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A two-point boundary value formulation of a mean-field crowd-averse game ⋆

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Abstract: We consider a population of "crowd-averse" dynamic agents controlling their states towards regions of low density. This represents a typical dissensus behavior in opinion dynamics. Assuming a quadratic density distribution, we first introduce a mean-field game formulation of the problem, and then we turn the game into a two-point boundary value problem. Such a result has a value in that it turns a set of coupled partial differential equations into ordinary differential equations.

Keywords: Game theory, optimal control, stochastic systems.

1. INTRODUCTION

We consider a population of "crowd-averse" dynamic agents. Crowd-averse means that, for a given density distribution of the states, the agents (also called players) seek to regulate their state to values characterized by a low density. Such a problem arises naturally in social science, where states are opinions, the dynamics represents opinions' propagations, and crowd-averse attitudes capture the tendency to escape consensus and seek dissensus (in the space of opinions). We assume that each agent's state evolves according to a linear stochastic differential equation (SDE) driven by a Brownian motion and under the influence of a control and an adversarial disturbance. The control of each player minimizes a cost functional which involves a quadratic penalty on the control and a mean-field term involving the density of the players. We analyze the case in which the initial distribution is (piecewise) quadratic.

The main result of the paper involves the reformulation of the mean-field game into a two-point boundary value problem. Such a problem includes two intertwined differential equations with boundary conditions at the beginning and at the end of the horizon for the two equations respectively. The first is a Riccati equation obtained from the Hamilton-Jacobi-Isaacs equation, and the second is a linear differential equation derived from the Fokker-Planck-Kolmogorov equation. Such a result has a value in that it turns a set of coupled partial differential equations into ordinary differential equations.

The theory of mean-field games was introduced in Lasry and Lions [2007] and independently in Huang et al. [2006, 2007]. Mean-field dynamical games represent a modeling framework at the interface of differential game theory, mathematical physics, and H_∞-optimal control that captures the interaction between a mass of players and each individual. Mean-field games arise in several application domains such as economics, physics, biology, and network engineering, see Achdou et al. [2012], Bagagiolo and Bauso [2014], Bauso et al. [2012b], Gueant et al. [2010], Huang et al. [2007], Lachapelle et al. [2010], Pesenti and Bauso [2013], Tembine et al. [2009].

A mean-field game is modelled by means of a system of two partial differential equations (PDEs). The first PDE is the Hamilton-Jacobi-Bellman equation. The second PDE is the Fokker-Planck-Kolmogorov equation which describes the density of the players, see Lasry and Lions [2007], Tembine et al. [2011]. Explicit solutions in terms of mean-field equilibria are not common unless the problem has a linear-quadratic structure, see Bardi [2012]. In this sense, a variety of solution schemes has been recently proposed based on discretization and/or numerical approximations, see for example Achdou et al. [2012]. More recently, robustness and risk-sensitivity have been brought into the picture of mean-field games see Bauso et al. [2012a], Tembine et al. [2011], where the first PDE has been replaced by a Hamilton-Jacobi-Bellman-Isaacs (HJBI) PDE.

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The rest of the paper is organized as follows. In Section 2 we illustrate the problem. In Section 3 we elaborate on the main motivations. In Section 4 we establish the main result. In Section 5 we discuss and interpret the results. In Section 6 we carry out some numerical studies. Finally in Section 7 we provide some conclusions.

Notation. We denote with $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. We let $\mathcal{B}$ be a finite-dimensional Brownian motion defined on this probability space. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be its natural filtration augmented by all $\mathbb{P}-$null sets (sets of measure-zero with respect to $\mathbb{P}$). We use ∂_x and ∂^2_{xx} to denote the first and second partial derivatives with respect to x, respectively.

2. PROBLEM SET-UP

Consider a game with an infinite number of homogeneous players. For each player let x_0 be its initial state, which is realized according to the probability distribution m_0. The state of the player at time t, denoted by $x_t \in \mathbb{R}$, evolves according to a controlled stochastic process over a finite horizon $T > 0$, i.e.

$$dx_t = [\alpha x_t + \beta u_t]dt + \sigma \left[x_t d\mathcal{B}_t + \zeta_t dt\right], \qquad (1)$$

where $u_t \in \mathbb{R}$ is the control input, $\mathcal{B}_t \in \mathbb{R}$ is a Brownian motion, which is independent of the initial state x_0, and independent across players and time. The constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are parameters, and $\zeta_t \in \mathbb{R}$ is an adversarial disturbance.

