \mathbf{E}_{r,i} &= \mathbf{W}_i^T \mathbf{E}_i \mathbf{V}_i, \\ \mathbf{A}_{r,i} &= \mathbf{W}_i^T \mathbf{A}_i \mathbf{V}_i, \\ \mathbf{B}_{r,i} &= \mathbf{W}_i^T \mathbf{B}_i, \\ \mathbf{C}_{r,i} &= \mathbf{C}_i \mathbf{V}_i. \end{aligned} \quad (3)$$

For detailed information on projection-based reduction methods, e.g. Krylov subspace methods, Truncated Balanced Realization (TBR) or Proper Orthogonal Decomposition (POD), please refer to Antoulas (2005) and references therein.

2.3 Pseudoinverse

As we will employ the pseudoinverse as introduced by Moore (1920), Bjerhammar (1951), and Penrose (1955) in this paper, the nomenclature is introduced in this section. Consider a matrix $\mathbf{Q} \in \mathbb{R}^{a \times b}$. If its columns are linearly independent for $a \geq b$, the left inverse $\mathbf{Q}^{(l)+} \in \mathbb{R}^{b \times a}$ with $\mathbf{Q}^{(l)+}\mathbf{Q} = \mathbf{I}$ is denoted with:

$$\mathbf{Q}^{(l)+} = (\mathbf{Q}^T \mathbf{Q})^{-1}\mathbf{Q}^T. \quad (4)$$

If matrix $\mathbf{Q}$ has linearly independent rows for $a \leq b$, the right inverse $\mathbf{Q}^{(r)+} \in \mathbb{R}^{b \times a}$ with $\mathbf{Q}\mathbf{Q}^{(r)+} = \mathbf{I}$ is:

$$\mathbf{Q}^{(r)+} = \mathbf{Q}^T (\mathbf{Q}\mathbf{Q}^T)^{-1}. \quad (5)$$

For the calculation of the pseudoinverse there exists a numerically more efficient approach using Singular Value Decomposition (SVD), see Shinozaki et al. (1972). It calculates the SVD of $\mathbf{Q} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{Z}^T$, where $\boldsymbol{\Sigma}$ is a diagonal matrix containing the singular values. Then $\boldsymbol{\Sigma}^{(r/l)+}$ is the transpose of $\boldsymbol{\Sigma}$ with reciprocal singular values leading to the (right/left) pseudoinverse $\mathbf{Q}^{(r/l)+}$:

$$\mathbf{Q}^{(r/l)+} = \mathbf{Z}\boldsymbol{\Sigma}^{(r/l)+}\mathbf{U}^T. \quad (6)$$

In this paper we use formula (6) for the calculation of the pseudoinverse. One application of the pseudoinverse is the solution of the matrix equality:

$$\mathbf{Q}\mathbf{X} = \mathbf{Y}, \quad (7)$$

with $\mathbf{X} \in \mathbb{R}^{b \times o}, \mathbf{Y} \in \mathbb{R}^{a \times o}$. A solution exists for matrix equation (7) if and only if the Penrose condition $\mathbf{Q}\mathbf{Q}^{(r/l)+}\mathbf{Y} = \mathbf{Y}$ is fulfilled, see Penrose (1955). Then, the solution with minimal $\|\mathbf{X}\|_F$ is $\mathbf{X} = \mathbf{Q}^{(r/l)+}\mathbf{Y}$, where $\|\cdot\|_F$ denotes the Frobnius norm. According to Penrose (1956), if the Penrose condition is not fulfilled, $\mathbf{X} = \mathbf{Q}^{(r/l)+}\mathbf{Y}$ is an approximate solution of equation (7) with minimal $\|\mathbf{X}\|_F$ in the sense of the method of least squares minimizing the optimization problem:

$$\mathbf{X} = \arg\min_{\mathbf{X} \in \mathbb{R}^{b \times o}} \|\mathbf{Q}\mathbf{X} - \mathbf{Y}\|_F. \quad (8)$$

2.4 Problem formulation

Assume a set of k low-order systems $G_{r,i}$ with different reduced orders q_i with $i = 1, \dots, k$. Let $\tilde{\mathbf{p}} \in \mathcal{D}$ be a value in the parameter domain. Then, the aim is to find a low-order system $G_r(\tilde{\mathbf{p}})$ with system matrices $\mathbf{E}_r(\tilde{\mathbf{p}}), \mathbf{A}_r(\tilde{\mathbf{p}}), \mathbf{B}_r(\tilde{\mathbf{p}}), \mathbf{C}_r(\tilde{\mathbf{p}})$ at $\tilde{\mathbf{p}} \in \mathcal{D}$ approximating the high-order system $G(\tilde{\mathbf{p}})$ by interpolating differently sized system matrices $\mathbf{E}_{r,i}, \mathbf{A}_{r,i}, \mathbf{B}_{r,i}, \mathbf{C}_{r,i}$ of systems $G_{r,i}$.

3. MAIN RESULTS

In this section pMOR by interpolating system matrices is presented where reduced systems at sampling points have different orders. The following approach is based on the six step procedure from Geuss et al. (2013) but generalizes it to differently sized local models. We introduce two approaches for generalized coordinates with pseudoinverses.

