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Single range observability for cooperative underactuated underwater vehicles.

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Abstract: The paper describes the single range observability issues related to a kinematics model of cooperating underwater vehicles. The paper extends previous results building on an augmented state technique allowing to reformulate the nonlinear observability problem in terms of a linear time varying one. As a result, all possible (globally) unobservable motions are characterized in terms of the systems initial conditions and velocity commands. These results are functional to the design of observers for the navigation of cooperating marine robots having an underactuated model as the one considered. The fundamental results reported are also illustrated by numerical simulations providing evidence of different motions generating the same output, i.e. lacking observability.

Keywords: Marine systems, Autonomous vehicles, Robot kinematics, Robot navigation, Observability.

1. INTRODUCTION

Multi robot systems in air, land and marine applications have received an increasing amount of attention in last years. Indeed many applications as sampling Antonelli et al. (2012) (and references therein), surveillance, mapping and exploration can benefit in robustness and coverage by exploiting cooperating teams of robots rather than single vehicle systems. In particular, research effort is targeting the issue of designing distributed and cooperative control schemes minimizing the need of a centralized team controller. Distributed motion control architectures for cooperative robots will require, in general, that the team members share *some* knowledge about their relative states: typically the relative positions (and eventually velocities) need to be known among neighboring vehicles in order to accomplish cooperative motion. In several applications, as underwater ones where sensors are mostly based on acoustics, team members can measure their relative distances only Soares et al. (2012). This poses a remarkable problem of observability (also known as single beacon navigation in the literature): given a kinematics model of, say, two vehicles, will their relative position and orientation (pose) be observable based on relative euclidean distance measurement only? As the euclidean distance is a nonlinear function of the relative position vector, the observability problem is nonlinear even for point mass (linear) kinematics model as in Arrichiello et al. (2011). Single beacon navigation problems have been addressed in the area of wheeled mobile robotics for relative localization (refer to Martinelli and Siegwart (2005) and Zhou and Roumeliotis (2008), for example). With reference to marine robotics applications, the issue of single beacon navigation (and observability analysis) has been addressed by several authors including Arrichiello et al. (2011), Batista et al. (2011), Gadre and Stilwell (2004), Olson et al. (2004) Fallon et al. (2010) and Webster et al. (2010), Webster et al. (2013). These studies focus on simple (point-mass like) kinematic models often in $2D$ only. The observability issues arising in single beacon navigation are similar to the observability properties of tracking systems. Although tracking systems are more often based on bearing only measurements, the problem of tracking through range-only measurements has received some attention also in oceanic engineering applications Song (1999), Maki et al. (2013).

Based on a recent approach to address the *global* observability of a system model made of two underactuated vehicles, this paper extends previous results Parlangeli et al. (2012) by including the case where both vehicles have constant, but non null, linear and angular velocities. From a methodological point of view, the proposed observability analysis is inspired by the work of Batista et al. (2010) and Batista et al. (2011) where a similar single range observability issue has been addressed for a different (point-mass) kinematics model.

In section 2 the system model is illustrated and the observability problem is defined. In section 3 the adopted observability tools and methods are described whereas the main results of the analysis are reported in section 4. Simulation results providing numerical evidence of unobservable trajectories are illustrated in section 5 and conclusions are finally addressed in section 6. An Appendix section with a few technical results is also included before the Bibliography.

2. SYSTEM MODEL

2.1 Notation

Vectors are denoted with lower case boldface fonts and matrices with capital roman letters. Reference frames are labeled as $<1>$, $<2>$, etc. Given a vector $\mathbf{p} \in \mathbb{R}^3$ its representation in frame $<1>$ will be denoted as ${}^1\mathbf{p}$ having components of $({}^1\mathbf{p})_1, ({}^1\mathbf{p})_2, ({}^1\mathbf{p})_3$. The norm of vector $\mathbf{p}$ will be equivalently indicated with $\|\mathbf{p}\|$ or p. The unit vector of $\mathbf{p} \neq \mathbf{0}$, namely $\mathbf{p}/\|\mathbf{p}\|$, will be indicated with $\check{\mathbf{p}}$. The set of all unit vectors in $\mathbb{R}^3$ will be denoted by $\mathbb{S}^2$. The special orthogonal group of $3D$ rotation matrices is $SO(3)$ and the rotation matrix between frames $<2>$ and $<1>$ will be indicated with ${}^1R_2$ such that ${}^1\mathbf{p} = {}^1R_2\,{}^2\mathbf{p}$. The $3D$ skew symmetric matrix associated to vector product will be indicated with $S(\mathbf{a})$, namely for any $\mathbf{a} = (a_1, a_2, a_3)^\top \in \mathbb{R}^3$

$$S(\mathbf{a}) := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3} \tag{1}$$

such that $S(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}$. A dot on a variable (either a scalar a vector or a matrix) indicates its time derivative. The symbol $\otimes$ will be used for the Kronecker product Laub (2005) between two matrices, namely given $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ the matrix $A \otimes B \in \mathbb{R}^{mp\times nq}$ is defined as:

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \tag{2}$$

and $\oplus$ denotes the Kronecker sum Laub (2005) such that for any two square matrices $C \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$ the matrix $(C \oplus D) \in \mathbb{R}^{mn\times mn}$ is defined as

$$C \oplus D = (I_{m\times m} \otimes C) + (D \otimes I_{n\times n}) \tag{3}$$

where $I_{l\times l} \in \mathbb{R}^{l\times l}$ is the $l-$dimensional identity matrix for any nonnegative integer l. The columns of $I_{l\times l}$ will be denoted with $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_l$. We denote with j the imaginary unit. The set of unobservable vectors is denoted with X_{no}.

2.2 Vehicles model

The kinematics model considered is a $3D$ underactuated vehicle (as a torpedo shaped submersible, a missile or airplane) having a linear velocity with an arbitrarily assigned norm and direction along a unit vector (usually the surge direction) that can be rotated with a desired angular velocity. Mathematically this simple model is captured by the following equations:

$$\dot{\mathbf{q}} = u\,\mathbf{h} \quad : \quad \|\mathbf{h}\| = 1 \tag{4}$$

$$\dot{\mathbf{h}} = \boldsymbol{\omega} \times \mathbf{h} \tag{5}$$

where $\mathbf{q}$ is the position (with respect to an earth-fixed frame) of the origin of a body-fixed frame, $\mathbf{h}$ is the unit vector of its linear velocity having norm $|u|$ and $\boldsymbol{\omega}$ is its angular velocity. Equations (4), (5) define a kinematics control system with state vector $\mathbf{x} = (\mathbf{q}^\top, \mathbf{h}^\top)^\top \in \mathbb{R}^3 \times \mathbb{S}^2$ and inputs $u \in \mathbb{R}$ and $\boldsymbol{\omega} \in \mathbb{R}^3$. This nonlinear model can be viewed as the $3D$ version of the classical planar unicycle nonholonomic model in $2D$. Notice that while many torpedo shaped AUVs cannot turn on the spot due to the use of control surfaces (only) for the angular velocity actuation, some AUVs (Caffaz et al. (2010)) are equipped with side thrusters that allow to actively control pitch and yaw velocities also at zero surge. Indeed the vehicles considered in this paper are of the latter kind (Caffaz et al. (2010)), hence the input angular velocity $\boldsymbol{\omega}$ in equation (5) will be assumed to be independent of the surge speed u. Moreover, although equation (5) does not pose constraints on the roll component $\boldsymbol{\omega}^\top\mathbf{h}$ the fact that such component might not be actuated does not limit the generality of the observability analysis developed in the reminder of the paper.

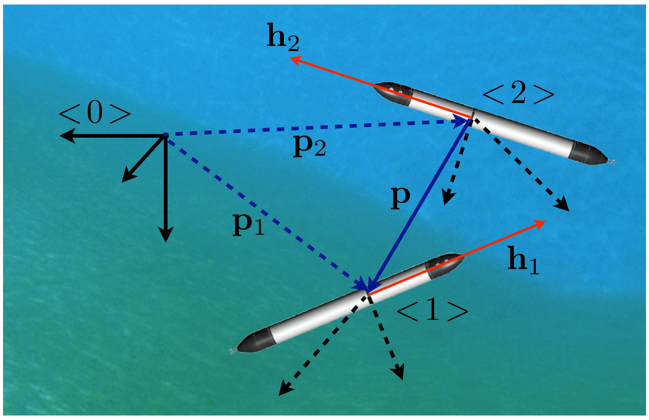

Fig. 1. Geometry of the problem.

With reference to figure 1 consider an earth fixed frame $<0>$ and two body fixed (moving) frames $<1>$ and $<2>$ having origin in $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. Frames $<1>$ and $<2>$ are assumed to move according to the kinematics equations:

$${}^0\dot{\mathbf{p}}_i(t) = u_i(t)\,{}^0\mathbf{h}_i(t)\,, \quad \left\|{}^0\mathbf{h}_i(t)\right\| = 1 \tag{6}$$

$${}^0\dot{\mathbf{h}}_i(t) = {}^0\boldsymbol{\omega}_{i/0}(t) \times {}^0\mathbf{h}_i(t) \tag{7}$$

for $i = 1, 2$. In accordance to the discussion of the model in equations (4 - 5), u_i and $\boldsymbol{\omega}_{i/0}$ are the linear and angular velocities respectively of the two systems and $\mathbf{h}_i$ are two unit vectors. We assume that $u_1(t)$ and $u_2(t)$ cannot be identically zero at the same time. Without loss of generality, in the following it will be assumed that ${}^i\mathbf{h}_i$ is the $x-$axis unit vector of frame $<i>$, namely ${}^i\mathbf{h}_i = {}^i\mathbf{e}_1 = (1, 0, 0)^\top$. Denoting with

$$\mathbf{p} := \mathbf{p}_2 - \mathbf{p}_1 \tag{8}$$

the relative position of frame $<2>$ with respect to $<1>$, we are interested in analyzing the motion of the vehicle $<2>$ as viewed by the observing vehicle $<1>$: standard kinematics calculations based on the projection of equations (6 - 7) on frame $<1>$ lead to the following

$${}^1\dot{\mathbf{p}}(t) = u_2(t)\,{}^1\mathbf{h}_2(t) - u_1(t)\,{}^1\mathbf{h}_1(t) - S({}^1\boldsymbol{\omega}_{1/0}(t))\,{}^1\mathbf{p}(t) \tag{9}$$

$${}^1\dot{\mathbf{h}}_1(t) = \mathbf{0} \tag{10}$$

$${}^1\dot{\mathbf{h}}_2(t) = S({}^1\boldsymbol{\omega}_{2/1}(t))\,{}^1\mathbf{h}_2(t). \tag{11}$$

For the sake of notation compactness and readability, in the following the left hand side superscript 1 will be omitted for vectors expressed in frame $<1>$. Assuming that the observing vehicle with body fixed frame $<1>$ can measure its relative distance to the other vehicle (namely the norm of $\mathbf{p}$), the above model can be written in state space form as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -S(\boldsymbol{\omega}_{1/0}) & u_2\, I_{3\times3} \\ \mathbf{0}_{3\times3} & S(\boldsymbol{\omega}_{2/1}) \end{bmatrix} \mathbf{x}(t) - \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{0}_3 \end{bmatrix} u_1 \tag{12}$$

$$y(t) = \frac{1}{2}\, \mathbf{x}_1^\top(t)\, \mathbf{x}_1(t) \tag{13}$$

being

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} := \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{h}_2(t) \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{S}^2 \tag{14}$$

the state vector and $y(t)$ the output of the system. Notice that in general the variables u_1, u_2, $\boldsymbol{\omega}_{1/0}$ and $\boldsymbol{\omega}_{2/1}$ may be time time-varying, although, unless otherwise stated, in the remaining of the paper they will be assumed constant.

