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An adaptive observer for hyperbolic systems with application to UnderBalanced Drilling

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Abstract: We present an adaptive observer design for a first-order hyperbolic system of Partial Differential Equations with uncertain boundary parameters. The design relies on boundary measurements only, and is based on a backstepping approach. Using a Gradient Descent technique, we prove exponential convergence of the distributed system and estimation of the parameter. This method is applied to the estimation of uncertain parameters during the process of oil well drilling.

1. INTRODUCTION

In this article, we propose a method to estimate boundary parameters for a linear first-order hyperbolic system of Partial Differential Equations (PDEs) using boundary measurements. The system is composed of n transport equations traveling in the same direction and one counterconvective transport equation. As highlighted below, it is representative of a wide class of multi-flow systems.

One of the most important drawbacks (see Bohm et al. [1998]) of most of the existing adaptive schemes for PDEs (see e.g. Bentsman and Orlov [2001], Duncan et al. [1992]) is that they require measurement of the full distributed state, which is seldom the case in applications. Over the past decade there has been a steady increase of interest in adaptive boundary control. Recently, output-feedback adaptive designs for parabolic PDEs have been developed in Krstic and Smyshlyaev [2008], Smyshlyaev and Krstic [2007a,b], Bekiaris-Liberis et al. [2013], Bresch-Pietri et al. [2012]. Similarly, in this paper, we extend the backstepping observer of Di Meglio et al. [2013a]. This output-feedback design is particularly relevant to the problem of UnderBalanced Drilling.

The drilling of an oil well consists of creating a borehole up to several thousand meters deep into the ground, until an oil reservoir is reached. The demand for automation of this process is increasing with the necessity to reach deeper and less accessible wells, and to improve safety and efficiency of the operations. One of the main challenges during drilling lies in the poor knowledge of the "downhole" conditions: pressure and temperature conditions, permeability and porosity of the reservoir, gas and oil ratios,... In this paper, we propose a method to estimate unknown parameters while drilling an UnderBalanced well.

While a well is being drilled, a fluid circulates through the drilling system. The drilling fluid cools down the drillbit, and evacuates rock cuttings. More importantly, it pressurizes the well. In conventional drilling, the pressure can only be changed by varying its density and viscosity, or the rate at which it circulates. Conversely, in Managed Pressure Drilling (MPD), the outflow of drilling fluid is controlled by a valve, which enables tighter control of the pressure. UnderBalanced Drilling (UBD) is a sub-category of MPD, where the goal is to maintain the downhole pressure below the reservoir pressure. This causes an influx of oil and gas into the well that needs to be closely controlled.

The benefits of UBD are manyfold, as described in Bennion and Thomas [1998]. Most importantly, it reduces formation damage by preventing fracture or clogging of the reservoir. Besides, compared with MPD, it allows for better control of the gas influxes, which, when unexpected, may be extremely damaging [2]. Finally, future production of a well may be inferred from the production during UBD. This makes estimation of the reservoir characteristics all the more important. Unfortunately, the dynamics of the flow during UBD are quite complicated, Rommetveit and Lage [2001],Petersen et al. [2008]. Up to four phases (drilling fluid, rock cuttings and produced liquid and gas) are convected along a several thousand meter-long well, yielding an inherently distributed configuration.

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[2] The Macondo blowout was caused by an undetected gas influx, referred to as a gas kick.

Here, we consider a nonlinear model of the two-phase (gas-liquid) flow in the well during drilling. When linearized, the model takes the form of a first-order hyperbolic system. The uncertainty on the downhole conditions yields uncertain boundary parameters in the linearized model. To address this difficulty, we extend the backstepping transformation introduced in Di Meglio et al. [2013a] by considering an adaptation law based on the Gradient Descent technique. We prove exponential convergence of both the parameter estimate and the distributed state, in $\mathcal{L}_2$-norm. Both the controller and the parameter estimators that we design employ only boundary measurements. This is the main achievement of this paper. The relevance of this design is illustrated by simulations *on the nonlinear model* with parameters from a realistic UBD scenario.

The paper is organized as follows. In Section 2, we describe the adaptive observer design and prove convergence of the proposed scheme. In Section 3, the process of UBD and the two-phase flow model are described. In Section 4, we apply the presented design to the specific problem of reservoir pressure estimation in UBD.

Notations In the sequel, we write $\|f(t)\|_\delta$, for $\delta \in \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}^m$ ($m \in \mathbb{N}^*$), the following modified $\mathcal{L}_2$-norm

$$\|f(t)\|_\delta = \sqrt{\int_0^1 e^{\delta x} f(x,t)^T f(x,t) dx}$$

and write $\|f(t)\| = \|f(t)\|_0$ the usual $\mathcal{L}_2$-norm.

