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One-layer robust MPC: a multi-model approach

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Abstract: In this paper, an MPC that explicitly integrates the RTO structure into the dynamic control layer is presented. In particular, a robust MPC is proposed, which takes into account the uncertainties that arise from the difference between nonlinear and linear models, by means of a multi-model approach: a finite family of linear models is considered, which operate appropriately in a moderate-to-large region around a given operating point. In this way, each linear model provides an enough accurate description of the system. Feasibility and stability conditions are preserved. Moreover, the real plant converges to the optimal point that optimizes the economic cost function.

Keywords: Model predictive control, economic optimization, robust control, multi-model uncertainties.

1. INTRODUCTION

Modern industrial application of model predictive control (MPC) requires a number of specific properties that have to be accounted for theoretic formulations (Rawlings and Mayne 2009). If petrochemical processes are considered, one of the main requirement is the economic optimization of the plant operation. In this context, the hierarchical control structure, in which an economic optimization level - usually referred as Real Time Optimizer (RTO) - sends setpoints to the MPC layer, is the usual strategy to account for the economic requirements (Engell 2007). However, the drawbacks of this strategy - i.e., communication problems between layers, different time scaling, model mismatches - motivate the development of the so-called one-stage strategies. The idea of this approach is to merge the RTO layer with the MPC layer, by designing controllers that integrates the RTO economic cost function as part of the MPC cost as in (Adetola and Guay 2010, Zanin et al. 2002, Biegler 2009, De Souza et al. 2010). Another approach, the so-called economic MPC, consists in using the RTO cost function directly as the MPC stage cost function (Rawlings et al. 2012, Müeller et al. 2013, Ferramosca et al. 2010b).
However, the proposed MPC is nominal and industrial applications requires the consideration of some kind of robustness. For the case of petrochemical processes, for instance, the plant to be controlled is nonlinear, but has sparse operation points with different economic behaviors. So, a convenient form of representing the models uncertainty is by considering a finite family of linear models (multi-model uncertainty), which operate appropriately in a moderate-to-large region around a given operating point (Badgwell 1997, González et al. 2009, Findeisen et al. 2000). In this context, each operating point defines a linear model sufficiently accurate to describe the system. Furthermore, since no many operating points are considered in the operation of this kind of systems, a few linear models could be required to describe the complete operation.
In this work, a one-layer economic MPC suitable for multi-model uncertainty is presented. To this end, a finite and countable family of model is considered, while the feasibility and stability conditions are preserved no matter which member of the family represents the true model. Furthermore, as required in petrochemical applications, the proposed controller can be easily adapted to the case of a gradient-based approximation of the economic cost (Alamo et al. 2012, Limon et al. 2013) and the zone control case, which consists in guiding the output to a economically optimal region, instead of a point (González et al. 2009, Ferramosca et al. 2010a).
The work is organized as follows. In Section 2 the problem is stated. In Section 3 the proposed multi-model one-layer MPC is presented. Finally, illustrative examples and conclusions of this study are provided in Sections 4 and 5.

2. PROBLEM STATEMENT

Consider a system described by an unknown linear discrete time-invariant model

$$x^+ = A_r x + B_r u, \quad y = C_r x \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, $y \in \mathbb{R}^p$ is the controlled output and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$ where $x = \phi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively. The system is subject to hard constraints on state and control:

$$x(k) \in \mathcal{X}, \quad u(k) \in \mathcal{U} \tag{2}$$

for all $k \geq 0$.

Assumption 1. $\mathcal{X}$ is convex and closed, $\mathcal{U}$ is convex and compact and both sets contain the origin in their interior.

The steady state, input and output of the plant (x_s, u_s, y_s) are such that (1) is fulfilled, i.e.

$$x_s = A_r x_s + B_r u_s, \quad y_s = C_r x_s \tag{3}$$

Consider now, a nonlinear function $f_{eco}(y, u, \rho)$ that takes into account the economic objectives of the plant. ρ is a parameter that takes into account prices, costs or production goals. Let us define the RTO problem as follows:

Definition 1. The optimal steady point, (x_s^*, u_s^*, y_s^*), satisfies

$$\begin{aligned}(x_s^*, u_s^*, y_s^*) \;=\; &\arg \min_{(x,u,y)} f_{eco}(y, u, \rho) \\ &s.t.\; x \in \mathcal{X}, \quad u \in \mathcal{U} \\ &\quad x = A_r x + B_r u, \quad y = C_r x\end{aligned} \tag{4}$$

Assumption 2. The economic cost function $f_{eco}(y, u, \rho)$ is convex in (y, u).

2.1 Multi-model description of the plant

It is assumed that the real model of the plant (1) is not known to the controller. However, a collection of L linear models of the form

$$x_i^+ = A_i x + B_i u, \quad y_i = C_i x \tag{5}$$

is supposed to be known (Badgwell 1997, González et al. 2009), in such a way that model uncertainties are represented. Let us define the set of possible linear plants as $\Pi = \{\pi_1, ..., \pi_L\}$, where π_i corresponds to the particular plant (A_i, B_i, C_i), $i \in \mathbb{I}_{1:L}$. Let us define as $\pi_r \in \Pi$, the model that represents the plant in its actual operation point, and as $\pi_{no} \in \Pi$, an average (or nominal) model.