Problem statement
With reference to the nonlinear state space model given by equations (12 - 13) determine all the unobservable states of the system subject to constant and known inputs u_1, u_2, $\boldsymbol{\omega}_{1/0}$ and $\boldsymbol{\omega}_{2/1}$ with $u_1 \neq 0$ and $u_2 \neq 0$.

Remark
The stated observability problem is not trivial in the light of the nonlinear nature of the model. The assumption of constant inputs is reasonable as this covers a wide range of typical maneuvers while simplifying the analysis. As for the assumption that the inputs are known, the linear and angular velocities u_1 and $\boldsymbol{\omega}_{1/0}$ can be considered to be known by vehicle 1 without major problems whereas, eventually, the terms u_2 and $\boldsymbol{\omega}_{2/1}$ might not always be simply accessible. Nevertheless, there are many scenarios where such assumption is not problematic. These include:

- Cooperative navigation: in a cooperative navigation scenario it can be assumed that the vehicles communicate to each other, that they have knowledge of their own attitudes and velocities thanks to on-board sensors and navigation filters, but they do not have a reliable (or any at all) self-position estimate (as in many underwater applications). In this case the two vehicles can transmit their velocity information. If, by example, vehicle 2 transmits its u_2 and ${}^0\boldsymbol{\omega}_{2/0}$ to vehicle 1 (that knows its own u_1, ${}^1\boldsymbol{\omega}_{1/0}$ and ${}^1R_0$), the velocity $\boldsymbol{\omega}_{2/1}$ can be computed as $\boldsymbol{\omega}_{2/1} = {}^1R_0\, {}^0\boldsymbol{\omega}_{2/0} - \boldsymbol{\omega}_{1/0}$ and all the needed inputs would be known.
- Cooperative mission: in a cooperative mission scenario, it can be assumed that the vehicles belong to a common mission and that, although, eventually, they do not transmit information to each other, they know *a priori* the velocities of the other vehicles as these are defined in the known mission plan.
- Straight line motion of $<2>$: if the vehicle 2 is known to move only along straight lines (i.e. ${}^0\boldsymbol{\omega}_{2/0} = \mathbf{0}$), the term $\boldsymbol{\omega}_{2/1}$ would results in $\boldsymbol{\omega}_{2/1} = {}^1R_0\, {}^0\boldsymbol{\omega}_{2/0} - \boldsymbol{\omega}_{1/0} = -\boldsymbol{\omega}_{1/0}$ that is known by 1. The term u_2 can be either communicated to vehicle 1 (cooperative case) or measured by 1 with on-board sensors (non-cooperative case).

Besides the above considerations, it should also be noticed that the stated observability problem has a theoretical interest of its own as it allows to determine an important structural property of the system at hand.

3. GLOBAL OBSERVABILITY ANALYSIS

Following the technique illustrated by Batista et al. (2011) for a point mass kinematics model, with reference to equation (12) we define an additional state

$$x_3 = y \tag{15}$$

having dynamics given by

$$\dot{x}_3 = \mathbf{x}_1^\top\, \dot{\mathbf{x}}_1. \tag{16}$$

By replacing $\dot{\mathbf{x}}_1$ as given by equation (12) in equation (16), the resulting augmented state equation for $(\mathbf{x}_1^\top, \mathbf{x}_2^\top, x_3)^\top$ will contain combinations of mixed terms

$$(\mathbf{x}_1)_i\, (\mathbf{x}_2)_j \quad \text{and} \quad (\mathbf{x}_2)_i\, (\mathbf{x}_2)_j \quad \forall\, i, j \in \{1, 2, 3\}. \tag{17}$$

Define the new state variables:

$$x_\ell = \mathbf{x}_1^\top(t)\left(\mathbf{e}_j \mathbf{e}_i^\top\right) \mathbf{x}_2(t) \quad \ell = 3i + j \tag{18}$$

$$x_\kappa = \mathbf{x}_2^\top(t)\left(\mathbf{e}_j \mathbf{e}_i^\top\right) \mathbf{x}_2(t)\; \kappa = 8 + 3i + j \tag{19}$$

$$i, j = 1, .., 3 \tag{20}$$

namely the components of $(\mathbf{x}_1 \otimes \mathbf{x}_2)$, $(\mathbf{x}_2 \otimes \mathbf{x}_2)$ and denote the augmented state as

$$\mathbf{x}(t) = \left[\mathbf{x}_1^\top(t)\;\; \mathbf{x}_2^\top(t)\;\; x_3(t)\;\; x_4(t)\;\; \ldots\;\; x_{21}(t)\right]^\top \in \mathbb{R}^{25}. \tag{21}$$

Taking the time derivative of $\mathbf{x}$ in equation (21) lengthy, but straightforward calculations lead to the following result for the dynamics of $\mathbf{x}$:

$$\begin{cases} \dot{\mathbf{x}}(t) = A\, \mathbf{x}(t) + B\, u_1(t) \\ y(t) = C\, \mathbf{x}(t) \end{cases} \tag{22}$$

where

$$A = \begin{bmatrix} -S_1 & u_2\, I_3 & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times9} & \mathbf{0}_{3\times9} \\ \mathbf{0}_{3\times3} & S_2 & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times9} & \mathbf{0}_{3\times9} \\ -\mathbf{e}_1^\top u_1 & \mathbf{0}_{1\times3} & 0 & u_2\, [\mathbf{e}_1^\top\; \mathbf{e}_2^\top\; \mathbf{e}_3^\top] & \mathbf{0}_{1\times9} \\ \mathbf{0}_{9\times3} & -u_1\, \mathbf{e}_1 \otimes I_{3\times3} & \mathbf{0}_{9\times1} & S_3 & u_2\, I_{9\times9} \\ \mathbf{0}_{9\times3} & \mathbf{0}_{9\times3} & \mathbf{0}_{9\times1} & \mathbf{0}_{9\times9} & S_4 \end{bmatrix} \tag{23}$$

being $S_1 = S(\boldsymbol{\omega}_{1/0})$, $S_2 = S(\boldsymbol{\omega}_{2/1})$, $S_3 = S(\boldsymbol{\omega}_{2/1}) \oplus S^\top(\boldsymbol{\omega}_{1/0})$ and $S(\boldsymbol{\omega}_{2/1}) \oplus S(\boldsymbol{\omega}_{2/1})$ while the matrices $B \in \mathbb{R}^{25}$ and $C \in \mathbb{R}^{1\times25}$ are given by

$$B = \begin{bmatrix} -\,\mathbf{e}_1 \\ \mathbf{0}_{22\times1} \end{bmatrix}, \quad C = [\mathbf{0}_{1\times3}\;\; \mathbf{0}_{1\times3}\;\; 1\;\; \mathbf{0}_{1\times9}\;\; \mathbf{0}_{1\times9}]. \tag{24}$$

In the above derivation, without loss of generality, it has been assumed that ${}^1\mathbf{h}_1 = \mathbf{e}_1 = (1, 0, 0)^\top$.

Equation (22) defines an algebraically augmented state systems with respect to the original system in equations (12) - (13). If the inputs u_1, u_2, $\boldsymbol{\omega}_{2/1}$ and $\boldsymbol{\omega}_{1/0}$ are constant the augmented state system in equation (22) is reduced to a linear time invariant (LTI) system and its observability set can be identified by standard LTI system tools.

By construction, all trajectories of system (12) - (13) are also trajectories of (22). The converse is not true, as a consequence of a much larger state space which allows a larger set of initial conditions. On the other hand, if the initial conditions are constrained to belong to the admissible initial conditions for the system (12) - (13), it is matter of simple computation to prove that the two systems have the same evolution.

Denote with $\mathcal{V}$ the set of initial conditions for (22) corresponding to admissible initial conditions for system (12), namely those vectors $\mathbf{x} = [\mathbf{x}_1^\top\, \mathbf{x}_2^\top\, x_3\, \mathbf{x}_4^\top\, \mathbf{x}_5^\top]^\top$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$ and $\mathbf{x}_4, \mathbf{x}_5 \in \mathbb{R}^9$ such that $\mathbf{x}_2^\top \mathbf{x}_2 = 1$, $\mathbf{x}_4 = \mathbf{x}_1 \otimes \mathbf{x}_2$, $\mathbf{x}_5 = \mathbf{x}_2 \otimes \mathbf{x}_2$ and finally $x_3 = \mathbf{x}_1^\top \mathbf{x}_1$.

Remark Notice that the dimension of the state space of system (22) is $\mathbb{R}^{25}$, though one could wonder to deal with a state space $\mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}^{19}$. This ambiguity comes from the fact that $\mathbf{x}_2(t)$ is a unitary vector. We capture this feature by imposing the initial condition $\mathbf{x}_2(0) : \|\mathbf{x}_2(0)\| = 1$ when we build the set $\mathcal{V}$ of the admissible initial conditions in conjunction with the fact that equations (22) are an isometry for $\mathbf{x}_2(t)$. Actually this point is crucial: the standard linear systems theory observability methods would not apply to system (22) as defined on $\mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}^{19}$, but indeed they can be applied to the same system as defined on $\mathbb{R}^{25}$. The whole point of the proposed approach is thus to identify the non observable states of system (22) on $\mathbb{R}^{25}$ using standard linear systems theory methods and then verify which of these states are also admissible as solutions of the original system, i.e. such that $\mathbf{x}_2(0) : \|\mathbf{x}_2(0)\| = 1$.

3.1 Determination of the indistinguishable states

From a practical point of view, the construction of the set of indistinguishable vectors from a given $\mathbf{x}_0$ is performed according to the following steps:

S-1) Compute the initial condition $\bar{\mathbf{x}}$ belonging to the augmented state space which corresponds to $\mathbf{x}_0$.
S-2) Compute the set of all the indistinguishable states for the augmented system $\mathcal{I}_{\bar{\mathbf{x}}} = \bar{\mathbf{x}} + ker(O)$, where O is the observability matrix of the linear system defined over the augmented state space.
S-3) Compute the intersection $\mathcal{I}_{\bar{\mathbf{x}}} \cap \mathcal{V}$. There is a one-to-one correspondence between one element of this set and the distinct states of system (12) - (13) that produce the same output of the given state $\mathbf{x}_0$. (Trivially, $\bar{\mathbf{x}}$ is observable if $\mathcal{I}_{\bar{\mathbf{x}}} \cap \mathcal{V}$ is singleton).