To introduce a macroscopic description of the game consider probability density functions on the state space, *i.e.*

$$\begin{cases} m : \mathbb{R} \times [0, +\infty[\to [0, +\infty[, \; (x, t) \mapsto m_t(x) \\ \displaystyle\int_{\mathbb{R}} m_t(x)dx = 1 \text{ for every } t\,. \end{cases}$$

Define now the average state distribution at time t as

$$\bar{m}_t := \int_{\mathbb{R}} x m_t(x)dx.$$

Finally consider a cost functional with penalty on the final state $g(\cdot)$, stage cost function $c(\cdot)$, and quadratic penalty on the unknown disturbance:

$$\begin{aligned} J(x_0, u, m, \zeta) = \mathbb{E}\Big(g(x_T, m_T) \\ + \int_0^T c(x_t, u_t, m_t)dt - \gamma^2 \int_0^T |\zeta_t|^2 dt\Big). \end{aligned} \qquad (2)$$

Players are crowd-averse and wish to drive their state towards state values where density distribution is minimal, and therefore we can select the stage cost

$$c(x_t, u_t, m) = m_t(x_t) + \frac{b}{2}u_t^2\,.$$

The term $m_t(x_t)$ represents the mean-field cost which is proportional to the density distribution in the same state x_t; $\frac{b}{2}u_t^2$, with $b > 0$, accounts for a penalty on the control energy. The penalty on the final state is

$$g(x_T, m_T(x_T)) = m_T(x_T),$$

namely it is a penalty on the state density distribution at the end of the horizon.

The cost functionals are such that players seek to reach less crowded states. It is however possible that the players seek to reach the same low-density state, as in Section 6.

The above preamble leads to the following robust mean-field game problem.

Problem 1. (**Robust mean-field game problem**) Let $\mathcal{B}$ be a one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the natural filtration generated by $\mathcal{B}$. Let x_0 be independent of $\mathcal{B}$ and with density $m_0(x)$. Let m_t^* be the optimal mean-field trajectory. The robust mean-field game problem in $\mathbb{R}$ and $[0, T[$ is given by

$$\begin{cases} \inf_{\{u_t\}_t} \sup_{\{\zeta_t\}_t} J(x, u, m^*, \zeta)\,, \\ dx_t = [\alpha x_t + \beta u_t + \sigma \zeta_t]\, dt \,+\, \sigma x_t d\mathcal{B}_t. \end{cases}$$

3. MOTIVATIONS

Motivations for the problem under study arise in the context of opinion dynamics in social networks. In this context crowd-averse attitudes on the part of the players means that the players tend to have very different opinions. This can be reviewed as the opposite phenomenon to the one of “emulation”, “mimicry” or “herd behavior”.

Example 1. **(Social networks)** Opinion dynamics has attracted the attention of many scientists over the past few years. The propagation of the opinions describe the time evolution of the beliefs of large population of agents as a consequence of repeated interactions among the agents, in many cases over a social network, see for example [Castellano et al., 2009, Sect. III] and Acemoğlu and Ozdaglar [2011]. In continuous models of opinion dynamics beliefs or opinions are represented by scalars or vectors, evolving according to some averaging process. The latter consists in each opinion moving towards a convex combinations of (a subset of) other agents’ current beliefs, thus modeling the attractive nature of social influence. There are many models that, under the assumption that the underlying social network is connected, prove that the agents’ opinions asymptotically reach consensus. Some exceptions can be found in the models by Krause [2000] where the authors introduce *homophily* in the form of “bounded confidence”, to mean that the agents are not influenced by far beliefs. A similar behavior can be found also in models with competing stubborn agents, see Acemoğlu et al. [2013], the latter being agents that do not change their opinions but try to influence others’ opinions. Such stubborn agents might represent leaders, political parties or media sources. For instance, Como and Fagnani [2011], provides scaling limit results showing that, if the agents’ population is homogeneous, the empirical belief distribution converges, as the population size grows large, towards the solution of a certain deterministic mean-field differential equation in the space of probability measures. Such results are in the spirit of the propagation of chaos, see Sznitman [1991], in interacting particle systems.