Assumption 3. Each plant A_i is stable. The pair (A_i, B_i) is controllable and each model π_i is subject to constraints (2). Moreover, it is assumed that the state of the real plant x_r is completely measurable ($C_i = I_n$).

Under Assumption 3, the set of steady states and inputs of system (5) is a m-dimensional linear subspace of $\mathbb{R}^{n+m}$ (Limon et al. 2008) given by

$$(x_{s,i}, u_s) = M_{\theta,i}\theta$$

Every pair $(x_{s,i}, u_s) \in \mathbb{R}^{n+m}$ is characterized by only one parameter $\theta \in \mathbb{R}^m$. The steady controlled outputs are given by

$$y_{s,i} = N_{\theta,i}\theta$$

where $N_{\theta,i} = C_i M_{\theta,i}$.

We define the sets of admissible equilibrium states, inputs and outputs as

$$\mathcal{Z}_{s,i} = \{(x_i, u) \in \mathcal{X} \times \mathcal{U} \mid x_i = A_i x_i + B_i u\} \tag{6}$$

$$\mathcal{X}_{s,i} = \{x_i \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ such that } (x_i, u) \in \mathcal{Z}_{s,i}\} \tag{7}$$

$$\mathcal{Y}_{s,i} = \{y = C_i x_i \mid (x_i, u) \in \mathcal{Z}_{s,i}\} \tag{8}$$

We assume that $(x_s^*, u_s^*) \in \mathcal{Z}_{s,i}$, for all $i \in \mathbb{I}_{1:L}$.

In this paper, we consider the control structure shown in Figure 1. The objective is to design a one-layer RTO+MPC controller that directly account for stationary economic objectives. In particular, a robust MPC is proposed, which takes into account the uncertainties that arise from the lack of knowledge of the real plant, by means of a multi-model approach.

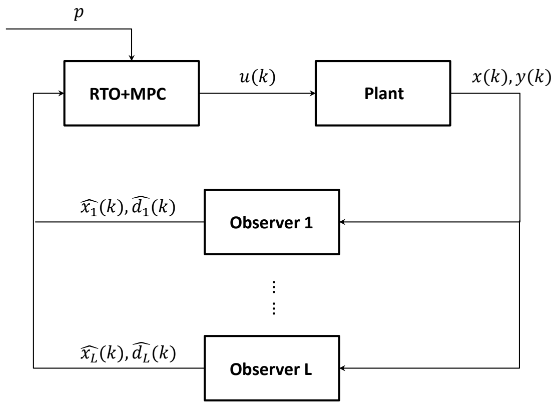

Figure 1. Control structure.

3. PROPOSED FORMULATION

Since we don't know *a priori* which is the real model of the plant, an augmented system, with an additional integrating disturbance is introduced:

$$\begin{bmatrix} x_i^+ \\ d_i^+ \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u \tag{9}$$

$$y_i = [\, C_i \;\; I_p \,] \begin{bmatrix} x_i \\ d_i \end{bmatrix}$$

where d_i represents an output disturbance corresponding to model π_i.

From Figure 1, it is clear that the control structure is equipped with one observer per each model, and the observer of the real plant is based on the real model π_r. Moreover, we want to estimate the output disturbances d_i. We propose an open-loop state observer and a closed-loop disturbance observer of the form:

$$\hat{x}_i(k+1) = A_i \hat{x}(k) + B_i u(k) \tag{10}$$

$$\hat{d}_i(k+1) = \hat{d}_i(k) + L_i^d (C_i \hat{x}_i(k) - y(k) + \hat{d}_i(k)) \tag{11}$$

where $\hat{x}_i(k) \in \mathbb{R}^p$ is the observed state at time k, with $\hat{x}_i(0) = x(0)$, that is the measured state at $k = 0$; $\hat{d}_i(k) \in \mathbb{R}^p$ is the estimated disturbance at time k, with $\hat{d}_i(0) = 0_p$ [1]; $y(k)$ is the measured output at time k. Furthermore, L_i^d is the observer gain of the disturbance estimation. Notice that, for the real plant π_r, the estimated disturbance will always be $\hat{d}_r = 0$.

For the sake of clarity let us define

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}, \; d = \begin{bmatrix} d_1 \\ \vdots \\ d_L \end{bmatrix}, \; h = \begin{bmatrix} y_1 \\ \vdots \\ y_L \end{bmatrix} \tag{12}$$

where $z \in \mathbb{R}^{Ln}$, $d \in \mathbb{R}^{Lp}$, and $h \in \mathbb{R}^{Lp}$.