3.1 Sampling of the parameter space

The parameter space is sampled for k vectors $\mathbf{p}_i \in \mathcal{D}$ with $i = 1, \dots, k$. The original systems are computed at the sampling points $\mathbf{p}_i$ which gives a set of k high-order systems $G_i := G(\mathbf{p}_i)$ with $i = 1, \dots, k$.

3.2 Reduction of the local systems

Every local system G_i is reduced to an individual reduced order $q_i \ll n$ applying an arbitrary projection-based reduction method according to Section 2.2. Thereby proper subspaces $\mathcal{V}_i$ and $\mathcal{W}_i$ are calculated which are spanned by $\mathbf{V}_i \in \mathbb{R}^{n \times q_i}$ and $\mathbf{W}_i \in \mathbb{R}^{n \times q_i}$ with linearly independent columns, respectively. This leads to a set of reduced systems $G_{r,i}$ with different reduced orders q_i, where $i = 1, \dots, k$.

3.3 Adjustment of the right ROBs

As projection matrices $\mathbf{V}_i \in \mathbb{R}^{n \times q_i}$ of systems $G_{r,i}$ do not span the same subspace, the reduced systems have to be expressed in a set of generalized coordinates $\hat{\mathbf{x}}_{r,i}(t) \in \mathbb{R}^{q_0}$ with respect to a reference subspace spanned by the columns of $\mathbf{R}_V \in \mathbb{R}^{n \times q_0}$. Generalized coordinates are introduced by state transformations $\mathbf{x}_{r,i}(t) = \mathbf{T}_i \hat{\mathbf{x}}_{r,i}(t)$ with $\mathbf{T}_i \in \mathbb{R}^{q_i \times q_0}$. This is equivalent to changing the right ROBs of systems $G_{r,i}$ to $\hat{\mathbf{V}}_i = \mathbf{V}_i \mathbf{T}_i \in \mathbb{R}^{n \times q_0}$ so that the right ROBs $\hat{\mathbf{V}}_i$ and the basis of the reference subspace $\mathbf{R}_V$ have a good correlation. We propose two approaches for evaluating the correlation of the basis vectors: a MAC approach and an equality approach:

3.3.1 MAC approach: The correlation between two vectors can be evaluated by the Modal Assurance Criterion (MAC), see Ewins (2000). We now demand that the corresponding vectors of $\hat{\mathbf{V}}_i$ and $\mathbf{R}_V$ are in good correlation with respect to the MAC. The maximal value of the MAC is 1, which corresponds to the best correlation, and the minimal value is 0. We define the MAC between the j-th vector of $\mathbf{R}_V$ and the l-th vector of $\hat{\mathbf{V}}_i$ as follows:

$$\text{MAC}(\mathbf{R}_V(:,j), \hat{\mathbf{V}}_i(:,l)) = |\mathbf{R}_V(:,j)^T \hat{\mathbf{V}}_i(:,l)|^2. \quad (9)$$

Then, diagonal elements of matrix $\mathbf{MAC} = \mathbf{R}_V^T \hat{\mathbf{V}}_i$ describe the square roots of the MACs between the corresponding vectors of $\hat{\mathbf{V}}_i$ and $\mathbf{R}_V$ and off-diagonal elements describe the square roots of the MACs between the non-corresponding vectors. Hence, we want the diagonal

elements of $\mathbf{R}_{\mathbf{V}}^T \hat{\mathbf{V}}_i$ to be maximal and the off-diagonal elements to be minimal by a proper choice of $\mathbf{T}_i$:

$$\mathbf{MAC}(\mathbf{R}_{\mathbf{V}}, \hat{\mathbf{V}}_i) = \mathbf{R}_{\mathbf{V}}^T \hat{\mathbf{V}}_i = \mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{T}_i \approx \mathbf{I}. \qquad (10)$$

Let the reference subspace spanned by the columns of $\mathbf{R}_{\mathbf{V}} \in \mathbb{R}^{n \times q_0}$ first be given. It is chosen in such a way that matrix $\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i$ has linearly independent rows for the case $q_i \geq q_0$ and accordingly linearly independent columns for the case $q_i \leq q_0$. Then, for every local system $G_{r,i}$ there exist three cases to solve formula (10) which depend on the reduced order q_i of the local system compared to the size q_0 of the reference subspace:

- Case 1: $q_i > q_0$

Consider the following proposition.

Proposition 1. If $q_i > q_0$ holds, the Penrose condition is fulfilled for formula (10).

Proof. As matrix $\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \in \mathbb{R}^{q_0 \times q_i}$ has linearly independent rows, its pseudoinverse is $(\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(r)+}$ according to formula (5). Then, the Penrose condition is fulfilled:

$$\underbrace{(\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)(\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(r)+}}_{\mathbf{I}} \mathbf{I} = \mathbf{I}.$$

$\square$

Therefore, a solution exists for matrix equation according to Section 2.3:

$$\mathbf{MAC}(\mathbf{R}_{\mathbf{V}}, \hat{\mathbf{V}}_i) = \mathbf{R}_{\mathbf{V}}^T \hat{\mathbf{V}}_i = \mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{T}_i \overset{!}{=} \mathbf{I}. \quad (11)$$

One can see that a solution with minimal $\|\mathbf{T}_i\|_F$ is obtained when the right pseudoinverse is used:

$$\mathbf{T}_i = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(r)+} \in \mathbb{R}^{q_i \times q_0}. \qquad (12)$$