4. MAIN RESULTS

Let $v_t(x)$ be the (upper) value of the robust optimization problem under worst-case disturbance starting at time t from state x. Let the corresponding Hamiltonian be given by

$$H(x, p, m) = \inf_u \left\{c(x, u, m) + p(\alpha x + \beta u)\right\},$$

where p is the co-state. Then the mean-field system associated to the robust mean-field game introduced in Problem 1 is given by

$$\begin{cases} \partial_t v_t + H(x,p,m_t) + \left(\dfrac{\sigma}{2\gamma}\right)^2 (\partial_x v_t)^2 \\ \qquad + \dfrac{1}{2}\sigma^2 x^2 \partial^2_{xx} v_t = 0, \text{ in } \mathbb{R} \times [0,T[, \\ v_T(x) = g(x,m), \text{ in } \mathbb{R}, \\ m_0(x) = d(x) \text{ in } \mathbb{R}, \\ \partial_t m_t + \partial_x \left(m_t \partial_p H(x,p,m)\right) + \dfrac{\sigma^2}{2\gamma^2}\partial_x(m_t \partial_x v_t) \\ -\dfrac{1}{2}\sigma^2 \partial^2_{xx}\left[x^2 m_t\right] = 0, \text{ in } \mathbb{R} \times [0,T[, \end{cases} \tag{3}$$

where d is the initial population distribution and g the terminal payoff. Any solution of the above system of equations is referred to as *worst-disturbance feedback mean-field equilibrium*.

We next assume that the density distribution is quadratic in the state.

Assumption 1. The density has compact support and, within its support, it is given by

$$\begin{cases} m_t(x) = \dfrac{1}{2} a_t x^2, \text{ in } \mathbb{R} \times [0,T[\,, \\ m_0(x) = d(x) = \dfrac{1}{2} a_0 x^2, \quad a_0 \text{ given.} \end{cases} \tag{4}$$

From the above assumption, as the density enters in the cost function, we can consider quadratic value functions

$$\begin{cases} v_t(x) = \dfrac{1}{2} q_t x^2, \text{ in } \mathbb{R} \times [0,T] \\ v_T(x) = g(x_T, m_T(x_T)) = m_T(x_T) = \dfrac{1}{2} q_T x^2. \end{cases} \tag{5}$$

We are ready to specialize the results obtained above to the case of a crowd-averse system in which the players seek to drive their state towards values characterized by a lower density.

Theorem 1. The mean-field system associated to the robust mean-field game for the crowd-averse system is described by the equations:

$$\begin{cases} \partial_t v_t + \left[-\dfrac{\beta^2}{2b} + \left(\dfrac{\sigma}{2\gamma}\right)^2\right](\partial_x v_t)^2 + \dfrac{a}{2} x_t^2 \\ +\alpha x_t \partial_x v_t + \dfrac{1}{2}\sigma^2 x^2 \partial^2_{xx} v_t = 0, \text{ in } \mathbb{R} \times [0,T[, \\ v_T(x) = \dfrac{1}{2} q_T x^2, \text{ in } \mathbb{R}, \\ \partial_t m_t + \dfrac{3}{2} a_t \Big(\alpha - \dfrac{\beta^2}{b} q_t + \dfrac{\sigma^2}{2\gamma^2} q_t\Big) x_t^2 \\ -\dfrac{1}{2}\sigma^2 \partial^2_{xx}\left(x^2 m_t\right) = 0, \text{ in } \mathbb{R} \times [0,T[, \\ m_0(x) = \dfrac{1}{2} a_0 x^2 \text{ in } \mathbb{R}. \end{cases} \tag{6}$$

Furthermore, the optimal control and worst disturbance are

$$\begin{cases} u_t^* = -\dfrac{\beta}{b}\partial_x v_t, \\ \zeta_t^* = \dfrac{\sigma}{2\gamma^2}\partial_x v_t. \end{cases} \tag{7}$$

The significance of the above result is that to find the optimal control input we need to solve the two coupled PDEs in (6) in v and m with given boundary conditions (the second and last conditions). This is usually done by iteratively solving the HJBI equation for fixed m and by entering the optimal u obtained from (7) in the FPK equation in (6), until a fixed point in v and m is reached.

The next theorem establishes that the mean-field system (6) can be replaced by a two-point boundary value problem.