In the following we apply the above procedure to perform a thorough observability analysis for a general configuration of constant inputs with nonzero linear velocity hence extending the preliminary results in Parlangeli et al. (2012) where one of the two vehicles had always zero linear speed. Notice that the described procedure allows to reformulate the initial nonlinear observability problem as an equivalent one involving a linear system with initial conditions constrained to a nonlinear manifold. This latter problem can be tackled with standard linear systems tools in the presence of some algebraic constraints.

With reference to step S-1 described above, consider two indistinguishable states $\mathbf{x}$ and $\bar{\mathbf{x}}$ in $\mathcal{V}$: their difference belongs to the unobservable subspace, namely $\mathbf{x}, \bar{\mathbf{x}} \in \mathcal{V}$, $\boldsymbol{\mu} = \mathbf{x} - \bar{\mathbf{x}} \in X_{no}$. Notice that given the structure of $\mathcal{V}$ in general $\boldsymbol{\mu} \notin \mathcal{V}$. Considering that $\bar{\mathbf{x}} = \mathbf{x} + \boldsymbol{\mu}$, we embed vector $\boldsymbol{\mu}$ into the structure of an admissible initial condition $\bar{\mathbf{x}}(0) \in \mathcal{V}$

$$\bar{\mathbf{x}}(0) = \begin{bmatrix} \mathbf{x}_1(0) + \boldsymbol{\nu}_a \\ \mathbf{x}_2(0) + \boldsymbol{\nu}_b \\ x_3(0) \\ (\mathbf{x}_1(0) + \boldsymbol{\nu}_a) \otimes (\mathbf{x}_2(0) + \boldsymbol{\nu}_b) \\ (\mathbf{x}_2(0) + \boldsymbol{\nu}_b) \otimes (\mathbf{x}_2(0) + \boldsymbol{\nu}_b) \end{bmatrix}, \tag{25}$$

with $\boldsymbol{\nu}_a, \boldsymbol{\nu}_b \in \mathbb{R}^3$. Finally, we impose the remaining conditions for a vector to belong to $\mathcal{V}$:

$$\begin{gathered} \frac{1}{2} (\mathbf{x}_1(0) + \boldsymbol{\nu}_a)^\top (\mathbf{x}_1(0) + \boldsymbol{\nu}_a) = \frac{1}{2} \mathbf{x}_1(0)^\top \mathbf{x}_1(0) \\ (\mathbf{x}_2(0) + \boldsymbol{\nu}_b)^\top (\mathbf{x}_2(0) + \boldsymbol{\nu}_b) = 1 \end{gathered} \tag{26}$$

which can be reduced to

$$\begin{aligned} \boldsymbol{\nu}_a^\top (2\,\mathbf{x}_1(0) + \boldsymbol{\nu}_a) &= 0 \\ \boldsymbol{\nu}_b^\top (2\,\mathbf{x}_2(0) + \boldsymbol{\nu}_b) &= 0\,. \end{aligned} \tag{27}$$

It is worth noticing that an unobservable vector $\boldsymbol{\mu}$ giving rise to a feasible trajectory starting from a $\bar{\mathbf{x}}(0)$ might depend on the initial condition $\mathbf{x}(0)$. Indeed the relation between of the unobservable $\boldsymbol{\mu}$ vectors satisfying the parametrization (25) and the uncertainty parameters $\boldsymbol{\nu}_a, \boldsymbol{\nu}_b$ is

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\nu}_a \\ \boldsymbol{\nu}_b \\ 0 \\ \mathbf{x}_1(0) \otimes \boldsymbol{\nu}_b + \boldsymbol{\nu}_a \otimes \mathbf{x}_2(0) + \boldsymbol{\nu}_a \otimes \boldsymbol{\nu}_b \\ \mathbf{x}_2(0) \otimes \boldsymbol{\nu}_b + \boldsymbol{\nu}_b \otimes \mathbf{x}_2(0) + \boldsymbol{\nu}_b \otimes \boldsymbol{\nu}_b \end{bmatrix}. \tag{28}$$

Notice that equations (27) depend on the value of the initial conditions, so it is not surprising (and it will be evident in the next analysis) that the observability itself can depend on the initial condition. This is a classical result for nonlinear systems: some initial conditions may be uniquely reconstructed while some others cannot. A goal of our analysis is the mathematical description of all indistinguishable (feasible) states for a given $\mathbf{x}_0$.

Inspired by the Popov - Belevitch - Hautus (PBH) Lemma Antsaklis and Michel (2007), we impose that $\boldsymbol{\mu} \in X_{no}$ by looking for those directions belonging to $\ker\{C\}$ that are A-invariant (i.e., eigenvectors of A). Notice that eigenvectors with nonzero imaginary component generate two (real) independent unobservable directions Antsaklis and Michel (2007). Using the parametrization (25), we look for nonzero $\boldsymbol{\nu}_a, \boldsymbol{\nu}_b$ such that

$$A\,\boldsymbol{\mu} = \lambda\,\boldsymbol{\mu} \tag{29}$$

being λ an eigenvalue of A in equation (23) and $\boldsymbol{\mu}$ defined in equation (28) that, by its very construction, satisfies $C\boldsymbol{\mu} = 0$. In particular after some algebraic manipulation, equation (29) can be expressed as follows:

$$\begin{aligned} & -S(\boldsymbol{\omega}_{1/0})\,\boldsymbol{\nu}_a + u_2\,\boldsymbol{\nu}_b = \lambda\,\boldsymbol{\nu}_a \\ & \quad S(\boldsymbol{\omega}_{2/1})\,\boldsymbol{\nu}_b = \lambda\,\boldsymbol{\nu}_b \\ & -u_1\,\mathbf{e}_1^\top \boldsymbol{\nu}_a + u_2\,\left(\mathbf{x}_1^\top(0)\,\boldsymbol{\nu}_b + \boldsymbol{\nu}_a^\top\,\mathbf{x}_2(0) + \boldsymbol{\nu}_a^\top\,\boldsymbol{\nu}_b\right) = 0 \\ & \boldsymbol{\nu}_a \otimes \left(S(\boldsymbol{\omega}_{2/1})\,\mathbf{x}_2(0)\right) + \\ & \quad + \left(S^\top(\boldsymbol{\omega}_{1/0})\,\mathbf{x}_1(0) + u_2\,\mathbf{x}_2(0) + \lambda\,\boldsymbol{\nu}_a - u_1\,\mathbf{e}_1\right) \otimes \boldsymbol{\nu}_b = \mathbf{0} \\ & \boldsymbol{\nu}_b \otimes \left(S(\boldsymbol{\omega}_{2/1})\,\mathbf{x}_2(0)\right) + \\ & \quad + \left(S(\boldsymbol{\omega}_{2/1})\,\mathbf{x}_2(0) + \lambda\,\boldsymbol{\nu}_b\right) \otimes \boldsymbol{\nu}_b = \mathbf{0}. \end{aligned} \tag{30}$$

4. MAIN RESULTS

With reference to step S-2 described in subsection 3.1, a key tool here adopted for the observability analysis is the PBH Lemma (Antsaklis and Michel (2007)): this requires a rank-based analysis on each eigenvalue of the system (equations (29) - (30) in our case). A first step toward this goal is the computation of the spectrum of the dynamical matrix. Here, things are simplified by the block-diagonal structure of such matrix; following the rules of the eigenstructure of kronecker product matrices (Laub (2005)), by direct computation the set of eigenvalues of (29) - (30) is

$$\begin{aligned}\{0,\ &\pm j\,\omega_{1/0}, \pm j\,\omega_{2/1}, \pm j\,(\omega_{1/0}+\omega_{2/1}),\\ &\pm j\,(\omega_{1/0}-\omega_{2/1}), \pm j\,2\,\omega_{2/1}\}.\end{aligned} \tag{31}$$

A first preliminary result is about the observability of all eigenvalues but $\lambda = 0, \pm j\,\omega_{1/0}, \pm j\,\omega_{2/1}$.

Proposition 4.1. *Given the system (22) - (24) with spectrum (31), the motion associated to the nonnull eigenvalues $\pm j\,(\omega_{1/0}+\omega_{2/1}), \pm j\,(\omega_{1/0}-\omega_{2/1}), \pm j\,2\,\omega_{2/1}$ is always observable from the output map (24).*

Proof. According to the PBH Lemma, the unobservable evolutions are directed along the eigenvectors of matrix A belonging to the kernel of the output map. Consider the first two blocks of equations in (30):

$$\begin{aligned} S(\boldsymbol{\omega}_{1/0})\,\boldsymbol{\nu}_a + u_2\,\boldsymbol{\nu}_b &= \lambda\boldsymbol{\nu}_a\\ S(\boldsymbol{\omega}_{2/1})\,\boldsymbol{\nu}_b &= \lambda\boldsymbol{\nu}_b.\end{aligned} \tag{32}$$

The last equation is satisfied either for $\boldsymbol{\nu}_b = \mathbf{0}$ or $\lambda \in \{0, \pm j\omega_{2/1}\}$. But, in turn, if we consider $\boldsymbol{\nu}_b = \mathbf{0}$ the first block has a nonzero solution either for $\boldsymbol{\nu}_a = \mathbf{0}$ or $\lambda \in \{0, \pm j\omega_{1/0}\}$. The solution $\boldsymbol{\nu}_b = \mathbf{0}$ and $\boldsymbol{\nu}_a = \mathbf{0}$ leads to a zero vector $\boldsymbol{\mu}$, so it must be excluded, and we conclude that nonzero solutions of (30) can exist only if $\lambda \in \{0, \pm j\omega_{2/1}, \pm j\omega_{1/0}\}$. □

The previous result is important because it allows to reduce the problem to the observability analysis of the motion associated to three eigenvectors. Another preliminary result having the same importance of reducing the initial problem is related to the following parametrization of all unobservable configurations corresponding to $\boldsymbol{\nu}_b = \mathbf{0}$.