Then, the new right ROB of system $G_{r,i}$ is:

$$\begin{aligned}
\hat{\mathbf{V}}_i &= \mathbf{V}_i \mathbf{T}_i = \mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(r)+} \\
&= \mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T \left[\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T \right]^{-1} \qquad (13) \\
&= \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \left[\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \right]^{-1}.
\end{aligned}$$

With orthogonal $\mathbf{R}_{\mathbf{V}}$ and hence $\mathbf{R}_{\mathbf{V}}^T \mathbf{R}_{\mathbf{V}} = \mathbf{I}$ one gets:

$$\hat{\mathbf{V}}_i = \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \left[\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \right]^{-1} \mathbf{R}_{\mathbf{V}}^T \mathbf{R}_{\mathbf{V}}. \qquad (14)$$

How can the new right ROB be interpreted? Let us define $\mathbf{V}_i^* = \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \in \mathbb{R}^{n \times q_0}$ to get:

$$\hat{\mathbf{V}}_i = \underbrace{\mathbf{V}_i^* (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i^*)^{-1} \mathbf{R}_{\mathbf{V}}^T}_{\mathbf{P}_{\mathbf{V}_i^* \perp \mathbf{R}_{\mathbf{V}}}} \mathbf{R}_{\mathbf{V}} \in \mathbb{R}^{n \times q_0}, \qquad (15)$$

where $\mathbf{P}_{\mathbf{V}_i^* \perp \mathbf{R}_{\mathbf{V}}}$ is a projector which projects the columns of $\mathbf{R}_{\mathbf{V}}$ orthogonally to the subspace spanned by $\mathbf{R}_{\mathbf{V}}$ into the subspace $\mathcal{V}_i^*$ spanned by $\mathbf{V}_i^*$. Therefore, we get a new reduced system $G_{r,i}^*$ with reduced order $q_0 < q_i$ and subspace $\mathcal{V}_i^*$. Consider the proposition:

Proposition 2. Formula (12) is equivalent to reducing system G_i to $G_{r,i}^*$ with subspace spanned by $\mathbf{V}_i^* = \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}}$ and using $\mathbf{T}_i^* = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i^*)^{-1}$ from Panzer et al. (2010).

Proof. Calculate $\mathbf{T}_i^*$ according to Panzer et al. (2010):

$$\mathbf{T}_i^* = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i^*)^{-1} = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}})^{-1}.$$

Determine the new right ROB for system $G_{r,i}^*$:

$$\hat{\mathbf{V}}_i = \mathbf{V}_i^* \mathbf{T}_i^* = \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \left[\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \right]^{-1},$$

which is the result from (13). $\square$

- Case 2: $q_i = q_0$

For this case consider the following proposition.

Proposition 3. If $q_i = q_0$ holds, the choice $\mathbf{T}_i = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{-1}$ from Panzer et al. (2010) is a special case of formula (12).

Proof. With $q_i = q_0$ we get for formula (12):

$$\begin{aligned}
\mathbf{T}_i &= (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(r)+} = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T \left[\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T \right]^{-1} \\
&= (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{-T} (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{-1} \\
&= (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{-1}.
\end{aligned}$$

$\square$

Then, the new right ROB of system $G_{r,i}$ is obtained by projecting the columns of $\mathbf{R}_{\mathbf{V}}$ into subspace $\mathcal{V}_i$ orthogonally to the subspace spanned by $\mathbf{R}_{\mathbf{V}}$:

$$\begin{aligned}
\hat{\mathbf{V}}_i &= \mathbf{V}_i \mathbf{T}_i = \mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{-1} \\
&= \underbrace{\mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{-1} \mathbf{R}_{\mathbf{V}}^T}_{\mathbf{P}_{\mathbf{V}_i \perp \mathbf{R}_{\mathbf{V}}}} \mathbf{R}_{\mathbf{V}}, \qquad (16)
\end{aligned}$$

where $\mathbf{P}_{\mathbf{V}_i \perp \mathbf{R}_i}$ is a projector.

- Case 3: $q_i < q_0$

As matrix $\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \in \mathbb{R}^{q_0 \times q_i}$ has more rows than columns, the Penrose condition is generally not fulfilled. Hence, we formulate for expression (10) the optimization problem:

$$\mathbf{T}_i = \arg \min_{\mathbf{T}_i \in \mathbb{R}^{q_i \times q_0}} \| \mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \mathbf{T}_i - \mathbf{I} \|_F. \qquad (17)$$

The approximate solution according to (8) is obtained by applying the left pseudoinverse:

$$\mathbf{T}_i = (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(l)+} \in \mathbb{R}^{q_i \times q_0}. \qquad (18)$$

The new right ROB of system $G_{r,i}$ is with $\mathbf{R}_{\mathbf{V}}^T \mathbf{R}_{\mathbf{V}} = \mathbf{I}$:

$$\begin{aligned}
\hat{\mathbf{V}}_i &= \mathbf{V}_i \mathbf{T}_i = \mathbf{V}_i (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^{(l)+} \\
&= \mathbf{V}_i \left[(\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T \mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i \right]^{-1} (\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i)^T \qquad (19) \\
&= \mathbf{V}_i \left[(\mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \mathbf{R}_{\mathbf{V}}^T) \mathbf{V}_i \right]^{-1} (\mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \mathbf{R}_{\mathbf{V}}^T) \mathbf{R}_{\mathbf{V}}.
\end{aligned}$$