Theorem 2. The mean-field system associated to the robust mean-field game for the crowd-averse system is equivalently described by the equations

$$\begin{cases} \dfrac{1}{2}\dot{q}_t + \left[-\dfrac{\beta^2}{2b} + \left(\dfrac{\sigma}{2\gamma}\right)^2\right] q_t^2 + \alpha q_t + \dfrac{a_t}{2} \\ +\dfrac{\sigma^2}{2} q_t = 0, \\ q_T = a_T, \\ \dfrac{1}{2}\dot{a}_t + \dfrac{3}{2} a_t \Big(\alpha - \dfrac{\beta^2}{b} q_t + \dfrac{\sigma^2}{2\gamma^2} q_t\Big) - 3\sigma^2 a_t = 0\,, \\ a_0 \text{ given.} \end{cases} \tag{8}$$

Furthermore, the optimal control and worst disturbance are

$$\begin{cases} \tilde{u}_t = -\dfrac{\beta}{b} q_t x_t, \\ \tilde{w}_t = \dfrac{\sigma}{2\gamma^2} q_t x_t. \end{cases} \tag{9}$$

In summary the above system of equations consists of two parts: the first is a Riccati equation in the variable q_t and the second is a linear differential equation in the variable a_t.

5. INTERPRETATION OF RESULTS

In this section we show that the stochastic differential equation describing the closed-loop system has an exponentially and asymptotically stable equilibrium. To see this use (9), rewrite the dynamics for x_t in (1) as

$$\begin{aligned} dx_t &= [\alpha x_t + \beta u_t^* + \sigma \zeta_t^*]\, dt + \sigma x_t d\mathcal{B}_t \\ &= \Big[\alpha + \Big(-\frac{\beta^2}{b} + \frac{\sigma^2}{2\gamma^2}\Big) q_t\Big] x_t dt + \sigma x_t d\mathcal{B}_t, \\ &\qquad\qquad t \in [0,T[,\ x_0 \in \mathbb{R}, \end{aligned}$$

and consider the following assumption.

Assumption 2. There exists $\kappa > 0$ such that

$$-\kappa x_t \geq \Big[\alpha + \Big(-\frac{\beta^2}{b} + \frac{\sigma^2}{2\gamma^2}\Big) q_t\Big] x_t \tag{10}$$

With the above assumption we can perform the analysis within the framework of stochastic stability theory, see Loparo and Feng [1996]. To this end consider the infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2 x_t^2 \frac{d^2}{dx_t^2} - \kappa x_t \frac{d}{dx_t}. \tag{11}$$

Consider the Lyapunov function $V(x) = x^2$. The stochastic derivative of $V(x)$ is obtained by applying the infinitesimal generator (11) to $V(x)$, which yields

$$\begin{aligned}\mathcal{L}V(x_t) &= \lim_{dt\to 0} \frac{\mathbb{E}V(x_{t+dt}) - V(x_t)}{dt} \\ &= [\sigma^2 - 2\kappa]x_t^2.\end{aligned}$$

Proposition 1. (Loparo and Feng [1996]). Let Assumption 2 hold. If $V(x) \geq 0$, $V(0) = 0$ and $\mathcal{L}V(x) \leq -\eta V(x)$ on $Q_\epsilon := \{x : V(x) \leq \epsilon\}$, for some $\eta > 0$ and for arbitrarily large ϵ, then the origin is asymptotically stable "with probability one", and

$$P_{x_0}\Big\{ \sup_{T\leq t<+\infty} x_t^2 \geq \lambda \Big\} \leq \frac{V(x_0)e^{-\psi T}}{\lambda}$$

for some $\psi > 0$.

From the above proposition we have the following result, which establishes exponential stochastic stability of the mean-field equilibrium.

Corollary 2. Let Assumption 2 hold. If $[\sigma^2 - 2\kappa] < 0$ then $\lim_{t\to\infty} x_t = 0$ almost surely and

$$P_{x_0}\Big\{ \sup_{T\leq t<+\infty} x_t^2 \geq \lambda \Big\} \leq \frac{V(x_0)e^{-\psi T}}{\lambda}$$

for some $\psi > 0$.