2. ADAPTIVE OBSERVER DESIGN

In this section, we derive an adaptive observer design for a somewhat general class of first-order linear hyperbolic systems. More precisely, we consider systems of $(n+1)$ equations of the form

$$u_t^i(x,t) + \lambda_i u_x^i(x,t) = \sum_{j=1}^n \sigma_{i,j} u^j(x,t) + \omega_i v, \quad i = 1, ..., n \tag{1}$$

$$v_t(x,t) - \mu v_x(x,t) = \sum_{j=1}^n \pi_j u^j(x,t) \tag{2}$$

that is to say, we consider systems of n transport equations convecting left to right, coupled with one transport equation convecting right to left.

Remark 1: For the sake of simplicity in the sequel, we have considered constant transport velocities λ_i, μ and coupling coefficients $\sigma_{i,j}$, ω_i, π_i. The results presented here straightforwardly extend to the case of spatially varying parameters.

Besides, we consider boundary conditions of the form

$$u^i(0,t) = q_i v(0,t) + \theta_i, \quad v(1,t) = \sum_{i=1}^n \rho_i u^i(1,t) + U(t) \tag{3}$$

where $U(t)$ is the control input and the $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$, $i = 1, ..., n$ are uncertain parameters. The system output is $y(t) = [u^1(1,t) \ \dots \ u^n(1,t) \ v(0,t)]^T \in \mathbb{R}^{n+1}$. We aim at estimating each of the uncertain parameter θ relying on boundary measurements only, i.e. $v(0,t)$ and $u^i(1,t)$ for $i = 1, ..., n$.

2.1 Adaptive observer equations

We introduce a parameter estimate $\hat{\theta}(t)$ and, following Di Meglio et al. [2013a], define distributed estimates $\hat{u}^i$ and $\hat{v}$ by the following observer equations

$$\begin{cases} \hat{u}_t^i + \lambda_i \hat{u}_x^i = \sum_{j=1}^n \sigma_{i,j} \hat{u}^j + \omega_i \hat{v} - p_i(x) \tilde{v}(0,t) \\ \hat{v}_t - \mu \hat{v}_x = \sum_{j=1}^n \pi_j \hat{u}^j - p_{n+1}(x) \tilde{v}(0,t) \\ \hat{u}^i(0,t) = q_i v(0,t) + \hat{\theta}_i, \ \hat{v}(1,t) = \sum_{i=1}^n \rho_i \hat{u}^i(1,t) + U(t) \end{cases} \tag{4}$$

where we have omitted the (x,t) arguments for the sake of brevity. The observer gains p_i, $i = 1, ..., n+1$ are chosen accordingly to the design presented in Di Meglio et al. [2013a]. According to this design, provided that the θ parameters were known, this would ensure that the errors $\tilde{u}^i$, v exponentially converge to zero. We can now state the main result of the paper.

Theorem 1. Define the distributed errors $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{u}^i = u^i - \hat{u}^i$ and $\tilde{v} = v - \hat{v}$ and introduce $(\bar{u}(\cdot, \tilde{\theta}), \bar{v}(\cdot, \tilde{\theta}))$ a solution of the following ODE

$$\begin{cases} \lambda_i \bar{u}_x^i = \sum_{j=1}^n \sigma_{i,j} \bar{u}^j + \omega_i \bar{v} - p_i(x) \bar{v}(0,t) \\ -\mu \bar{v}_x = \sum_{j=1}^n \pi_j \bar{u}^j - p_{n+1}(x) \bar{v}(0,t) \\ \bar{u}^i(0,t) = \tilde{\theta}_i, \quad \bar{v}(1,t) = \sum_{i=1}^n \rho_i \bar{u}^i(1,t) \end{cases} \tag{5}$$

Consider the observer (4) with the following adaptation law

$$\dot{\hat{\theta}}(t) = -\gamma \Phi(1)^T \tilde{u}(1,t) \tag{6}$$

where $\gamma > 0$ is an adaptation gain and the transition matrix $\Phi(\cdot)$ is such that $\bar{u}(x, \tilde{\theta}) = \Phi(x) \tilde{\theta}$. Finally, define the functional

$$\Gamma(t) = \|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2 + \|\bar{u}(t)\|^2 + \|\bar{v}(t)\|^2 + |\tilde{\theta}|^2 \tag{7}$$