For the interpretation of the new basis $\hat{\mathbf{V}}_i$ of subspace $\mathcal{V}_i$ for system $G_{r,i}$ we define $\mathbf{R}_i^* = \mathbf{V}_i^T \mathbf{R}_{\mathbf{V}} \mathbf{R}_{\mathbf{V}}^T$ to get:

$$\hat{\mathbf{V}}_i = \underbrace{\mathbf{V}_i (\mathbf{R}_i^{*T} \mathbf{V}_i)^{-1} \mathbf{R}_i^{*T}}_{\mathbf{P}_{\mathbf{V}_i \perp \mathbf{R}_i^*}} \mathbf{R}_{\mathbf{V}} \in \mathbb{R}^{n \times q_0}, \qquad (20)$$

where $\mathbf{P}_{\mathbf{V}_i \perp \mathbf{R}_i^*}$ is a projector which projects the columns of $\mathbf{R}_{\mathbf{V}}$ orthogonally to the subspace spanned by $\mathbf{R}_i^*$ into the subspace $\mathcal{V}_i$. For the new right ROB $\hat{\mathbf{V}}_i$ holds $\text{span}(\hat{\mathbf{V}}_i) = \text{span}(\mathbf{V}_i) = \mathcal{V}_i$ with $q_0 - q_i$ linear dependent columns.

The algorithm given below for calculating $\mathbf{T}_i$ is the same for the three different cases because we employ the pseudoinverse using the economy size SVD with (6). As $\mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i$ has linearly independent rows or columns there is no singular value at zero and hence we get $\mathbf{\Sigma}_i^+ = \mathbf{\Sigma}_i^{-1}$.

Algorithm 1 Calculation of $\mathbf{T}_i$ with MAC approach

Input: k matrices $\mathbf{V}_i$ and $\mathbf{R}_{\mathbf{V}}$
Output: k matrices $\mathbf{T}_i$
 1: **for** $i = 1$ to k **do**
 2:     Compute $\mathbf{S}_i = \mathbf{R}_{\mathbf{V}}^T \mathbf{V}_i$
 3:     Compute $\mathbf{S}_i = \mathbf{U}_i \mathbf{\Sigma}_i \mathbf{Z}_i^T$ // SVD
 4:     Compute $\mathbf{T}_i = \mathbf{Z}_i \mathbf{\Sigma}_i^{-1} \mathbf{U}_i^T$
 5: **end for**

The complexity of the algorithm is dominated by matrix multiplication in line 2, which is proportional to $\mathcal{O}(nq_iq_0)$. Calculations in line 3 and 4 are independent of the original order n with $q_i, q_0 \ll n$ and hence are of low complexity.

3.3.2 Equality approach: We motivate this approach by demanding that the new right ROBs are supposed to resemble the reference basis $\mathbf{R_V}$:

$$\hat{\mathbf{V}}_i \approx \mathbf{R_V} \Rightarrow \mathbf{V}_i\mathbf{T}_i \approx \mathbf{R_V}. \tag{21}$$

As matrix $\mathbf{V}_i \in \mathbb{R}^{n \times q_i}$ has more rows than columns, Penrose condition is generally not fulfilled. Therefore, we formulate for expression (21) the minimization problem:

$$\mathbf{T}_i = \arg\min_{\mathbf{T}_i \in \mathbb{R}^{q_i \times q_0}} \|\mathbf{V}_i\mathbf{T}_i - \mathbf{R_V}\|_F. \tag{22}$$

Hence, equality between $\hat{\mathbf{V}}_i$ and $\mathbf{R_V}$ is seen as minimum norm least squares solution. The approximate solution according to (8) is obtained using the left pseudoinverse:

$$\mathbf{T}_i = \mathbf{V}_i^{(l)+}\mathbf{R_V}.$$

Then, the new basis $\hat{\mathbf{V}}_i$ of subspace $\mathcal{V}_i$ for system $G_{r,i}$ is:

$$\hat{\mathbf{V}}_i = \mathbf{V}_i\mathbf{T}_i = \mathbf{V}_i\mathbf{V}_i^{(l)+}\mathbf{R_V}$$
$$= \underbrace{\mathbf{V}_i(\mathbf{V}_i^T\mathbf{V}_i)^{-1}\mathbf{V}_i^T}_{\mathbf{P}_{\mathbf{V}_i \perp \mathbf{V}_i}}\mathbf{R_V}, \tag{23}$$

where $\mathbf{P}_{\mathbf{V}_i \perp \mathbf{V}_i}$ is a projector which projects the columns of $\mathbf{R_V}$ orthogonally to the subspace $\mathcal{V}_i$ into the subspace $\mathcal{V}_i$. Again there are three cases:

- Case 1: $q_i > q_0$

As $\text{span}(\hat{\mathbf{V}}_i) \subset \text{span}(\mathbf{V}_i)$, one gets a new reduced system $G_{r,i}^*$ with right subspace spanned by the columns $\hat{\mathbf{V}}_i$ and reduced order $q_0 < q_i$.