To propose a robust MPC controller, based on a multi-model approach, that integrates the RTO into the MPC problem, and capable of ensuring feasibility for any economic objective f_{eco}, the following cost function is proposed:

$$\begin{aligned} V_N(x, \hat{d}, \rho; \mathbf{u}, \theta) = &\sum_{j=0}^{N-1} \|x_{no}(j) - x_{s,no}\|_Q^2 + \|u(j) - u_s\|_R^2 \\ &+ V_O(h_s, u_s, \rho) \end{aligned} \tag{13}$$

[1] 0_p is a vector in $\mathbb{R}^p$ with all elements equal to 0.

where x is the real plant measured state at time k, $x_{no}(j)$ is the prediction based on the nominal model and is such that $x_{no}(0) = x$, $\mathbf{u}$ is the control sequence to be calculate, which is unique for all models in Π, and Q and R are positive definite matrices. The variable $(x_{s,no}, u_s, h_s)$ represents an admissible equilibrium point - the so-called artificial reference - and are all parameterized by θ. Notice that u_s and θ are unique for all models in Π. $V_O(h_s, u_s, \rho)$ represents the economic cost, which is given by:

$$V_O(h_s, u_s, \rho) = \sum_{i=1}^{L} f_{eco}(y_{s,i}, u_s, \rho) \qquad (14)$$

At the time k, the optimization problem $P_N(x, \hat{z}, \hat{d}, \rho, \tilde{\mathbf{u}}, \tilde{\theta})$ to be solved is given by:

Problem $P_N(x, \hat{z}, \hat{d}, \rho, \tilde{\mathbf{u}}, \tilde{\theta})$

$$\begin{array}{lll}
\min\limits_{\mathbf{u},\theta} & V_N(x, \hat{d}, \rho; \mathbf{u}, \theta) & \\
s.t. & x_i(0) = \hat{x}_i, & i \in \mathbb{I}_{1:L} \\
& x_i(j+1) = A_i x_i(j) + B_i u(j), \quad j \in \mathbb{I}_{0:N-1}, & i \in \mathbb{I}_{1:L} \\
& x_i(j) \in \mathcal{X},\ u(j) \in \mathcal{U}, \quad j \in \mathbb{I}_{0:N-1}, & i \in \mathbb{I}_{1:L} \\
& (x_{s,i}, u_s) = M_{\theta,i}\theta, & i \in \mathbb{I}_{1:L} \\
& y_{s,i} = N_{\theta,i}\theta + \hat{d}_i, & i \in \mathbb{I}_{1:L} \\
& x_i(N) \in \mathcal{X}_{s,i} & i \in \mathbb{I}_{1:L} \\
& V_N^i(\hat{x}_i, \hat{d}_i, \rho; \mathbf{u}, \theta) \leq V_N^i(\hat{x}_i, \hat{d}_i, \rho; \tilde{\mathbf{u}}, \tilde{\theta}), & i \in \mathbb{I}_{1:L}
\end{array} \qquad (15)$$

where the last constraint, is a robustness constraint, added for stability reasons (Badgwell 1997), and

$$V_N^i(\hat{x}_i, \hat{d}_i, \rho; \mathbf{u}, \theta) = \sum_{j=0}^{N-1} \|x_i(j) - x_{s,i}\|_Q^2 + \|u(j) - u_s\|_R^2 + f_{eco}(y_{s,i}, u_{s,i}, \rho) \qquad (16)$$

Remark 2. Notice that $\tilde{\mathbf{u}}$ and $\tilde{\theta}$ are feasible solutions to problem (15), based on a solution of the same problem at time $k-1$. Moreover, at time $k = 0$, $z(0)$ is such that $x_i(0) = x(0)$.

The optimal cost and the optimal decision variables will be denoted as $V_N^0(x_r, \hat{d}, \rho)$ and $(\mathbf{u}^0, \theta^0)$ respectively. Based on this, at time k, we define $\tilde{\mathbf{u}}(k) = \{u^0(1; k-1), u^0(2; k-1), ..., u^0(N-1; k-1), \tilde{u}\}$, where $\tilde{u}$ is a feasible control action, and $\tilde{\theta}(k) = \theta^0(k-1)$.

Considering the receding horizon policy, the control law is given by

$$\kappa_N(x, \hat{z}, \hat{d}, \rho, \tilde{\mathbf{u}}, \tilde{\theta}) = u^0(0; x, \hat{z}, \hat{d}, \rho, \tilde{\mathbf{u}}, \tilde{\theta}) \qquad (17)$$

Since the set of constraints of problem (15) does not depend on p, its feasibility region does not depend on the economic objective. Then, for any plant π_i there exists a region $\mathcal{X}_{N,i} \subseteq \mathcal{X}$, which represents the set of initial states x_i that can be admissibly steered to $\mathcal{X}_{s,i}$ in N steps. The domain of attraction of the proposes controller can then be defined as

$$\mathcal{X}_N = \bigcap_{i=1}^{L} \mathcal{X}_{N,i}$$

Consider the following assumption on the controller parameters:

Assumption 4. The prediction horizon N is such that

$$rank(Co_{N,i}) \geq n, \quad i \in \mathbb{I}_{1:L}$$

where $Co_{N,i} = [A_i^{N-1} B_i \ldots A_i B_i\ B_i]$ is the N-controllability matrix of system (A_i, B_i). Moreover, there exists a dead-beat control gain K_i, such that $A_i + B_i K_i$, $i \in \mathbb{I}_{1:L}$, has null eigenvalues.