4.1 Unobservable configurations with $\boldsymbol{\nu}_b = \mathbf{0}$

The next two results are on the unobservability conditions and characterization for the extended-space system; after that, the same results are adjusted for the original nonlinear system

Proposition 4.2. *Consider system (22) - (24) with spectrum (31); set $\boldsymbol{\nu}_b = \mathbf{0}$ in (28) and consider $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$.*

Then, system (22) - (24) is completely observable if and only if neither

$$\begin{cases}\boldsymbol{\omega}_{1/0} \perp \left(\pm\, u_2\,\breve{\boldsymbol{\omega}}_{2/1} - u_1\,\mathbf{e}_1\right)\\ \mathbf{x}_2(0) = \pm\,\breve{\boldsymbol{\omega}}_{2/1}\end{cases} \quad \textit{(case a)} \tag{33}$$

nor

$$\begin{cases}\boldsymbol{\omega}_{1/0} \parallel \left(\pm\, u_2\,\breve{\boldsymbol{\omega}}_{2/1} - u_1\,\mathbf{e}_1\right)\\ \mathbf{x}_2(0) = \pm\,\breve{\boldsymbol{\omega}}_{2/1}\end{cases} \quad \textit{(case b)} \tag{34}$$

holds. Moreover, in the case of loss of observability, all the indistinguishable states of (22) - (24) with respect to the initial conditions

$$\begin{aligned}&\mathbf{x}_1(0)\ \textit{arbitrary}\\ &\mathbf{x}_2(0) = \pm\,\breve{\boldsymbol{\omega}}_{2/1}\end{aligned} \tag{35}$$

are given by the following configurations:

$$\begin{cases}\mathbf{x}_1^*(0) = \mathbf{x}_1(0) + \alpha\,\boldsymbol{\omega}_{1/0}\ :\ \alpha \neq 0\\ \mathbf{x}_2^*(0) = \mathbf{x}_2(0)\end{cases} \tag{36}$$

$$\begin{cases}\mathbf{x}_1^*(0) = \mathbf{x}_1(0) + \boldsymbol{\gamma}\ :\ \boldsymbol{\gamma}^\top\boldsymbol{\omega}_{1/0} = 0\\ \mathbf{x}_2^*(0) = \mathbf{x}_2(0)\end{cases} \tag{37}$$

in other words, the initial conditions

$$(\mathbf{x}_1(0)^\top, \mathbf{x}_2(0)^\top)^\top \ \textit{and}\ (\mathbf{x}_1^*(0)^\top, \mathbf{x}_2^*(0)^\top)^\top$$

generate the same output.

Proof. Set $\boldsymbol{\nu}_b = \mathbf{0}$. As a direct consequence of this choice, equations (29) - (30) take the form:

$$-S(\boldsymbol{\omega}_{1/0})\,\boldsymbol{\nu}_a = \lambda\,\boldsymbol{\nu}_a \tag{38}$$

$$-u_1\,\mathbf{e}_1^\top\boldsymbol{\nu}_a + u_2\boldsymbol{\nu}_a^\top\,\mathbf{x}_2(0) = 0 \tag{39}$$

$$\boldsymbol{\nu}_a \otimes \left(S(\boldsymbol{\omega}_{2/1})\,\mathbf{x}_2(0)\right) = \mathbf{0} \tag{40}$$

for a nonzero $\boldsymbol{\nu}_a$. Equation (38) implies $\lambda \in \{0, \pm j\omega_{1/0}\}$. As for equation (40), recall that we must consider $\boldsymbol{\nu}_a \neq \mathbf{0}$ to guarantee (28) $\boldsymbol{\mu} \neq \mathbf{0}$: hence, assuming $\boldsymbol{\nu}_a \neq \mathbf{0}$, equation (40) is satisfied either by $\mathbf{x}_2(0) = \pm\,\breve{\boldsymbol{\omega}}_{2/1}$ if $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$ or by an arbitrary unit vector $\mathbf{x}_2(0)$ if $\boldsymbol{\omega}_{2/1} = \mathbf{0}$.

Equation (39) can be always satisfied by some values of u_1 and u_2. In view of the previous results on $\mathbf{x}_2(0)$, equation (39) can be rewritten as

$$\boldsymbol{\nu}_a^\top\,\tilde{\mathbf{v}}_\pm = 0 \tag{41}$$

$$\tilde{\mathbf{v}}_\pm := \left(\pm\, u_2\,\breve{\boldsymbol{\omega}}_{2/1} - u_1\,\mathbf{e}_1\right). \tag{42}$$

Defining the set

$$\mathcal{P}_\pm = \left\{\mathbf{x} \in \mathbb{R}^3\ :\ \mathbf{x}^\top\left(\pm\, u_2\,\breve{\boldsymbol{\omega}}_{2/1} - u_1\,\mathbf{e}_1\right) = 0\right\} \tag{43}$$

(one for the + sign and one for the − sign), the condition in equation (41) can be alternatively formulated stating that

$$\boldsymbol{\nu}_a \in \mathcal{P}_\pm. \tag{44}$$

Now, $\boldsymbol{\nu}_a$ is an eigenvector of $S(\boldsymbol{\omega}_{1/0})$ with eigenvalue $\lambda = 0$ or $\lambda = \pm\, j\omega_{1/0}$. Consider $\lambda = 0$ first; then $\boldsymbol{\nu}_a = \alpha\,\boldsymbol{\omega}_{1/0}$ with $\alpha \neq 0$ and equation (41) is satisfied if $\boldsymbol{\omega}_{1/0}^\top\left(u_1 \mp \breve{\boldsymbol{\omega}}_{2/1}u_2\right) = 0$ as in equation (33). Notice that the unobservable states in equation (35), (36) are initial conditions of unobservable motions associated to the eigenvalue $\lambda = 0$.

Consider now the loss of observability of eigenvalues $\lambda = \pm\, j\omega_{1/0}$: then, the unobservable subspace is generated by the real and imaginary parts of its eigenvectors (see Appendix). On the other hand, the real and imaginary part of any complex eigenvector of $S(\boldsymbol{\omega}_{1/0})$ span the plane orthogonal to $\boldsymbol{\omega}_{1/0}$, so relation (41) is indeed always satisfied if $\tilde{\mathbf{v}}_\pm$ is orthogonal to such plane or, equivalently, if $\tilde{\mathbf{v}}_\pm$ is parallel to $\boldsymbol{\omega}_{1/0}$, i.e.

$$\boldsymbol{\omega}_{1/0} \parallel \left(\pm\, u_2\,\breve{\boldsymbol{\omega}}_{2/1} - u_1\,\mathbf{e}_1\right), \tag{45}$$

namely

$$\boldsymbol{\omega}_{1/0} \perp \mathcal{P}_{\pm}.$$

The condition derived above can be seen as an existence condition for unobservability in terms of motion parameters: if the vehicles' velocities satisfy (45) then some unobservable states are expected to exist, otherwise not. In the former case, the unobservable states are those orthogonal to $\boldsymbol{\omega}_{1/0}$, and (37) follows. □

Following the same logic of the previous proof it is possible to find an analogous result related to the case of $\omega_{2/1} = 0$. We report the statement hereafter and we give omit the proof for the sake of brevity.

Proposition 4.3. *Consider system (22) - (24) with spectrum (31). If $\boldsymbol{\nu}_b = \mathbf{0}$ in (28) and $\boldsymbol{\omega}_{2/1} = \mathbf{0}$, then all the indistinguishable states of (22) - (24) are given by the following configurations:*

$$\mathbf{x}_1(0) \text{ and } \mathbf{x}_2(0) \text{ arbitrary} \tag{46}$$

$$\text{if } \boldsymbol{\omega}_{1/0} \perp (\pm u_2\, \mathbf{x}_2(0) - u_1\, \mathbf{e}_1) \text{ holds then} \tag{47}$$

$$\begin{cases} \mathbf{x}_1^*(0) = \mathbf{x}_1(0) + \alpha\, \boldsymbol{\omega}_{1/0} \;:\; \alpha \neq 0 \\ \mathbf{x}_2^*(0) = \mathbf{x}_2(0) \end{cases} \tag{48}$$

$$\text{or, if } \boldsymbol{\omega}_{1/0} \parallel (\pm u_2\, \mathbf{x}_2(0) - u_1\, \mathbf{e}_1) \text{ holds then} \tag{49}$$

$$\begin{cases} \mathbf{x}_1^*(0) = \mathbf{x}_1(0) + \boldsymbol{\gamma} \;:\; \boldsymbol{\gamma}^\top \boldsymbol{\omega}_{1/0} = 0 \\ \mathbf{x}_2^*(0) = \mathbf{x}_2(0) \end{cases} \tag{50}$$

where the initial conditions

$$(\mathbf{x}_1(0)^\top, \mathbf{x}_2(0)^\top)^\top \text{ and } (\mathbf{x}_1^*(0)^\top, \mathbf{x}_2^*(0)^\top)^\top$$

generate the same output of the model in equations (22) - (24).

The main difference between the case $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$ and $\boldsymbol{\omega}_{2/1} = \mathbf{0}$ relates to the existence condition of unobservable configurations. In the latter case, the two unobservability conditions (equation (45) and first line of (33)) involve system parameters that can be verified in advance. On the contrary, in the latter case equations (47) and (49) depend on the value $x_2(0)$ which is not known and so there is no way to ensure in advance complete observability by choosing suitable motion parameters.

With reference to the three steps procedure described in subsection 3.1, we are now ready to implement step S-3, namely to identify which unobservable states of the augmented system (22) - (24) are also feasible states for the original system (12), (13). In particular, consider the case $\boldsymbol{\nu}_b = \mathbf{0}$ addressed in Propositions 4.8 and 4.3.

Proposition 4.4. *Consider system (12), (13). If $\boldsymbol{\nu}_b = \mathbf{0}$ in (28) and $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$, then all the indistinguishable states of (12), (13) are given the following configurations:*

- *Initial conditions as in equation (35) and (36) with*

$$\alpha = -\frac{2\,\mathbf{x}_1(0)^\top \boldsymbol{\omega}_{1/0}}{\|\,\boldsymbol{\omega}_{1/0}\,\|^2}. \tag{51}$$

- *Initial conditions as in equation (35) and (37) with*

$$\begin{aligned} \boldsymbol{\gamma} = \mathbf{x}_1(0) - d(\mathbf{x}_1(0), \mathcal{P}_{\pm})\check{\boldsymbol{\omega}}_{1/0} + \\ + \sqrt{\|\mathbf{x}_1(0)\|^2 - d(\mathbf{x}_1(0), \mathcal{P}_{\pm})^2}\, (\check{\mathbf{v}}_1 \cos(\theta) + \check{\mathbf{v}}_2 \sin(\theta)) \end{aligned} \tag{52}$$

where: i) $d(\mathbf{x}_1(0), \mathcal{P}_{\pm})$ is the euclidean distance of $\mathbf{x}_1(0)$ from $\mathcal{P}_{\pm}$, ii) $\check{\mathbf{v}}_1$ and $\check{\mathbf{v}}_2$ are two orthonormal vectors belonging to $\mathcal{P}_{\pm}$ and iii) θ is a generic angle in $[-\pi, \pi]$.

Proof. Consider the first case of Proposition 4.4, namely the one leading to equation (51). The indistinguishable states for the original system in equations (12) - (13) will satisfy equation (33) while also being compatible with the output map (13). In particular, the indistinguishable states with respect to a given $\mathbf{x}_1(0)$ are those

$$\mathbf{x}_1^*(0) = \mathbf{x}_1(0) + \alpha\, \boldsymbol{\omega}_{1/0}$$

such that

$$\mathbf{x}_1^*(0)^\top \mathbf{x}_1^*(0) = \mathbf{x}_1(0)^\top \mathbf{x}_1(0)$$

that immediately implies equation (57) for α.