Figure 1 represents a graphical illustrations of the main results of the paper. In particular, Figure 1(a) depicts the initial density distribution $m_0(x)$ on the vertical axis and the state space on the horizontal axis. According to Assumption 1, the density $m_0(x)$ is quadratic in x (see the grey area). Now, if the vector field is converging to zero, the density function shrinks towards zero and becomes "more convex", which corresponds to a_t increasing with t. This occurs when Assumption 2 holds true, as all players are drawn towards the origin by the linear feedback. This is illustrated in Fig. 1(b), in which the grey area is closer to zero. On the other hand, if the vector field is diverging from zero, the density function is drawn apart from zero and becomes "less convex" and "more flat", which corresponds to a_t decreasing with t. This is due to a higher influence on the part of the disturbances (both the stochastic one, namely the Brownian motion, and the adversarial one ζ). This case is illustrated in Fig. 1(c), in which the grey area is more dispersed.

The main results of the paper can also be extended to piece-wise quadratic density functions as the one illustrated in Figure 2(a). The underlying idea is to partition the state space into different regions, for instance two in the figure (the negative and positive orthants) and analyze them separately and independently. In the first scenario we consider a vector field, which is converging to the local minimum on the left in the negative orthant, and is converging to the local minimum on the right in the positive orthant. The same field can be seen as diverging from zero. The density splits into two separate areas thus creating two independent clusters. This is illustrated in Figure 2(b)where the grey area turns into two separate areas far from zero as more and more players move towards the local minima.

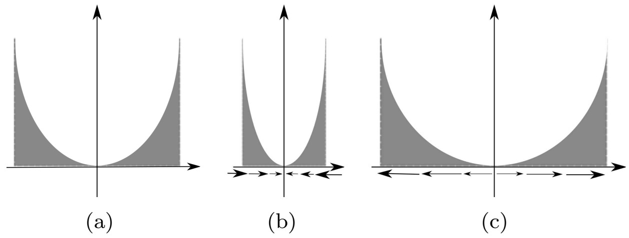

Fig. 1. Initial distribution m_0 (a) and final distribution m_T for converging (b) and diverging vector field (c). Graphs are not to scale.

On the other hand, if the vector field is diverging from the local minima and converging to zero, the density is discontinuous in zero and gives rise to a Dirac impulse at the origin. This means that the density accumulates a mass in zero and is no longer regular as illustrated in 2(c).

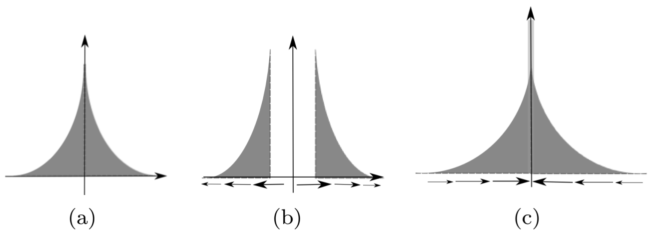

Fig. 2. Initial distribution m_0 (a) and final distribution m_T for converging (b) and diverging vector field (c).

6. NUMERICAL STUDIES

The theoretical resulsts are illustrated by a numerical study in this section. Consider a system consisting of $n = 7700$ indistinguishable players with dynamics (1) and suppose each of the players seek to minimise the cost functionals (2) subject to an adversary disturbance. Furthermore, suppose that the initial distribution is quadratic in line with Assumption 1. It follows from Theorem 1 that optimal control and the worst-case disturbance are given by (9), which relies on the solution of the two-point boundary value problem (8). The numerical results are obtained by solving the coupled ODEs (8) numerically before using the solution to simulate the closed-loop system (1) for a discretised set of states, namely $x \in [-1, 1]$. The states of the n agents are initially within this set of states. The state trajectories are computed over the period $[0, 5]$ using the sample time 0.01. The parameters used are $a_0 = 0.2597$, $\alpha = -0.1$, $\beta = 0.1$, $b = \gamma = 1$ The simulations have been run for $\sigma = 0$, *i.e.* without noise, and $\sigma = 0.1$.

Figure 3 illustrates the solution to the coupled ODEs (8). The solid lines show the time history of a_t, whereas the dashed lines show the time history of q_t for $\sigma = 0$ (top) and $\sigma = 0.1$ (bottom). Note that the boundary conditions are satisfied, *i.e.* $a(0) = a_0$ and $q_T = a_T$. Furthermore, a_t is monotonically increasing with time. Figures 4 and 5 show the initial (black, dashed line) and final (black, solid line) distribution of the agents' states for $\sigma = 0$ and $\sigma = 0.1$, respectively. The black dash-dotted lines indicate the distribution of the agents at $t = 2.5$. The grey lines indicate the distribution "predicted" by the solution to the ODEs (8), shown in Figure 3 for $t = 2.5$ (dash-dotted line) and at the final time (solid line). The distribution computed based on the evolution of the states coincides well with the solution of (8).