Theorem 1. Consider that Assumptions 1-4 hold, and consider a given parameter p for the economic cost $f_{eco}(y, u, \rho)$. Then, for any $x_r \in \mathcal{X}_{N,r}$, the system controlled by the MPC control law $\kappa_N(\mathrm{x}_r, \hat{d}, \rho)$ at each time step k is stable and fulfills the constraints throughout the time. Furthermore, the closed-loop system converges asymptotically to a steady point (x_s^*, u_s^*, y_s^*) that satisfies (4).

Proof. Consider the measured output at time k, $y(k)$, the state of the real plant, $x_r(k)$, and the observed state and disturbance $\hat{z}(k)$ and $\hat{d}(k)$. Consider also the solution to Problem (15), given by $(\mathbf{u}^0(k), \theta^0(k))$, where

$$\mathbf{u}^0(k) = \{u^0(0; k), u^0(1; k), ..., u^0(N-1; k)\}$$

From Problem (15), this sequence is a feasible sequence that accounts for the constraints of all models $\pi_i \in \Pi$. Since we know that the real plant model $\pi_r \in \Pi$, then its state sequence corresponding to applying $\mathbf{u}^0(k)$ is

$$\mathbf{x}_r^0(k) = \left\{x_r^0(k), x_r^0(1; k) \cdots, x_r^0(N; x)\right\}$$

where $x_r^0(N; k) = x_{s,r}^0(k)$. This comes from the constraint $x_i(N) \in \mathcal{X}_{s,i}$.

Now, consider the successor states at time $k+1$

$$x_r^+ = A_r x_r(k) + B_r u^0(0; k) = x_r^0(1; k)$$

which is obtained by implementing the control law (17), and define the following feasible solution to problem (15), at time $k+1$, $\tilde{\mathbf{u}}(k+1) = \left\{u^0(1; k), ..., u^0(N-1; k), \tilde{u}_s(k+1)\right\}$ and $\tilde{\theta}(k+1) = \theta^0(k)$.

Notice that $\tilde{\mathbf{u}}(k+1)$ is a sequence made by shifting one step ahead the sequence $\mathbf{u}^0(k)$ and adding the admissible equilibrium input at time k. In fact $(\tilde{x}_{s,i}(k+1), \tilde{u}_s(k+1)) = M_{\theta,i}\tilde{\theta}(k+1) = M_{\theta,i}\theta^0(k) = (x_{s,i}^0(k), u_s^0(k))$, for all $i \in \mathbb{I}_{1:L}$, and so for $i = r$. Notice also that this equilibrium input $u_s^0(k)$ is unique for all models in Π. Define also the associated real plant state sequence, $\tilde{\mathbf{x}}_r = \left\{x_r^0(1; k), ..., x_{s,r}^0(k), x_{s,r}^0(k)\right\}$, where the last state is given by $x_{s,r}^0(k) = A_r x_{s,r}^0(k) + B_r u_s^0(k)$.

The observed real plant disturbance at time $k+1$ will be $\hat{d}_r(k+1) = \hat{d}_r(k) = 0_p$, and hence $\tilde{y}_{s,r}(k+1) = N_{\theta,r}\tilde{\theta}(k+1) + \hat{d}_r(k+1) = y_{s,r}^0$.

Now, following standard arguments in MPC literature (Rawlings and Mayne 2009), the cost function of the real plant corresponding to $\mathbf{u}^0(k)$, $V_N^{r0}(x_r, \hat{d}_r, \rho; \mathbf{u}^0(k), \theta^0(k))$ will be compared to the one given by $\tilde{\mathbf{u}}(k+1)$. We get

$$\begin{aligned}
\Delta V_N^r &= V_N^r(x_r^+, \hat{d}_r(k+1), \rho; \tilde{\mathbf{u}}(k+1), \tilde{\theta}(k+1)) - V_N^{r0}(x_r, \hat{d}_r(k), \rho) \\
&= -\|x_r(k) - x_{s,r}^0(k)\|_Q^2 - \|u^0(0; k) - u_s^0(k)\|_R^2
\end{aligned}$$

From the last constraint of problem (15), we get that, the optimal cost at time $k+1$ cannot exceed, $V_N^r(x_r^+, \hat{d}(k+1), \rho; \tilde{\mathbf{u}}(k+1), \tilde{u}_s(k+1))$. Hence, we can state that

$$\begin{aligned}
\Delta V_N^{r0} &= V_N^{r0}(x_r(k+1), \hat{d}_r(k+1), \rho) - V_N^{r0}(x_r(k), \hat{d}_r(k), \rho) \\
&\leq -\|x_r(k) - x_{s,r}^0(k)\|_Q^2 + \|u^0(0; k) - u_s^0(k)\|_R^2
\end{aligned}$$