Consider now the set (37) according to which an unobservable vector $\boldsymbol{\gamma} = \boldsymbol{\nu}_a$ must belong to $\mathcal{P}_{\pm}$ as defined in equation (43). Moreover, among these vectors $\boldsymbol{\gamma} = \boldsymbol{\nu}_a$, those associated to indistinguishable states of system (12), (13) must satisfy equations (26) and (27) that is

$$\boldsymbol{\nu}_a^\top \boldsymbol{\nu}_a = -2\,\mathbf{x}_1(0)^\top \boldsymbol{\nu}_a.$$

Relation (26) for a given $\mathbf{x}_1(0)$ and $\boldsymbol{\nu}_a \in \mathbb{R}^3$ can be seen as a sphere $\mathcal{S}_{\mathbf{x}_1(0)}$ centered in $-\mathbf{x}_1(0)$ with radius $\|\mathbf{x}_1(0)\|$. The values $\boldsymbol{\nu}_a \in \mathbb{R}^3$ that belong to $\mathcal{P}_{\pm}$ and concurrently verify (26) belong to the intersection set $\mathcal{S}_{\mathbf{x}_1(0)} \bigcap \mathcal{P}_{\pm}$, i.e. to a circumference (that we denote with $\mathcal{C}_{\mathbf{x}_1(0)}$ from now on). If the set $\mathcal{C}_{\mathbf{x}_1(0)}$ contains nonzero real points, then these points are associated to unobservable initial conditions. In order to check whether there exist real points, one may verify if the distance between the center of the circumference and the center of the sphere is smaller than the sphere radius or not, i.e.

$$d(\mathbf{x}_1(0), \mathcal{P}_{\pm})^2 \leq \|\mathbf{x}_1(0)\|^2. \tag{53}$$

Interestingly, by direct calculation it can be shown that equation (53) is always true. In particular, the sphere $\mathcal{S}_{\mathbf{x}_1(0)}$ and the plane $\mathcal{P}_{\pm}$ both contain the origin: the condition of full observability will hence correspond to the situation where the plane $\mathcal{P}_{\pm}$ and the sphere $\mathcal{S}_{\mathbf{x}_1(0)}$ are tangent being the origin the tangent point.

As for the condition expressed by equation (58), the points $\boldsymbol{\gamma}$ are precisely the points of $\mathcal{C}_{\mathbf{x}_1(0)}$, i.e. of $\mathcal{S}_{\mathbf{x}_1(0)} \bigcap \mathcal{P}_{\pm}$. Such points can be parametrized identifying the center of $\mathcal{C}_{\mathbf{x}_1(0)}$, namely $\mathbf{c}$ the closest point of $\mathcal{P}_{\pm}$ from $\mathbf{x}_1(0)$:

$$\mathbf{c} := \mathbf{x}_1(0) - d(\mathbf{x}_1(0), \mathcal{P}_{\pm})\check{\boldsymbol{\omega}}_{1/0}. \tag{54}$$

The radius r of $\mathcal{C}_{\mathbf{x}_1(0)}$ is given by

$$r = \sqrt{\|\mathbf{x}_1(0)\|^2 - d(\mathbf{x}_1(0), \mathcal{P}_{\pm})^2}. \tag{55}$$

From equations (54) and (55) the condition (52) follows immediately. □

Finally, the indistinguishable states of (12), (13) can be identified in the case $\boldsymbol{\nu}_b = \mathbf{0}$ and $\boldsymbol{\omega}_{2/1} = \mathbf{0}$. In particular, this case is addressed in the following Proposition. Consider the plane

$$\tilde{\mathcal{P}}_{\mathbf{x}_2(0)} = \left\{ \mathbf{x} \in \mathbb{R}^3 \;:\; \mathbf{x}^\top (u_2\, \mathbf{x}_2(0) + u_1\, \mathbf{e}_1) = 0 \right\}. \tag{56}$$

Proposition 4.5. *Consider system (12), (13). If $\boldsymbol{\nu}_b = \mathbf{0}$ in (28) and $\boldsymbol{\omega}_{2/1} = \mathbf{0}$, then all the indistinguishable states of (12), (13) are given the following configurations:*

- *Initial conditions as in equation (47) and (48) with*
$$\alpha = -\frac{2\,\mathbf{x}_1(0)^\top \boldsymbol{\omega}_{1/0}}{\|\,\boldsymbol{\omega}_{1/0}\,\|^2}\,. \tag{57}$$
- *Initial conditions as in equation (49) and (50) with*
$$\begin{aligned}\gamma &= \mathbf{x}_1(0) - d(\mathbf{x}_1(0), \tilde{\mathcal{P}}_{\mathbf{x}_2(0)})\check{\boldsymbol{\omega}}_{1/0}\ + \\ &+\sqrt{\|\mathbf{x}_1(0)\|^2 - d(\mathbf{x}_1(0), \tilde{\mathcal{P}}_{\mathbf{x}_2(0)})^2}\ (\check{\mathbf{v}}_1\cos(\theta) + \check{\mathbf{v}}_2\sin(\theta))\end{aligned} \tag{58}$$
where: i) $d(\mathbf{x}_1(0), \tilde{\mathcal{P}}_{\mathbf{x}_2(0)})$ is the euclidean distance of $\mathbf{x}_1(0)$ from $\tilde{\mathcal{P}}_{\mathbf{x}_2(0)}$, ii) $\check{\mathbf{v}}_1$ and $\check{\mathbf{v}}_2$ are two orthonormal vectors belonging to $\tilde{\mathcal{P}}_{\mathbf{x}_2(0)}$ and iii) θ is a generic angle in $[-\pi, \pi]$.

The proof of Proposition 4.5 is omitted for the sake of brevity. It can be derived following exactly the same procedure used to demonstrate Proposition 4.9 the only difference being to replace the plane $\mathcal{P}_\pm$ defined in (43) with the plane $\tilde{\mathcal{P}}_{\mathbf{x}_2(0)}$ defined in equation (56).

We are now ready to derive the set of solutions associated to $\boldsymbol{\nu}_b \neq \mathbf{0}$.

4.2 Unobservable configurations with $\boldsymbol{\nu}_b \neq \mathbf{0}$

As far as the case $\boldsymbol{\nu}_b \neq 0$ is concerned, some preliminary remarks are useful to decompose the problem into a set of easier problems. A first consequence of equation (30) is that a nonzero solution $\boldsymbol{\nu}$ can exist only if λ is an eigenvalue of $S(\boldsymbol{\omega}_{2/1})$, so that a loss of observability with a nonzero $\boldsymbol{\nu}_b$ is possible only if $\lambda \in \{0, \pm j\omega_{2/1}\}$.

First, a technical lemma is reported. We skip the proof for the sake of brevity, the interested reader may refer to Laub (2005).

Lemma 4.6. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^n$;*
$$\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha} = \mathbf{0} \tag{59}$$
if and only if either $\boldsymbol{\alpha}$ or/and $\boldsymbol{\beta}$ is the zero vector.

Remark If vectors $\alpha = \alpha_R + j\alpha_I$ and $\beta = \beta_R + j\beta_I$ have nonzero real part, relation (59) can be rewritten as
$$\alpha_R \otimes \beta_R + \beta_R \otimes \alpha_R - \alpha_I \otimes \beta_I - \beta_I \otimes \alpha_I = 0 \tag{60}$$
$$\alpha_I \otimes \beta_R + \beta_R \otimes \alpha_I + \alpha_R \otimes \beta_I + \beta_I \otimes \alpha_R = 0. \tag{61}$$
and the above relations are satisfied if and only if either $\alpha_R = \alpha_I = 0$ or $\beta_R = \beta_I = 0$.

We now try and seek all possible unobservable configurations associated to $\boldsymbol{\nu}_b \neq \mathbf{0}$. We start with a preliminary result giving some necessary conditions for unobservability involving a relation between $\mathbf{x}_0$ and $\boldsymbol{\nu}_b$ when $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$.

Lemma 4.7. *Consider the system (12)-(13). For any given initial condition $\mathbf{x}_0 \in \mathbb{R}^6$, $\mathbf{x}_0 = \left[\mathbf{x}_{01}^\top\, \mathbf{x}_{02}^\top\right]^\top$ all unobservable configurations associated to $\boldsymbol{\nu}_b \neq \mathbf{0}$ and $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$ are relative to*
$$\boldsymbol{\nu}_b = -2\,\mathbf{x}_{02}. \tag{62}$$

Proof. According to the proof of Proposition 4.1, a necessary condition to hold when $\boldsymbol{\nu}_b \neq 0$ is $S(\boldsymbol{\omega}_{2/1})\,\boldsymbol{\nu}_b = \lambda\,\boldsymbol{\nu}_b$ being $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$ by hypothesis. In order to find indistinguishable initial conditions, we compare the evolution of two initial conditions $\mathbf{x}_0 = [\,\mathbf{x}_{01}^\top\ \mathbf{x}_{02}^\top\ y_0\ (\mathbf{x}_{01} \otimes \mathbf{x}_{02})^\top\ (\mathbf{x}_{02} \otimes \mathbf{x}_{02})^\top\,]^\top$ and $\bar{\mathbf{x}}_0 = \mathbf{x}_0 + \boldsymbol{\mu}$ where $\boldsymbol{\mu}$ is a vector structured as in (28). Consider the last block of equations of (30); the left-hand side can be rearranged as follows:
$$\begin{aligned}&S(\boldsymbol{\omega}_{2/1})\boldsymbol{\nu}_b \otimes \mathbf{x}_{02} + \boldsymbol{\nu}_b \otimes S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} + S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} \otimes \boldsymbol{\nu}_b + \\ &+\mathbf{x}_{02} \otimes S(\boldsymbol{\omega}_{2/1})\boldsymbol{\nu}_b + S(\boldsymbol{\omega}_{2/1})\boldsymbol{\nu}_b \otimes \boldsymbol{\nu}_b + \boldsymbol{\nu}_b \otimes S(\boldsymbol{\omega}_{2/1})\boldsymbol{\nu}_b.\end{aligned}$$
allowing to rewrite the last equation in (30) as:
$$\boldsymbol{\nu}_b \otimes \left(S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} + \frac{\lambda}{2}\boldsymbol{\nu}_b\right) + \left(S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} + \frac{\lambda}{2}\boldsymbol{\nu}_b\right) \otimes \boldsymbol{\nu}_b = \mathbf{0}. \tag{63}$$

Consider $\lambda = 0$. In view of Lemma (4.6), the above relation holds if and only if $S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} = \mathbf{0}$. Considering the algebraic constraint (26) and $\boldsymbol{\nu}_b = \beta\boldsymbol{\omega}_{2/1}$, the only solution is $\boldsymbol{\nu}_b = -2\mathbf{x}_{02}$.