- Case 2: $q_i = q_0$

As $\text{span}(\hat{\mathbf{V}}_i) = \text{span}(\mathbf{V}_i) = \mathcal{V}_i$ holds, the columns of the new right ROB $\hat{\mathbf{V}}_i$ of system $G_{r,i}$ are linear independent.

- Case 3: $q_i < q_0$

The new right ROB $\hat{\mathbf{V}}_i$ of system $G_{r,i}$ has $q_0 - q_i$ linear dependent columns with $\text{span}(\hat{\mathbf{V}}_i) = \text{span}(\mathbf{V}_i) = \mathcal{V}_i$.

The algorithm using the economy size SVD for the calculation of $\mathbf{T}_i$ is given below. As $\mathbf{V}_i$ has linearly independent columns according to Section 3.2 it has no singular values at zero and hence we get $\mathbf{\Sigma}_i^+ = \mathbf{\Sigma}_i^{-1}$.

Algorithm 2 Calculation of $\mathbf{T}_i$ with equality approach
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Input: k matrices $\mathbf{V}_i$ and $\mathbf{R_V}$
Output: k matrices $\mathbf{T}_i$
 1: **for** $i = 1$ to k **do**
 2:    Compute $\mathbf{V}_i = \mathbf{U}_i\mathbf{\Sigma}_i\mathbf{Z}_i^T$ // SVD
 3:    Compute $\mathbf{T}_i = \mathbf{Z}_i\mathbf{\Sigma}_i^{-1}\mathbf{U}_i^T\mathbf{R_V}$
 4: **end for**

The economy size SVD in line 2 has the complexity $\mathcal{O}(nq_i^2)$ with $q_i \ll n$. The complexity of line 3 is $\mathcal{O}(nq_iq_0)$. Hence, algorithm 2 is in general computationally more expensive than algorithm 1.

3.3.3 Relation between the two approaches Note that the MAC and equality approach have in common that the

basis of the reference subspace $\mathbf{R_V}$ is projected in order to get the new right ROB $\hat{\mathbf{V}}_i$. However, they differ in the kind of projection. In general it is an open question which of the both approaches leads to more accurate generalized coordinates. We will give a numerical example in Section 4. However, a connection between the two approaches is given in the next proposition.

Proposition 4. Both approaches are equivalent for $q_0 = n$.

Proof. If $\mathbf{R}_V$ is chosen to be orthogonal with $q_0 = n$, one gets $\mathbf{R}_V\mathbf{R}_V^T = \mathbf{I}$. For $q_i < q_0$ consider formula (19):

$$\hat{\mathbf{V}}_i = \mathbf{V}_i\left[(\mathbf{R_V}^T\mathbf{V}_i)^T\mathbf{R_V}^T\mathbf{V}_i\right]^{-1}(\mathbf{R_V}^T\mathbf{V}_i)^T$$
$$= \mathbf{V}_i\left[\mathbf{V}_i^T\mathbf{R_V}\mathbf{R_V}^T\mathbf{V}_i\right]^{-1}\mathbf{V}_i^T\mathbf{R_V} \tag{24}$$
$$= \mathbf{V}_i(\mathbf{V}_i^T\mathbf{V}_i)^{-1}\mathbf{V}_i^T\mathbf{R_V},$$

which is the result from (23). For the (unrealistic) case $q_0 = q_i = n$ formulas (24) and (19) additionally simplify to $\hat{\mathbf{V}}_i = \mathbf{R_V}$ as $\mathbf{V}_i \in \mathbb{R}^{n \times n}$ is invertible. The case $q_i > q_0$ cannot occur as n is the maximal order. $\square$

Another link between the two approaches exists for the case of orthogonal $\mathbf{T}_i$, where an orthogonal Procrustes problem is solved, see Amsallem, and Farhat (2011).

3.3.4 Reference subspace With the explanation above we can now motivate the choice of the reference subspace which is spanned by $\mathbf{R}_V \in \mathbb{R}^{n \times q_0}$. It should comprise the directions which describe the most important dynamics of all reduced models. We assume $\mathbf{R}_V$ to be orthogonal. An approach inspired by Amsallem, and Farhat (2011) chooses $\mathbf{R}_V$ as right ROB $\mathbf{V}_{i_0}$ of one reduced system G_{r,i_0}:

$$\mathbf{R}_V = \mathbf{V}_{i_0} \in \mathbb{R}^{n \times q_{i_0}}. \tag{25}$$

Another approach based on Panzer et al. (2010) sums up all subspaces $\mathbf{V}_i \in \mathbb{R}^{n \times q_i}$ and takes q_0 directions by calculating the economy version of the Singular Value Decomposition (SVD):

$$\mathbf{U}\mathbf{\Sigma}\mathbf{N}^T = \text{svd}([\mathbf{V}_1 \ldots \mathbf{V}_k], \text{'econ'})$$
$$\Rightarrow \mathbf{R}_V = \mathbf{U}(:, 1:q_0). \tag{26}$$

Another possibility is the weighted SVD approach where weighting factors $\omega_i(\mathbf{p})$ for bases $\mathbf{V}_i$ are introduced:

$$\mathbf{U}\mathbf{\Sigma}\mathbf{N}^T = \text{svd}([\omega_1(\mathbf{p})\mathbf{V}_1 \ldots \omega_k(\mathbf{p})\mathbf{V}_k], \text{'econ'})$$
$$\Rightarrow \mathbf{R}_V = \mathbf{U}(:, 1:q_0). \tag{27}$$

In contrast to Panzer et al. (2010) now every order q_0 from $q_0 = 1$ to $q_0 = n$ can be chosen. However, there are two main strategies which seem reasonable:

- Strategy 1:

One captures all important directions of the subspaces. For this a tolerance for the singular values is defined and q_0 is chosen to get all important singular values. As it will be pointed out in Section 3.5, the matrices of systems at the sampling points with $q_i < q_0$ will be singular. Hence, the advantageous interpolation in the manifold of nonsingular matrices cannot by applied. However, at the sample points this strategy achieves the accuracy of the local models $G_{r,i}$.