Since Q and R are positive definite, the previous inequality implies that there exists a $\mathcal{K}$-function α such that:

$$\Delta V_N^{r0} \leq -\alpha(\|x_r(k) - x_{s,r}^0(k)\|) \qquad (18)$$

Define the function $J(x_r,\rho) = V_N^{r0}(x_r,\hat{d},\rho) - f_{eco}(y_s^0,u_s^0,\rho)$. This function is well defined in $\mathcal{X}_{N,r} = proj(\mathcal{Z}_N)_{x_r}$. Define also $e(x_r) = x_r - x_{s,r}^0$. Notice that, since Q and R are positive definite, $J(x_r,\rho) \geq \alpha(\|e(x_r)\|)$, for all $x_r \in \mathcal{X}_{N,r}$; due to (18), we have that $J(x_r^+,\rho) - J(x_r,\rho) \leq -\alpha(\|e(x_r)\|)$, for all $x_r \in \mathcal{X}_{N,r}$.
From Lemma 5 in the Appendix, it follows that

$$\alpha(\|e(x_r)\|) \geq \alpha(\alpha_e(\|x_r - x_{s,r}^*\|)) = \alpha_J(\|x_r - x_{s,r}^*\|)$$

where α_e and α_J are $\mathcal{K}$-functions. Then, we can conclude that:

(i) $J(x_r,\rho) \geq \alpha_J(\|x_r - x_{s,r}^*\|)$, for all $x_r \in \mathcal{X}_{N,r}$.
(ii) $J(x_r^+,\rho) - J(x_r,\rho) \leq -\alpha_J(\|x_r - x_{s,r}^*\|)$, for all $x_r \in \mathcal{X}_{N,r}$.
(iii) Since $\mathcal{X}_{N,r}$ is compact, $J(x_{s,r}^*,\rho) = 0$, and $J(x_r,\rho)$ is continuous in $x_r = x_{s,r}^*$, then there exists a $\mathcal{K}$-function β_J such that $J(x_r,\rho) \leq \beta_J(\|x_r - x_{s,r}^*\|)$, for all $x_r \in \mathcal{X}_{N,r}$, Rawlings and Mayne (2009).

Hence $J(x_r,\rho)$ is a Lyapunov function and $x_{s,r}^*$ is an asymptotically stable equilibrium point for the closed-loop system, that is, there exists a $\mathcal{KL}$-function ϑ such that

$$\|x_r(k) - x_{s,r}^*\| \leq \vartheta(\|x_r(0) - x_{s,r}^*\|, k)$$

for all $x_r(0) \in \mathcal{X}_{N,r}$. ■

Remark 3. If the economic cost function f_{eco} is highly nonlinear, but still convex, the results proposed in (Alamo et al. 2012) and (Limon et al. 2013) can be applied to the proposed controller, in order to reduce computational complexity. Moreover, a zone control strategy (González et al. 2009, Ferramosca et al. 2010a), which consists in guiding the output to an economically optimal region instead of a point, can also be applied to the proposed controller.

Remark 4. Notice that, if the real plant (1) is a nonlinear plant, $\hat{d}_r$ will not be identically null, but it will converge to a constant value, as any other estimated disturbance $\hat{d}_i$. In this case, the results presented above are still valid, and the multi-model approach will provide not only robustness, but offset cancelation as well.

4. ILLUSTRATIVE EXAMPLE

The proposed controller has been tested in simulation on a 4 tanks system.

The four tanks plant (Johansson 2000) is a multivariable laboratory plant of interconnected tanks with nonlinear dynamics and subject to state and input constraints. The inputs are the flows of the two pumps and the outputs are the water levels in the lower tanks. The nonlinear continuous time model of this process can be derived from first principles as follows (Johansson 2000)

$$\frac{dh_1}{dt} = -\frac{a_1}{A}\sqrt{2gh_1} + \frac{a_3}{A}\sqrt{2gh_3} + \frac{\gamma_a}{A}\frac{q_a}{3600} \quad (19a)$$
$$\frac{dh_2}{dt} = -\frac{a_2}{A}\sqrt{2gh_2} + \frac{a_4}{A}\sqrt{2gh_4} + \frac{\gamma_b}{A}\frac{q_b}{3600} \quad (19b)$$
$$\frac{dh_3}{dt} = -\frac{a_3}{A}\sqrt{2gh_3} + \frac{(1-\gamma_b)}{A}\frac{q_b}{3600} \quad (19c)$$
$$\frac{dh_4}{dt} = -\frac{a_4}{A}\sqrt{2gh_4} + \frac{(1-\gamma_a)}{A}\frac{q_a}{3600} \quad (19d)$$

The linearized model is given by:

Table 1. Linearization points.