On the other hand, if $\lambda = j\omega_{2/1}$ then we look for two directions $\boldsymbol{\nu}_{b_R}$, $\boldsymbol{\nu}_{b_I}$ orthogonal to $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$ satisfying $S(\boldsymbol{\omega}_{2/1})\boldsymbol{\nu}_{b_R} = \omega_{2/1}\boldsymbol{\nu}_{b_I}$ and $S(\boldsymbol{\omega}_{2/1})\boldsymbol{\nu}_{b_I} = -\omega_{2/1}\boldsymbol{\nu}_{b_R}$.

By direct inspection it is seen that equation (63) takes the form
$$\begin{aligned}&\omega_{2/1}\boldsymbol{\nu}_{b_I} \otimes \left(\mathbf{x}_{02} + \frac{1}{2}\boldsymbol{\nu}_{b_R}\right) + \left(\mathbf{x}_{02} + \frac{1}{2}\boldsymbol{\nu}_{b_R}\right) \otimes \omega_{2/1}\boldsymbol{\nu}_{b_I} + \\ &\quad + \boldsymbol{\nu}_{b_R} \otimes \left(S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} + \frac{\omega_{2/1}}{2}\boldsymbol{\nu}_{b_I}\right) + \\ &\quad + \left(S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} + \frac{\omega_{2/1}}{2}\boldsymbol{\nu}_{b_I}\right) \otimes \boldsymbol{\nu}_{b_R} = \mathbf{0}.\end{aligned}$$
In view of equation (61) the above relation is satisfied if and only if
$$\begin{cases}\omega_{2/1}\left(\mathbf{x}_{02} + \frac{1}{2}\boldsymbol{\nu}_{b_R}\right) &= 0\\ S(\boldsymbol{\omega}_{2/1})\mathbf{x}_{02} + \frac{\omega_{2/1}}{2}\boldsymbol{\nu}_{b_I} &= 0\end{cases} \tag{64}$$
which are simultaneously satisfied choosing $\boldsymbol{\nu}_{b_R} = -2\mathbf{x}_{02}$. □

Remark Notice that a direct consequence of the above result is that the second constraint of (26) is satisfied.

Having determined above that $\boldsymbol{\nu}_b$ is either null or (62) $\boldsymbol{\nu}_b = -2\mathbf{x}_{02}$, in case $\boldsymbol{\nu}_b \neq \mathbf{0}$ (and $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$) we can now study the solution for $\boldsymbol{\nu}_a$. For the ease of presentation, we separate the observability analysis related to $\lambda = 0$ from $\lambda = \pm j\omega_{2/1}$.

Proposition 4.8. *Consider system (22) - (24), $\boldsymbol{\nu}_b \neq \mathbf{0}$ in (28) and consider $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$.*

Then, $\lambda = 0$ is not observable if and only if
$$\begin{cases}\boldsymbol{\omega}_{1/0}^\top\boldsymbol{\omega}_{2/1} = 0\\ \boldsymbol{\omega}_{1/0}^\top\mathbf{e}_1 = 0\\ \mathbf{x}_2(0) = \kappa\,\check{\boldsymbol{\omega}}_{2/1} \qquad \kappa \in \mathbb{R}\end{cases} \tag{65}$$
In the case of loss of observability, all the indistinguishable states of (22) - (24) with respect to the initial conditions
$$\begin{aligned}\mathbf{x}_1(0) &= \left(\frac{\kappa u_2}{2\omega_{1/0}^2}\boldsymbol{\omega}_{2/1} + \frac{u_1}{\omega_{1/0}^2}\mathbf{e}_1\right) \times \boldsymbol{\omega}_{1/0} + \gamma\boldsymbol{\omega}_{1/0},\\ \mathbf{x}_2(0) &= \kappa\,\check{\boldsymbol{\omega}}_{2/1}\end{aligned} \tag{66}$$
where γ and κ are real parameters, are given by the following configurations:

$$\mathbf{x}_1^*(0) = \left(-\frac{\kappa u_2}{2\omega_{1/0}^2}\boldsymbol{\omega}_{2/1} + \frac{u_1}{\omega_{1/0}^2}\mathbf{e}_1\right) \times \boldsymbol{\omega}_{1/0} + \delta\boldsymbol{\omega}_{1/0},$$
$$\mathbf{x}_2^*(0) = -\kappa\breve{\boldsymbol{\omega}}_{2/1} \tag{67}$$

in other words, the initial conditions

$$(\mathbf{x}_1(0)^\top, \mathbf{x}_2(0)^\top)^\top \text{ and } (\mathbf{x}_1^*(0)^\top, \mathbf{x}_2^*(0)^\top)^\top$$

generate the same output.

Proof. From Lemma 4.7, we now look for those solutions of (30) with $\boldsymbol{\nu}_b = -2x_2(0)$ in the case of $\lambda = 0$. A consequence of the second line of equations (30) is that $\boldsymbol{\nu}_b = \kappa\,\boldsymbol{\omega}_{2/1}$, $\kappa \in \mathbb{R}$, so that also $x_2(0) = -\frac{\kappa\,\boldsymbol{\omega}_{2/1}}{2}$. The first block of equations in (30) for $\lambda = 0$ is

$$-\,S(\boldsymbol{\omega}_{1/0})\,\boldsymbol{\nu}_a = -u_2\boldsymbol{\nu}_b \tag{68}$$

and, in view of the above results and that $S(\boldsymbol{\omega}_{1/0})\boldsymbol{\omega}_{1/0} = \mathbf{0}$, it is easily seen that a nonzero unobservable configuration with $\boldsymbol{\nu}_b \neq \mathbf{0}$ might exist only if $\boldsymbol{\omega}_{1/0}^\top\boldsymbol{\omega}_{2/1} = 0$. In this case, all the solutions of (68) are

$$\boldsymbol{\nu}_a = \frac{u_2\kappa}{\omega_{1/0}^2}\boldsymbol{\omega}_{1/0} \times \boldsymbol{\omega}_{2/1} + \beta\,\boldsymbol{\omega}_{1/0}, \qquad \beta \in \mathbb{R} \tag{69}$$

Consider now the fourth block of equations (30); in view of $S(\boldsymbol{\omega}_{2/1})\mathbf{x}_2(0) = \mathbf{0}$ and $\boldsymbol{\nu}_b \neq \mathbf{0}$, it is reduced to

$$S^\top(\boldsymbol{\omega}_{1/0})\mathbf{x}_1(0) = \frac{\kappa u_2}{2}\boldsymbol{\omega}_{2/1} + u_1\mathbf{e}_1. \tag{70}$$

The above equation is fit to compute the initial conditions $\mathbf{x}_1(0)$ compatible with some unobservable configurations. Equation (70) does not always admit solutions and this means that there exist some configurations that are always observable. Again, in view of $S(\boldsymbol{\omega}_{1/0})\boldsymbol{\omega}_{1/0} = \mathbf{0}$, by premultiplying equation (70) with $\boldsymbol{\omega}_{1/0}^\top$, the existence condition for equation (70) to hold is

$$\boldsymbol{\omega}_{1/0}^\top\mathbf{e}_1 = 0 \tag{71}$$

and, when (71) holds, any $\mathbf{x}_{10}$ satisfying (70) can be written as

$$\left(\frac{\kappa u_2}{\boldsymbol{\omega}\ \ _{2/1}} + \frac{u_1}{\omega_{1/0}^2}\mathbf{e}_1\right) \times \boldsymbol{\omega}_{1/0}$$

$$\mathbf{x}_1(0) = \left(\frac{\kappa u_2}{2\omega_{1/0}^2}\boldsymbol{\omega}_{2/1} + \frac{u_1}{\omega_{1/0}^2}\mathbf{e}_1\right) \times \boldsymbol{\omega}_{1/0} + \gamma\boldsymbol{\omega}_{1/0}, \qquad \gamma \in \mathbb{R} \tag{72}$$

Finally, last equation of (30) to be satisfied is

$$-\,u_1\,\mathbf{e}_1^\top\boldsymbol{\nu}_a + u_2\,\left(\mathbf{x}_1^\top(0)\,\boldsymbol{\nu}_b + \boldsymbol{\nu}_a^\top\,\mathbf{x}_2(0) + \boldsymbol{\nu}_a^\top\,\boldsymbol{\nu}_b\right) = 0$$

and it is a matter of algebraic manipulations to see that it is always satisfied. □

Proposition 4.9. *Consider system (12), (13). If $\boldsymbol{\nu}_b \neq \mathbf{0}$ in (28) and $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$, then $\lambda = 0$ is not observable if conditions (65) hold for $\kappa = 1$ and all the indistinguishable states of (12), (13) are given by (66), (67) and (69) with β and γ satisfying*

$$\beta(\beta + 2\gamma)\omega_{1/0}^2 = \frac{4u_1u_2}{\omega_{1/0}^2}\boldsymbol{\omega}_{2/1}^\top\mathbf{e}_1. \tag{73}$$

Proof. Consider the algebraic constraints (27) and the parametrizations (66) and (69); the quantity $\boldsymbol{\nu}_a + 2\mathbf{x}_1(0)$ is easily computed

$$\boldsymbol{\nu}_a + 2\mathbf{x}_1(0) = \frac{2u_1}{\omega_{1/0}^2}\mathbf{e}_1 \times \boldsymbol{\omega}_{1/0} + (\beta + 2\gamma)\,\boldsymbol{\omega}_{1/0}$$

and (27) takes the form

$$\boldsymbol{\nu}_a^\top(\boldsymbol{\nu}_a + 2\mathbf{x}_{10}) = \frac{2\kappa u_1u_2}{\omega_{1/0}^4}(\boldsymbol{\omega}_{2/1} \times \boldsymbol{\omega}_{1/0})^\top(\mathbf{e}_1 \times \boldsymbol{\omega}_{1/0}) + + \beta(\beta + 2\gamma)\omega_{1/0}^2 \tag{74}$$

Considering the equality $(\boldsymbol{\omega}_{2/1} \times \boldsymbol{\omega}_{1/0})^\top(\mathbf{e}_1 \times \boldsymbol{\omega}_{1/0}) = \|\boldsymbol{\omega}_{1/0}\|^2\boldsymbol{\omega}_{2/1}^\top\mathbf{e}_1$, the algebraic constraint (27) can finally be reduced to the following expression

$$\frac{2\kappa u_1u_2}{\omega_{1/0}^2}\boldsymbol{\omega}_{2/1}^\top\mathbf{e}_1 + \beta(\beta + 2\gamma)\omega_{1/0}^2 = 0 \tag{75}$$

□

We now consider the PBH criterion to find some configurations related to unobservable sinusoidal motions, i.e. we consider the case of $\lambda = \pm j\omega_{2/1}$. If $\omega_{2/1} \neq \pm\omega_{1/0}$ then λ is a simple eigenvalue of A in (23). In this case, the eigenspace associated is monodimensional and, denoting with $\boldsymbol{\mu}_R, \boldsymbol{\mu}_I \in \mathbb{R}^3$ such that $S(\boldsymbol{\omega}_{2/1})\boldsymbol{\mu}_R = \omega_{2/1}\boldsymbol{\mu}_I$, $S(\boldsymbol{\omega}_{2/1})\boldsymbol{\mu}_I = -\omega_{2/1}\boldsymbol{\mu}_R$, it can be parametrized as

$$\begin{aligned} \boldsymbol{\nu}_{aR} &= -u_2\left(S^2(\boldsymbol{\omega}_{2/1}) + \omega_{2/1}^2 I)\right)^{-1} S(\boldsymbol{\omega}_{1/0})\boldsymbol{\mu}_R \\ \boldsymbol{\nu}_{aI} &= -u_2\left(S^2(\boldsymbol{\omega}_{2/1}) + \omega_{2/1}^2 I)\right)^{-1} S(\boldsymbol{\omega}_{1/0})\boldsymbol{\mu}_I \end{aligned} \tag{76}$$

Notice that in this case the solutions $\boldsymbol{\nu}_a$, $\boldsymbol{\nu}_b$ of (32) do not contain any parameter that can be adjusted in order to fulfill the other equations in (30) together with (26) and (27).