- Strategy 2:

One chooses the smallest reduced order $q_0 = \min(q_i)$ with $i = 1, \ldots, k$. Then, system matrices are in general nonsingular and this strategy benefits from the interpolation

on the manifold of nonsingular matrices. However, for the systems at the sampling points the accuracy deteriorates compared to $G_{r,i}$ as they are approximated by smaller systems $G_{r,i}^*$. Therefore, this strategy leads to an approximation method instead of an interpolation method.

3.4 Adjustment of the left ROBs

After the adaption of the right ROBs, the left ROBs $\mathbf{W}_i$ have to be adjusted with respect to a reference subspace spanned by $\mathbf{R}_W$ for a meaningful interpolation of the system matrices of the reduced systems $G_{r,i}$. We use the concept of duality between the left and right subspace, which was introduced for pMOR in Geuss et al. (2013).

3.4.1 Reference subspace for the dual systems Due to duality, the reference subspace spanned by $\mathbf{R}_W$ can be calculated in the same way as $\mathbf{R}_V$ just by replacing $\mathbf{V}_i$ by $\mathbf{W}_i$. However, the size of $\mathbf{R}_W \in \mathbb{R}^{n \times q_0}$ is determined by q_0 from Section 3.3.4. Different sizes of $\mathbf{R}_W$ and $\mathbf{R}_V$ would lead to under- or overdetermined systems of equations.

3.4.2 Generalized coordinates for the dual systems The adjustment of the left ROBs is done with:

$$\hat{\mathbf{W}}_i = \mathbf{W}_i \mathbf{M}_i, \tag{28}$$

where—due to duality—matrices $\mathbf{M}_i \in \mathbb{R}^{q_i \times q_0}$ can be obtained using explications for the right ROBs in Section 3.3. The formulas for matrices $\mathbf{M}_i$ are shown in Table 1.

Table 1. Formulas for calculation of $\mathbf{M}_i$

	MAC approach	Equality approach
$q_i < q_0$	$\mathbf{M}_i = (\mathbf{R}_W^T \mathbf{W}_i)^{(r)+}$	
$q_i = q_0$	$\mathbf{M}_i = (\mathbf{R}_W^T \mathbf{W}_i)^{-1}$	$\mathbf{M}_i = \mathbf{W}_i^{(l)+} \mathbf{R}_W$
$q_i > q_0$	$\mathbf{M}_i = (\mathbf{R}_W^T \mathbf{W}_i)^{(l)+}$	

For the calculation of $\mathbf{M}_i$ the two algorithms from Section 3.3 can be used changing $\mathbf{V}_i$ to $\mathbf{W}_i$ and $\mathbf{R}_V$ to $\mathbf{R}_W$.

3.5 Choice of the interpolation manifold

After adjusting the right ROBs and left ROBs one finally gets the set of reduced, compatible systems $\hat{G}_{r,i}$:

$$\hat{G}_{r,i} : \begin{cases} \hat{\mathbf{E}}_{r,i} \dot{\hat{\mathbf{x}}}_{r,i}(t) = \hat{\mathbf{A}}_{r,i} \hat{\mathbf{x}}_{r,i}(t) + \hat{\mathbf{B}}_{r,i} \mathbf{u}(t), \\ \mathbf{y}_{r,i}(t) = \hat{\mathbf{C}}_{r,i} \hat{\mathbf{x}}_{r,i}(t), \end{cases} \tag{29}$$

where $\hat{\mathbf{x}}_{r,i}(t) \in \mathbb{R}^{q_0}$ is the reduced state vector and

$$\begin{aligned} \hat{\mathbf{E}}_{r,i} &= \mathbf{M}_i^T \mathbf{E}_{r,i} \mathbf{T}_i \in \mathbb{R}^{q_0 \times q_0}, \\ \hat{\mathbf{A}}_{r,i} &= \mathbf{M}_i^T \mathbf{A}_{r,i} \mathbf{T}_i \in \mathbb{R}^{q_0 \times q_0}, \\ \hat{\mathbf{B}}_{r,i} &= \mathbf{M}_i^T \mathbf{B}_{r,i} \in \mathbb{R}^{q_0 \times r}, \\ \hat{\mathbf{C}}_{r,i} &= \mathbf{C}_{r,i} \mathbf{T}_i \in \mathbb{R}^{m \times q_0}. \end{aligned} \tag{30}$$

We want to point out two important cases for systems $\hat{G}_{r,i}$:

- Case 1: $q_i < q_0$

The transformation matrices $\mathbf{T}_i$ and $\mathbf{M}_i$ leave the input-output behavior of systems $\hat{G}_{r,i}$ unchanged in relation to systems $G_{r,i}$. However, consider the following proposition.