Model	h_1	h_2	q_a	q_b
π_1	0.4210	0.4678	1.4802	1.5197
π_2	0.2977	0.3308	1.2447	1.2779
π_3	0.8550	0.5672	1.0444	2.6980
π_{no}	0.6487	0.6636	1.63	2

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{\tau_1} & 0 & \frac{A}{A\tau_3} & 0 \\ 0 & \frac{-1}{\tau_2} & 0 & \frac{A}{A\tau_4} \\ 0 & 0 & \frac{-1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{-1}{\tau_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_a}{3600A} & 0 \\ 0 & \frac{\gamma_b}{3600A} \\ 0 & \frac{(1-\gamma_b)}{3600A} \\ \frac{(1-\gamma_a)}{3600A} & 0 \end{bmatrix} u$$

where $x_i = h_i - h_i^o$, $u_j = q_j - q_j^o$, $j = a,b$ and $i = 1,\cdots,4$. $\tau_i = \frac{A}{a_i}\sqrt{\frac{2h_i^0}{g}} \geq 0$, $i = 1,\cdots,4$, are the time constants of each tank. The plant parameters, estimated on a real experimental plant developed at the University of Seville, are given in (Alvarado et al. 2011, Table 1). The constraints on the state are given by $0.2 \leq h_i \leq 1.2$ $[m]$, while the constraints on the inputs are $0 \leq q_a \leq 3.26$ $[m^3/h]$ and $0 \leq q_b \leq 4$ $[m^3/h]$.
The multi-model MPC has been applied by linearizing the plant in 4 different operation points, one of them - given by the pumps at one half of their power - has been taken has the nominal model. These linearization points are shown in Table 1. The linearized models have been discretized using the zero-order hold method with a sampling time of 15 seconds.

The economic objective is to minimize the plant energetic consumption, by minimizing the voltage of the two pumps, and at the same time to maximize the volume of water in the tanks 1 and 2. The economic cost function reads:

$$f_{eco}(y,u,\rho) = (q_a^2 + \rho_1 q_b^2) + \rho_2\frac{V_{min}}{A(h_1+h_2)} \quad (20)$$

where $y = (h_1,h_2)$, $u = (q_a,q_b)$, $\rho = (\rho_1,\rho_2)$ are the prices on the cost function, and V_{min} is the minimum volume to be accumulated.
The controller has been tested in two simulations, in order to compare the multi-model MPC with a mono-model MPC. Model π_1 is been taken as the real plant model. In the mono-model controller, the nominal model π_{no} is used. Three changes of the economic cost have been considered, based on the following prices: $\rho^{[1]} = (1,20)$, $\rho^{[2]} = (1,10)$ and $\rho^{[3]} = (0.4,30)$. The MPC controller has been setup with $Q = I_4$, $R = 0.01I_2$, and $N = 6$. In both simulations, the plant is assumed to start from the linearization point of the nominal model. The optimizations have been executed using the Matlab function *fmincon*.
The results of these simulations are shown in Figure 2. The solid lines represent the multi-model controller, while the dashed line the nominal MPC. In both cases, the controller is capable to drive the plant to the optimal point that optimizes the economic cost function. However, the system is driven to different setpoints. This result becomes clearer in Figure 3. In this figure, the solid line represents the multi-model controller, while the dashed line the nominal MPC. The dotted line represents the optimal value of the economic cost, provided by a stationary optimization of f_{eco}. Notice how, the multi-model approach ensures convergence to the optimal cost, while the nominal MPC always provides a larger cost.

5. CONCLUSIONS

In this paper, a robust MPC that integrates a Real Time Optimizer (RTO), based on a multi-model strategy, has been presented: a finite family of linear models has been considered (multi-model uncertainty), which operates appropriately in a moderate-to-large region around a given operating point. In this way, each operating point defines a linear model, providing an enough accurate description of the system. It has been shown that feasibility and stability conditions are preserved. Moreover, the real plant converges to the optimal point that optimizes the economic cost function. The proposed controller has been tested in simulation on a 4 tanks system.

6. APPENDIX

Lemma 5. Consider system (1) subject to constraints (2). Consider that Assumptions 1-4 hold. Let $x^*_{s,r}$ be the optimal steady state defined in Definition 1. For all $x_r \in \mathcal{X}_{N_r}$ and $x^0_{s,r} \in \mathcal{X}_{s,r}$ such that $x^0_{s,r}$ is a fixed point of the closed-loop system, define the function $e(x_i) = x_i - x^0_{s,r}$. Then, there exists a $\mathcal{K}$-function α_e such that

$$\|e(x_r)\| \geq \alpha_e(\|x_r - x^*_{s,r}\|) \tag{21}$$

Proof. Notice that, due to convexity, $e(x_r)$ is a continuos function (Rawlings and Mayne 2009). Moreover, let us consider these two cases.