On the contrary, in the case of $\omega_{2/1} = \pm\omega_{1/0}$, a larger set of solutions might exist if some additional conditions are satisfied, as it is stated in the following Lemma.

Lemma 4.10. *Let $S(\boldsymbol{\omega}_i)$, $\boldsymbol{\omega}_i \in \mathbb{R}^3$, be the matrix defined in* (1) *and consider matrix $\Omega \in \mathbb{R}^{6\times 6}$ defined as*

$$\Omega = \begin{bmatrix} -S(\boldsymbol{\omega}_i) & \kappa\,I_{3\times 3} \\ \mathbf{0}_{3\times 3} & S(\boldsymbol{\omega}_j) \end{bmatrix}. \tag{77}$$

Ω has complex eigenvalues with geometric multiplicity equal to 2 if and only if $\boldsymbol{\omega}_i = \pm\boldsymbol{\omega}_j$.

□

Proof. First, since the eigenvalues of $S(\boldsymbol{\omega}_i)$ are $\{0, j\omega_i, -j\omega_i\}$, then Ω has complex eigenvalues with algebraic multiplicity equal to 2 if and only if $\omega_i = \pm\omega_j$. Now, consider two matrices T_i, T_j representing the coefficients of two orthogonal bases to describe T_i and T_j in their real canonical form Antsaklis and Michel (2007). More specifically, if $T_i = [\boldsymbol{\omega}_i\ \mathbf{v}_{Ri}\ \mathbf{v}_{Ii}]$ where $\mathbf{v}_{Ri}$ and $\mathbf{v}_{Ii}$ are the real and complex parts of any eigenvector of $S(\boldsymbol{\omega}_i)$ associated to $\lambda = j\omega_i$, then $T_i S(\boldsymbol{\omega}_i) T_i^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_i \\ 0 & -\omega_i & 0 \end{bmatrix}$ and it can be easily seen that $\mathbf{v}_{Ri}$ and $\mathbf{v}_{Ii}$ are orthogonal with respect themselves and $\boldsymbol{\omega}_i$.

Now, denote with μ an eigenvector of $S(\boldsymbol{\omega}_i)$ associated to $\lambda = j\omega_i$; it is straight to see that matrix Ω has an eigenvalue $\lambda = j\omega_i$ with geometric multiplicity equal to 2 if and only if both $\Omega \mathbf{v}_a = \lambda \mathbf{v}_a$ and $\Omega \mathbf{v}_b = \lambda \mathbf{v}_b$ hold, where $\mathbf{v}_a = \begin{bmatrix} \mu \\ 0 \end{bmatrix}$ and $\mathbf{v}_b = \begin{bmatrix} v_\mu \\ \mu \end{bmatrix}$, v_μ a suitable vector. Moreover, it is easily seen that vectors with the structure of $\mathbf{v}_a$ are always eigenvector of Ω (even in the case of simple eigenvalue λ) so that λ has geometric multiplicity equal to 2 if and only if Ω has eigenvectors with the structure of $\mathbf{v}_b$.

Consider the transformation $T\Omega T^{-1}$ with $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$; direct computation shows that

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} -S(\boldsymbol{\omega}_i) & \kappa I_{3\times 3} \\ \mathbf{0}_{3\times 3} & S(\boldsymbol{\omega}_j) \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}^{-1} = \begin{bmatrix} M(\omega_i) & L \\ 0_{3\times 3} & M(\omega_i) \end{bmatrix}$$

where $M(\omega_i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_i \\ 0 & -\omega_i & 0 \end{bmatrix}$ and $L = \kappa {T_1}^{-1} T_2$.

As a final step, considering this representation, a solution to $\Omega \mathbf{v}_b = \lambda \mathbf{v}_b$ exists if and only if

$$M(\omega_i)\mathbf{v}_\mu = L\boldsymbol{\mu}$$

admits solution. In view of the first row of $M(\omega_i)$ this is possible if and only if $(L)_{12} = L_{13} = 0$; since $L = \kappa {T_1}^{-1} T_2$ and ${T_1}^{-1} = T^\top$, this holds only if the first column of T_1 is orthogonal to the second and third column of T_2. This, in turn, implies that $\boldsymbol{\omega}_i = \pm\boldsymbol{\omega}_j$, thus proving the Lemma.

□

Remark In view of the previous result, it follows that, in the case of multiple complex eigenvalues, either $S(\boldsymbol{\omega}_j) = S(\boldsymbol{\omega}_i)$ or $S(\boldsymbol{\omega}_j) = -S(\boldsymbol{\omega}_i)$ must hold. Moreover, it is a matter of simple computation to see that the vectors $\mathbf{v}_b$ used in the prevous proof can be parametrized as

$$\mathbf{v}_b = \begin{bmatrix} v_\mu \\ \mu \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_R + \frac{u_2}{\omega}\boldsymbol{\mu}_I \\ \boldsymbol{\mu}_R \end{bmatrix} + j \begin{bmatrix} \boldsymbol{\mu}_I \\ \boldsymbol{\mu}_I \end{bmatrix}. \tag{78}$$

Consider now equation (30); in view of Lemma 4.7 and after some algebraic manipulations it can be put in the more convenient form

$$S^\top(\boldsymbol{\omega}_{1/0})\left(\mathbf{x}_1(0) + \frac{\boldsymbol{\nu}_a}{2}\right) = u_1\mathbf{e}_1 \tag{79}$$

so that it is easy to see that equation (79) has solution only if $\boldsymbol{\omega}_{1/0}^\top \mathbf{e}_1 = 0$ and it is equal to

$$\mathbf{x}_1(0) = u_1\boldsymbol{\omega}_{1/0} \times \mathbf{e}_1 + \eta\,\boldsymbol{\omega}_{1/0} - \frac{\boldsymbol{\nu}_a}{2}. \tag{80}$$

Finally, last equation to consider can be reorganized as

$$u_1\mathbf{e}_1^\top \boldsymbol{\nu}_a = -u_2\mathbf{x}_2(0)^\top(\mathbf{x}_1(0) + \boldsymbol{\nu}_a) \tag{81}$$

which must be satisfied together with (27) to have an unobservable configuration.

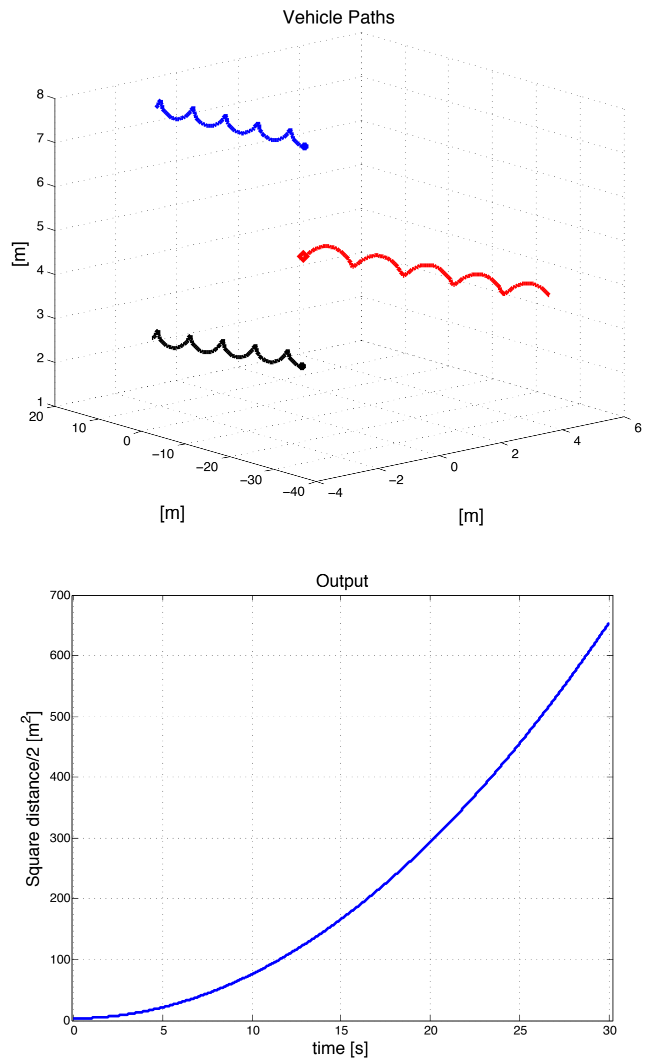

Fig. 2. Indistinguishable motions and corresponding output signal relative to the case in equation (37). Refer to the text for details.

5. SIMULATION EXAMPLES

Numerical simulations have been performed to plot and visualize unobservable motions generated according to the developed theory. Examples are depicted in figures 2 - 5 where the vehicle's paths and the system output $y(t)$ are reported. As apparent from the numerical results, in agreement with the derived observability conditions, we have derived different initial conditions for the two vehicles generating the same output (i.e. unobservable poses) for given fixed velocity inputs. In all simulations the linear angular velocities of the two vehicles are $u_1 = 0.75$[m/s] and $u_2 = 0.5$[m/s].