Proposition 5. The matrices $\mathbf{T}_i$ introduce a common kernel of dimension $q_0 - q_i$ for matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$.

Proof. Matrix $\mathbf{T}_i \in \mathbb{R}^{q_i \times q_0}$ has a kernel of dimension $q_0 - q_i$. Then, for all vectors $\mathbf{v} \in \ker(\mathbf{T}_i)$ we get:

$$\hat{\mathbf{E}}_{r,i} \mathbf{v} = \mathbf{M}_i^T \mathbf{E}_{r,i} \mathbf{T}_i \mathbf{v} = \mathbf{M}_i^T \mathbf{E}_{r,i} \mathbf{0} = \mathbf{0},$$
$$\hat{\mathbf{A}}_{r,i} \mathbf{v} = \mathbf{M}_i^T \mathbf{A}_{r,i} \mathbf{T}_i \mathbf{v} = \mathbf{M}_i^T \mathbf{A}_{r,i} \mathbf{0} = \mathbf{0}.$$

$\square$

Hence, the matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ are in any case singular.

- Case 2: $q_i \geq q_0$

The matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ can be singular due to two reasons. Firstly $\mathbf{E}_{r,i}$ and $\mathbf{A}_{r,i}$ itself can be singular and secondly an improper choice of $\mathbf{T}_i$ and $\mathbf{M}_i$ can cause singularities. Otherwise, matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ are nonsingular.

The system matrices of $\hat{G}_{r,i}$ are elements of Riemannian manifolds. Therefore, the interpolated matrices should as well be elements of the respective manifold, see Amsallem et al. (2009); Amsallem, and Farhat (2011); Degroote et al. (2010). Hence, one chooses a reference system $\hat{G}_{i_0}$ and maps the matrices $\hat{\mathbf{E}}_i$, $\hat{\mathbf{A}}_i$, $\hat{\mathbf{B}}_i$ and $\hat{\mathbf{C}}_i$ to the tangent space by the logarithmic mapping (Log). For example, take matrices $\hat{\mathbf{A}}_i$ and let $\hat{\mathbf{A}}_{i_0}$ belong to the reference system. The mapped matrices are $\mathbf{\Gamma}_{\hat{\mathbf{A}}_i} = \mathrm{Log}_{\hat{\mathbf{A}}_{i_0}}(\hat{\mathbf{A}}_i)$. In analogy matrices $\hat{\mathbf{E}}_i$, $\hat{\mathbf{B}}_i$ and $\hat{\mathbf{C}}_i$ are mapped to $\mathbf{\Gamma}_{\hat{\mathbf{E}}_i}$, $\mathbf{\Gamma}_{\hat{\mathbf{B}}_i}$ and $\mathbf{\Gamma}_{\hat{\mathbf{C}}_i}$. Formulas for the logarithmic mapping can e.g. be look up in Amsallem, and Farhat (2011); Degroote et al. (2010).

In the present paper the system matrices $\hat{\mathbf{B}}_{r,i}$ belong to the manifold of real matrices $\mathbb{R}^{q_0 \times r}$ and the matrices $\hat{\mathbf{C}}_{r,i}$ to the manifold of real matrices $\mathbb{R}^{m \times q_0}$. For the manifold of the matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ there exist two cases:

- Case 1: $q_i < q_0$

As system matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ are singular, they are interpolated in the manifold of real matrices $\mathbb{R}^{q_0 \times q_0}$.

- Case 2: $q_i \geq q_0$

If matrices $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ are nonsingular, they belong to the manifold of nonsingular matrices. Otherwise, they are interpolated in the manifold of real matrices $\mathbb{R}^{q_0 \times q_0}$.

3.6 Interpolation process

Let $\tilde{\mathbf{p}} \in \mathcal{D}$ be a value in the parameter domain. In order to obtain a reduced system $G_r(\tilde{\mathbf{p}})$ we first interpolate system matrices in the tangent space to the respective manifold:

$$\begin{aligned} \mathbf{\Gamma}_{\hat{\mathbf{E}}}(\tilde{\mathbf{p}}) &= \sum_{i=1}^{k} \omega_i(\tilde{\mathbf{p}}) \mathbf{\Gamma}_{\hat{\mathbf{E}}_i}, \\ \mathbf{\Gamma}_{\hat{\mathbf{A}}}(\tilde{\mathbf{p}}) &= \sum_{i=1}^{k} \omega_i(\tilde{\mathbf{p}}) \mathbf{\Gamma}_{\hat{\mathbf{A}}_i}, \\ \mathbf{\Gamma}_{\hat{\mathbf{B}}}(\tilde{\mathbf{p}}) &= \sum_{i=1}^{k} \omega_i(\tilde{\mathbf{p}}) \mathbf{\Gamma}_{\hat{\mathbf{B}}_i}, \\ \mathbf{\Gamma}_{\hat{\mathbf{C}}}(\tilde{\mathbf{p}}) &= \sum_{i=1}^{k} \omega_i(\tilde{\mathbf{p}}) \mathbf{\Gamma}_{\hat{\mathbf{C}}_i}, \end{aligned} \tag{31}$$