(1) $\|e(x_r)\| = 0$ iff $x_r = x^*_{s,r}$. In fact, (i) if $e(x_r) = 0$, then $x_r = x^0_{s,r}$, and from Lemma 6, this implies that $x^0_{s,r} = x^*_{s,r}$; (ii) if $x_r = x^*_{s,r}$, then by optimality $x^0_{s,r} = x^*_{s,r}$, and then $x_r = x^0_{s,r}$. Then, $\|e(x_r)\| = 0$.
(2) $\|e(x_r)\| > 0$ for all $\|x_r - x^*_{s,r}\| > 0$. In fact, for any $x_r \neq x^*_{s,r}$, $\|e(x_r)\| \neq 0$ and moreover $\|x_r - x^*_{s,r}\| > 0$. Then, $\|e(x_r)\| > 0$.

Then, since $\mathcal{X}_{N,r}$ is compact, in virtue of (Vidyasagar 1993, Ch. 5, Lemma 6, pag. 148), there exists a $\mathcal{K}$-function α_e such that $\|e(x_r)\| \geq \alpha_e(\|x_r - x^*_{s,r}\|)$ on $\mathcal{X}_{N,r}$. ■

Lemma 6. Consider system (1) subject to constraints (2). Consider that Assumptions 1-4 hold, and consider a given parameter ρ for the economic cost $f_{eco}(y, u, \rho)$. Consider that, at time k, the disturbance reaches a stationary value d^∞_s and the optimal solution to Problem (15) is such that $y_i(k) = y^0_{s,i}(k)$, $x_i(k) = x^0_{s,i}(k)$ and $u(k) = u^0_s(k)$, and that $x_i(k+1) = x^0_{s,i}(k)$. Then $y_r(k) = y(k) = y^*_s$ and $u_r(k) = u(k) = u^*_s$.

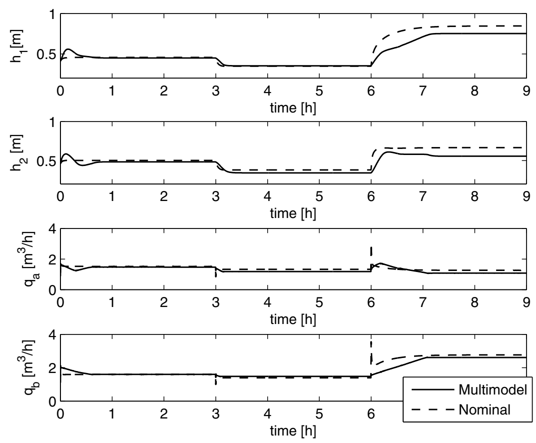

Figure 2. Comparison of multimodel approach and standard MPC: time evolution of outputs and inputs when the controller is applied to a linear plant given by model π_1.

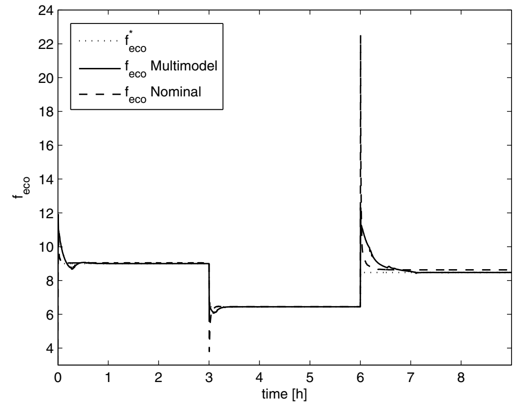

Figure 3. Comparison of multimodel approach and standard MPC: evolutions of the economic cost, f_{eco}.

Proof. Consider that $(x^0_{s,i}(k), u^0_s(k), y^0_{s,i}(k))$ is the optimal solution to (15) at time k. Then

$$V^0_N(x^0_r(k), \hat{d}(k), \rho) = V_O(h^0_s(k), u^0_s(k), \rho)$$

Moreover, $(x^0_{s,i}(k), u^0_s(k), y^0_{s,i}(k))$ is a stationary point.
If the real plant reaches a stationary point, the disturbance observer equation at stationary conditions reads (the time dependence is removed for the sake of clarity):

$$\hat{d}^\infty_{s,i} = \hat{d}^\infty_{s,i} + L^d_i(C_i x^0_{s,i} - y_s + \hat{d}^\infty_{s,i}), \tag{22}$$

where $x^0_{s,i} = A_i x^0_{s,i} + B_i u^0_s$, and u^0_s is the stationary input that correspond to the measured stationary output y_s. Notice that all $\hat{d}^\infty_{s,i}$ are constant, and in particular, $\hat{d}^\infty_{s,r} = 0_p$. If the gain L^d_i is such that the eigenvalues of $(I_p + L^d_i)$ are strictly inside the unite circle, then we have

$$y_s = C_i x^0_{s,i} + d^\infty_{s,i} = y^0_{s,i}, \; \forall i \in \mathbb{I}_{1:L} \tag{23}$$