The first simulation refers to the case in equation (37), namely of complex unobservable eigenvalue λ with $\boldsymbol{\omega}_{2/1} \neq \mathbf{0}$. In particular the vehicle velocities were

$$\boldsymbol{\omega}_{2/1} = (-0.4215, -0.2621, 0.0603)^\top [rad/s]$$

$$\boldsymbol{\omega}_{1/0} = (-0.9746, -0.2181, 0.0502)^\top [rad/s],$$

The top of figure 2 refers to paths generating the same output depicted in the bottom of figure 2. In particular, the relative distance among the vehicles with trajectory plotted in red and blue and the ones plotted in red and black generate the same output (blue plot in the bottom of figure 2). The second simulation refers to the case

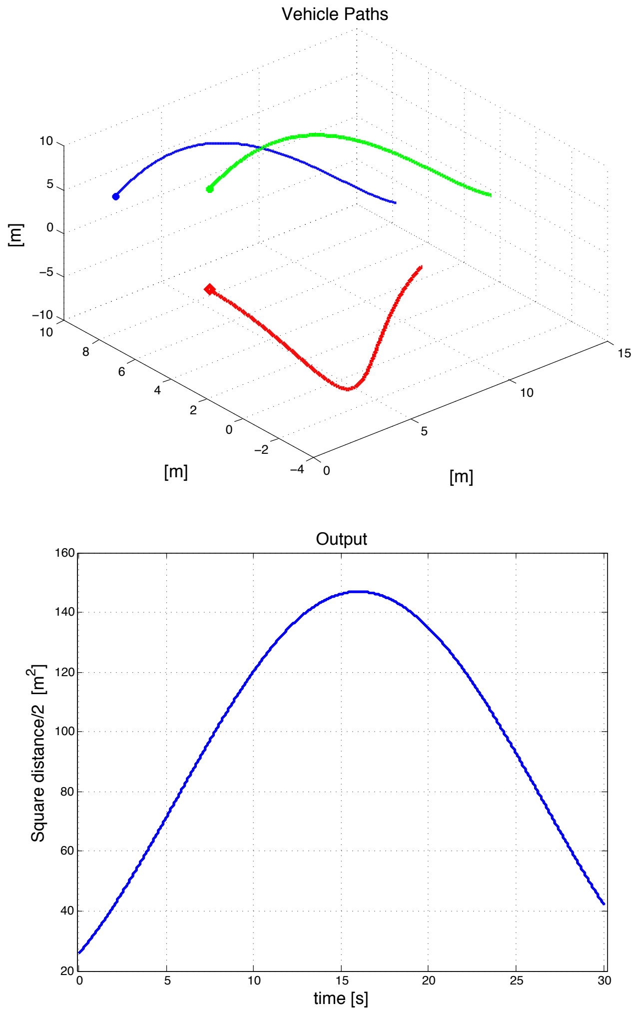

Fig. 3. Case (33): the vehicles moving along the red and blue paths (top figure) have a relative distance (bottom plot) that is identical to the one between the red and green paths.

in equation (33), namely to the $\lambda = 0$ unobservable eigenvalue. The angular velocities were

$$\boldsymbol{\omega}_{2/1} = (-0.0210, 0.0625, 0.0183)^\top [rad/s]$$

$$\boldsymbol{\omega}_{1/0} = (0.0664, 0.1342, -0.0098)^\top [rad/s]$$

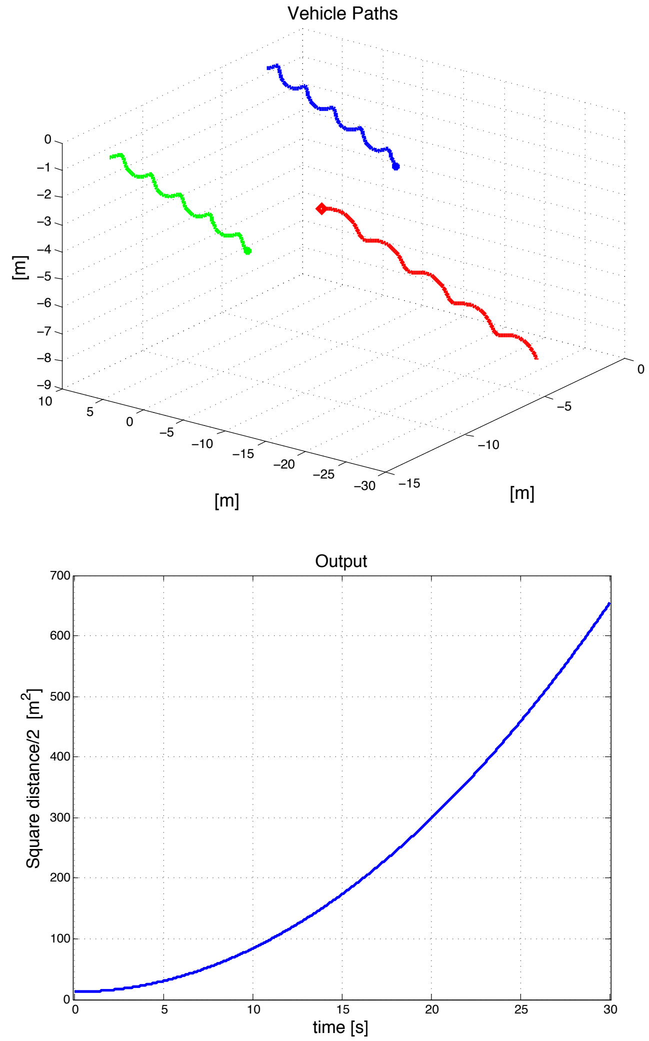

Fig. 4. Indistinguishable movements in the case of a complex unobservable eigenvalue with $\boldsymbol{\omega}_{2/1} = \mathbf{0}$. Refer to the text for further details.

The third example reported in figure 4 refers to the case of complex unobservable eigenvalue described in equation (58), namely with $\boldsymbol{\omega}_{2/1} = \mathbf{0}$. In this case the angular velocity vector $\boldsymbol{\omega}_{1/0}$ was

$$\boldsymbol{\omega}_{1/0} = (-0.9699, -0.1197, 0.2119)^\top [rad/s].$$

Finally the simulation results in figure 5 refer to the case of null unobservable eigenvalue with $\boldsymbol{\omega}_{2/1} = \mathbf{0}$. In particular this case is addressed in equation (57): the angular velocity $\boldsymbol{\omega}_{1/0}$ in this simulation was

$$\boldsymbol{\omega}_{1/0} = (-0.0333, 0.1463, 0.0003)^\top [rad/s].$$

6. CONCLUSIONS

The problem of range only localization has received increasing attention in the last years given its potential impact on marine (and aerospace) application. To the best of the author's knowledge, most previous studies have focused on the analysis of weak observability (for simple kinematic models) rather than on observability as

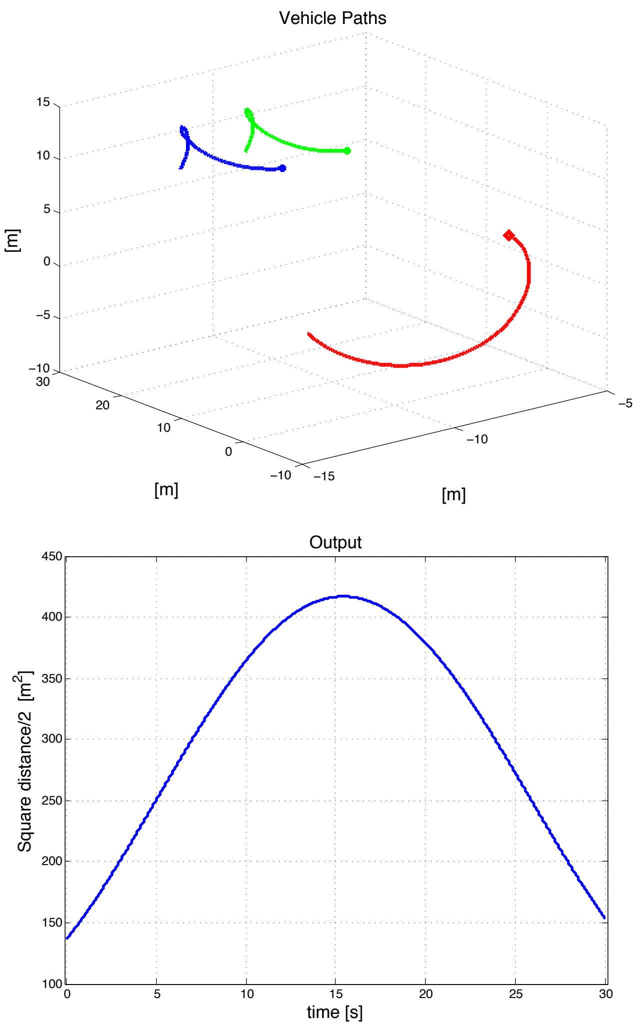

Fig. 5. Indistinguishable motions: the red and blue curves in the top plot refer to vehicles having the same relative distance as the green and red curves. The blue and green trajectories are indistinguishable on behalf of the relative distance measurement from the vehicle moving on the red path (bottom plot).

usually defined for linear systems. As clearly illustrated by Jouffroy and Ross (2005), Gadre and Stilwell (2004) and Hermann and Krener (1977), the very definition of weak observability does not allow to capture and describe indistinguishable states at a global level. This implies that any observer designed on the basis of weak observability (only) will fail to guarantee global convergence. The need for tools allowing to assess (global) observability appears thus to be a fundamental pre-requisite to design globally convergent observers. The proposed method exploits and extends a state augmentation approach described in Batista et al. (2011) Batista et al. (2010): in particular, building on the preliminary results in Parlangeli et al. (2012), a method to compute globally indistinguishable states for a non-holonomic kinematic model capturing the main features of an AUV Caffaz et al. (2010) has been described in the general case of non-null linear velocity. Future work will aim at exploiting the provided observability analysis to design observer filters for pose estimation.

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APPENDIX

An effective tool for the observability analysis of linear time invariant systems is the PBH Lemma: it consists in computing the state matrix eigenvectors (i.e. invariant directions of the dynamical system) belonging to the kernel of the output map.

Given that complex eigenvalues are associated to complex eigenvectors, if complex eigenvalues are present in the spectrum of the dynamical matrix, the connection between the complex eigenvectors and the real valued system trajectories needs to be analyzed. A brief summary of the results about this issue as applied to the case at hand is here reported. Refer to Antsaklis and Michel (2007) for a more general discussion about the subject.

Indicating with $\bar{z}$ the complex conjugate of z, given a square matrix $A \in \mathbb{R}^{n\times n}$, if $\lambda \in \mathbb{C}$ is an eigenvalue with $Im(\lambda) \neq 0$ and $\mathbf{v} \in \mathbb{C}^n$ its corresponding eigenvector, then $\bar{\lambda} \in \mathbb{C}$ is also an eigenvector of A and its corresponding eigenvector is $\bar{\mathbf{v}}$. Moreover, if we denote with $\lambda = \sigma + j\omega$ and $\mathbf{v} = \mathbf{v}_R + j\mathbf{v}_I$ being $\sigma, \omega \in \mathbb{R}$ and $\mathbf{v}_R, \mathbf{v}_I \in \mathbb{R}^n$, it follows that $A\mathbf{v}_R = \sigma\,\mathbf{v}_R - \omega\,\mathbf{v}_I$ and $A\mathbf{v}_I = \omega\,\mathbf{v}_R + \sigma\,\mathbf{v}_I$. The observability condition for the eigenvalue λ is equivalent to the observability for the eigenvalue $\bar{\lambda}$, namely they are both either observable or unobservable at the same time. Consequently the directions $\mathbf{v}$ and $\bar{\mathbf{v}}$, or equivalently, $\mathbf{v}_R$ and $\mathbf{v}_I$, define a plane of unobservable or observable states, respectively, for unobservable or observable eigenvalues λ.

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