where $\omega_i(\tilde{\mathbf{p}})$ are weighting functions of an arbitrary interpolation method with $\sum_{i=1}^{k} \omega_i(\tilde{\mathbf{p}}) = 1$. Alternatively, system matrices can be interpolated coefficient-wise with every interpolation method. The interpolated matrices (31) are mapped back to the original space by the exponential mapping which results in the system matrices $\mathbf{E}_r(\tilde{\mathbf{p}})$, $\mathbf{A}_r(\tilde{\mathbf{p}})$, $\mathbf{B}_r(\tilde{\mathbf{p}})$, and $\mathbf{C}_r(\tilde{\mathbf{p}})$. Formulas for the exponential

mapping can e.g. be look up in Amsallem, and Farhat (2011); Degroote et al. (2010). The system matrices have size q_0 which is the size of the basis of the reference subspace. Finally, we get the interpolated, reduced system:

$$G_r(\tilde{\mathbf{p}}) : \begin{cases} \mathbf{E}_r(\tilde{\mathbf{p}})\dot{\mathbf{x}}_r(t) = \mathbf{A}_r(\tilde{\mathbf{p}})\mathbf{x}_r(t) + \mathbf{B}_r(\tilde{\mathbf{p}})\mathbf{u}(t), \\ \qquad \mathbf{y}_r(t) = \mathbf{C}_r(\tilde{\mathbf{p}})\mathbf{x}_r(t). \end{cases} \quad (32)$$

Note that for the case $q_i < q_0$, although $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$ share a common kernel, interpolation generally leads to nonsingular matrices as every local system adds a part to the dynamics of the interpolated system.

4. NUMERICAL RESULTS

The considered example is a FE model of a Timoshenko beam from Panzer et al. (2009) with $n = 2400$ degrees of freedom. The system parameter is the length L of the beam which varies between $L = 0.8$m and $L = 1.4$m. The model input is a vertical force which is applied at the tip of the beam and the model output is the vertical displacement at this point. We compute four high-order systems $G_1, \ldots, G_4$ for lengths $L_1 = 0.8$m, $L_2 = 1$m, $L_3 = 1.2$m, and $L_4 = 1.4$m. Systems G_i are reduced using a two-sided Krylov subspace method with expansion points $s_0 = 0$ and reduced orders $q_1 = 14$, $q_2 = 12$, $q_3 = 12$, and $q_4 = 10$ in order to get local systems $G_{r,i}$ with $i = 1, \ldots, 4$. We choose decreasing reduced orders so that systems $G_{r,i}$ are more accurate for small lengths. For calculating the reference subspace we first apply strategy 1 and choose q_0 to capture all relevant singular values with the non-weighted SVD up to relative tolerance of 10^{-1} which gives $q_0 = q_1 = 14$. For strategy 2 we choose $q_0 = q_4 = 10$ and use the manifold of nonsingular matrices for interpolating $\hat{\mathbf{E}}_{r,i}$ and $\hat{\mathbf{A}}_{r,i}$. For both strategies we applied linear interpolation. In Figure 1 the relative error between the interpolated and the original systems for the two strategies and the MAC and equality approach is shown. It can be seen that there is only a small difference between the MAC and equality approach. For the plots with strategy 2 there is a loss of accuracy at the sampling points L_1, L_2, L_3 as the systems are approximated by reduced systems of order 10—compare Section 3.3—although systems $G_{r,1}, G_{r,2}, G_{r,3}$ have order $q_1 = 14$, $q_2 = 12$, $q_3 = 12$. The error rises with growing discrepancy $q_0 - q_i$. As $G_{r,4}$ at L_4 also possesses order $q_4 = 10$ there is no loss of accuracy. Hence, it can be seen that strategy 2 is an approximation method, whereas strategy 1 is an interpolation method as systems $G_{r,i}$ are obtained at the sampling points. In this example, strategy 1 performs better than strategy 2 except for a small parameter interval.

5. CONCLUSION

In this paper a method for pMOR by matrix interpolation was proposed for the case where local systems have different reduced orders. In order to interpolate the resulting differently sized system matrices we introduced generalized coordinates with two different approaches using pseudoinverses. The generalized coordinates are computed with respect to a reference subspace. We also proposed two reasonable strategies how to choose the size of the reference subspace. Then, the interpolated system possesses the size of the basis of the reference subspace.
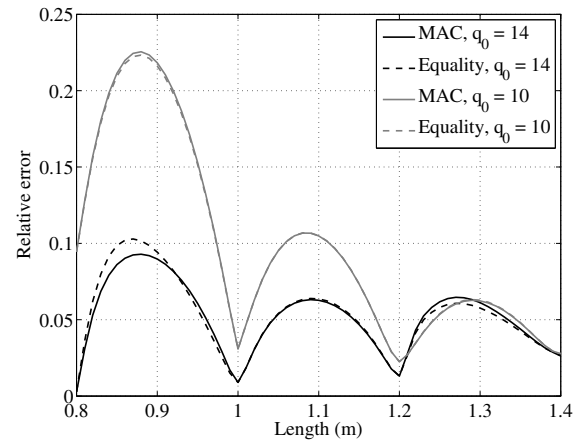

Fig. 1. Relative error in $\mathcal{H}_2$-norm at various lengths L for the cantilever beam.

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