The time evolution of the distribution function is a shrinking quadratic function in accordance with 1(b). Since the initial distribution is such that $x = 0$ is the state with the lowest density, it is expected that the players move towards this point, which is consistent with Figures 4 and 5.

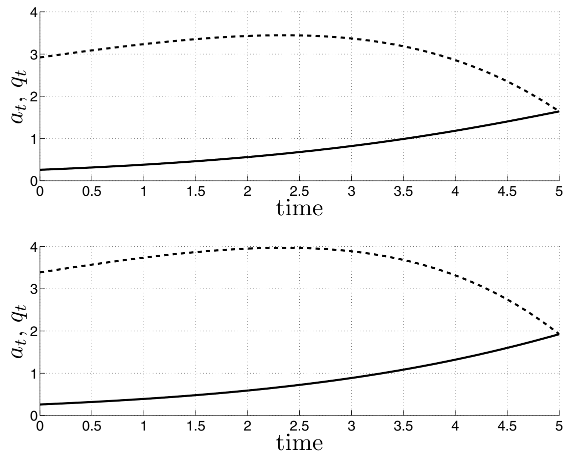

Fig. 3. Time histories of a_t (solid lines) and q_t (dashed lines) for $\sigma = 0$ (top) and $\sigma = 0.1$ (bottom).

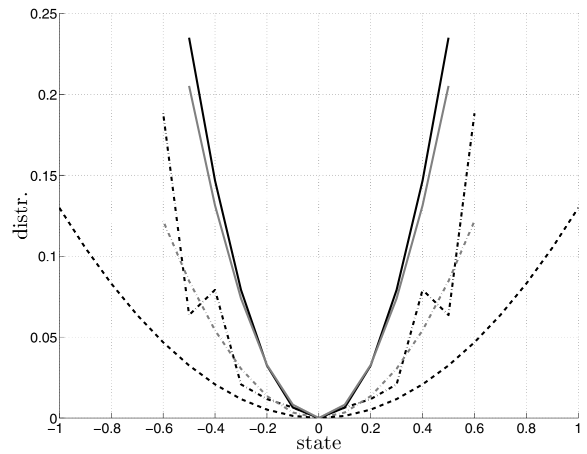

Fig. 4. The initial (dashed line), final (solid line) and intermediate (dash-dotted lines) distributions of the agents's states for $\sigma = 0$. Grey lines indicate the distribution "predicted" by the solution to (8).

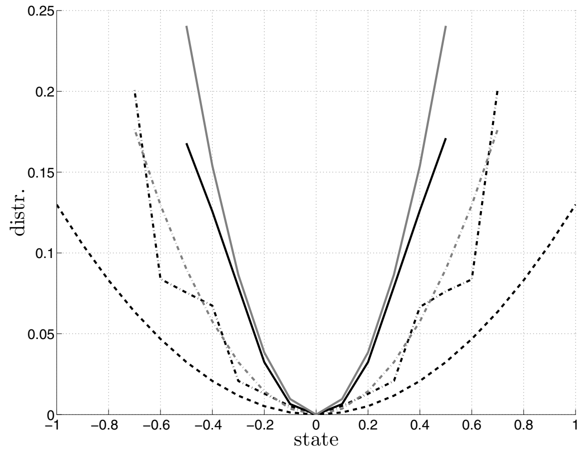

Fig. 5. The initial (dashed line), final (solid line) and intermediate (dash-dotted lines) distributions of the agents's states for $\sigma = 0.1$. Grey lines indicate the distribution "predicted" by the solution to (8).

7. CONCLUDING REMARKS

We have discussed robust mean-field games as a paradigm for crowd-averse systems. Future directions include i) the extension of the approximation method to more general cost functionals, ii) the study of the case with "local" mean-field interactions rather than "global" as in the current scenario, and iii) the analysis of crowd-seeking scenarios in contrast to the crowd-averse case analyzed in the paper.

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