This implies that, the output of all models in Π converges to the same stationary point $y^0_{s,i} = y_s$. Hence, from (14):

$$V_O(h^0_s, u^0_s, \rho) = L f_{eco}(y_s, u^0_s, \rho) \tag{24}$$

Assume now that, the stationary point at time k is not the optimal one, that is $(y^0_s, u^0_s) \neq (y^*_s, u^*_s)$. Then, by convexity, there exists a $\gamma \in [0, 1]$ such that such that

$$\tilde{\theta} = \gamma\theta^0 + (1-\gamma)\theta^*$$

characterizes a stationary point and moreover

$$V_O(\tilde{h}_s, \tilde{u}^0_s, \rho) \leq V_O(h^0_s, u^0_s, \rho) \tag{25}$$

with $(\tilde{z}_s, \tilde{u}_s) = M_\theta\tilde{\theta}$ and $\tilde{h}_s = N_\theta\tilde{\theta} + d^\infty_s$. That is, since the system is not at the optimal point (y^*_s, u^*_s), it is more convenient to move towards $(\tilde{y}_s, \tilde{u}_s)$, than to remain in (y^0_s, u^0_s). Define as $\tilde{\mathbf{u}} = \{\tilde{u}(0), \tilde{u}(i), ..., \tilde{u}(N-1)\}$ a feasible sequence, to Problem (15), that drives the system from (y^0_s, u^0_s) to $(\tilde{y}_s, \tilde{u}_s)$. This sequence is such that, the j-th element is given by $\tilde{u}(j) = K_i(\tilde{x}_i(j) - \tilde{x}_{s,i}) + \tilde{u}_s$, and $\tilde{x}_i(j+1) = A_i\tilde{x}_i(j) + B_i\tilde{u}(j)$, $\tilde{x}_i(0) = x^0_{s,i}$, for all $i \in \mathbb{I}_{1:L}$. Let us consider the real plant π_r. Then, the cost to drive the system from (y^0_s, u^0_s) to $(\tilde{y}_s, \tilde{u}_s)$ is given by

$$
\begin{aligned}
V_N(x^0_{s,r}, \hat{d}^0_s, \rho; \tilde{\mathbf{u}}, \tilde{\theta}) =& \sum_{j=0}^{N-1} \|\tilde{x}_r(j) - \tilde{x}_{s,r}\|^2_Q + \|\tilde{u}(j) - \tilde{u}_s\|^2_R \\
&+ V_O(\tilde{h}_s, \tilde{u}_s, \rho) \\
=& \|x^0_{s,r} - \tilde{x}_{s,r}\|^2_{P_r} + V_O(\tilde{h}_s, \tilde{u}_s, \rho) \\
=& (1-\gamma)^2 \|\theta^0 - \theta^*\|^2_{H_r} + V_O(\tilde{h}_s, \tilde{u}_s, \rho)
\end{aligned}
$$

where $H_r = M'_{x,r} P_r M_{x,r}$, P_r is given by

$$
P_r = \sum_{j=0}^{N-1} (A_r + B_r K_r)^j (Q + K'_r R K_r)(A_r + B_r K_r)
$$

andK_r is a dead beat controller for the real plant π_r.
Now define $W(\gamma) = (1-\gamma)^2 \|\theta^0 - \theta^*\|^2_{H_r} + V_O(\tilde{h}_s, \tilde{u}_s, \rho)$ and notice that for $\gamma = 1$, $W(1) = V_O(h^0_s, u^0_s, \rho)$. Taking the partial of this function with respect to γ, and evaluating it for $\gamma = 1$ we obtain:

$$
\left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=1} = g^{0'}(h^0_s, u^0_s, \rho)
$$

where $g^{0'} \in \partial V_O(h^0_s, u^0_s, \rho)$, defining $\partial V_O(h^0_s, u^0_s, \rho)$ as the subdifferential of $V_O(h^0_s, u^0_s, \rho)$. From convexity and from (25),

$$
\left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=1} = g^{0'}(h^0_s, u^0_s, \rho) \geq V_O(h^0_s, u^0_s, \rho) - V_O(\tilde{h}_s, \tilde{u}_s, \rho) > 0
$$

This means that there exists a value of $\gamma \in [0, 1)$ such that $V_N(x^0_{s,r}, \hat{d}^0_s, \rho; \tilde{\mathbf{u}}, \tilde{\theta})$ is smaller than the value of the cost $V_N(x^0_{s,r}, \hat{d}^0_s, \rho; \tilde{\mathbf{u}}, \tilde{\theta})$ for $\gamma = 1$, which is $V_O(h^0_s, u^0_s, \rho)$. This contradicts the optimality of the solution to Problem (15) at time k, and the assumption that $(y^0_s(k), u^0_s(k))$ is a fixed point, that is the optimal solution to Problem (15) at time $k+1$ is still $(y^0_s(k), u^0_s(k))$. Then it has to be that $(y^0_s(k), u^0_s(k)) = (y^*_s, u^*_s)$. Moreover, from (24), we can state that this point is the one that optimizes the economic function $f_{eco}(y, u, \rho)$. So the Lemma is proved. ■

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