There exists γ^* such that, provided $0 < \gamma < \gamma^*$, then there exist $R, \tau > 0$ such that

$$\Gamma(t) \leq R\Gamma(0) e^{-t/\tau}, \quad t \geq 0 \tag{8}$$

According to this result, exponential estimation of the unknown parameter θ is provided based on a Gradient Descent algorithm exploiting the boundary measurement of $u(1,t)$ and comparing it to the estimated value $\hat{u}(1,t)$. As detailed in the proof, this algorithm uses a known transition matrix Φ which relates the state trajectory $\bar{u}(\cdot, \tilde{\theta})$ to the estimation error $\tilde{\theta}$. On the other hand, the observer design is grounded on the measurement of the bottom variable $v(0,t)$.

Note that it is possible to obtain pointwise exponential convergence of the considered distributed quantities. For the sake of clarity of the exposition, we do not provide this result here. This is a direction for future work.

2.2 Proof of Theorem 1

To prove convergence of the scheme, we first introduce a tailored backstepping transformation. The stability of the corresponding target system is then proved via a Lypunov analysis which gives rise to Theorem 1.

Backstepping transformation and target system We rely on the following backstepping transformation

$$T \;:\; \left(L^2([0,1],\mathbb{R})\right)^{n+1} \to \left(L^2([0,1],\mathbb{R})\right)^{n+1} \tag{9}$$

$$(\tilde{u}^i, \tilde{v}) \mapsto (\tilde{\alpha}^i, \tilde{\beta}) \tag{10}$$

with

$$\tilde{\alpha}^i(x,t) = \tilde{u}^i(x,t) + \int_0^x r^i(x,y)\tilde{v}(y,t)dy \tag{11}$$

$$\tilde{\beta}(x,t) = \tilde{v}(x,t) + \int_0^x r^{n+1}(x,y)\tilde{v}(y,t)dy \tag{12}$$

where the kernels $r^i(\cdot,\cdot)$, $i = 1, ..., n+1$ are given in Di Meglio et al. [2013a] and are such that (α^i, β) satisfy the following system

$$\begin{cases} \tilde{\alpha}^i_t + \lambda_i \tilde{\alpha}^i_x = \sum_{j=1}^n \sigma_{i,j}\tilde{\alpha}^j + \sum_{j=1}^n \int_0^x g_{i,j}(x,y)\tilde{\alpha}^j(y,t)dy \\ \tilde{\beta}_t - \mu\tilde{\beta}_x = \sum_{j=1}^n \pi_j \tilde{\alpha}^j + \sum_{j=1}^n \int_0^x h_j(x,y)\tilde{\alpha}^j(y,t)dy \\ \tilde{\alpha}^i(0,t) = \tilde{\theta}_i\,, \quad \tilde{\beta}(1,t) = \sum_{j=1}^n \rho_j \tilde{\alpha}^j(1,t) \end{cases} \tag{13}$$

The existence of such a transformation has been demonstrated in Di Meglio et al. [2013b]. Similarly, we define $(\bar{\alpha}^i, \bar{\beta}) = T(\bar{u}^i, \bar{v})$ which satisfy

$$\begin{cases} \lambda_i \bar{\alpha}^i_x(x,\tilde{\theta}) = \sum_{j=1}^n \sigma_{i,j}\bar{\alpha}^j(x,\tilde{\theta}) + \sum_{j=1}^n \int_0^x g_{i,j}(x,y)\bar{\alpha}^j(y,\tilde{\theta})dy \\ -\mu\bar{\beta}_x(x,\tilde{\theta}) = \sum_{j=1}^n \pi_j \bar{\alpha}^j(x,\tilde{\theta}), + \sum_{j=1}^n \int_0^x h_j(x,y)\bar{\alpha}^j(y,\tilde{\theta})dy \\ \bar{\alpha}^i(0,\tilde{\theta}) = \tilde{\theta}_i\,, \quad \bar{\beta}(1,\tilde{\theta}) = \sum_{j=1}^n \rho_j \bar{\alpha}^j(1,\tilde{\theta}) \end{cases} \tag{14}$$

Lemma 2. There exists a transition matrix $\Phi(\cdot)$ such that $\bar{u}(x,\tilde{\theta}) = \Phi(x)\tilde{\theta}$, which only depends on the constants involved in the dynamics (1)–(2). Further, this matrix has no zero eigenvalues.

Proof: In the sequel, we omit the argument $\tilde{\theta}$ for the sake of clarity. Consider the equations governing $\bar{\alpha}^i$ (for $i = 1, \ldots, n$) in (14). One can rewrite these equations as one integro-differential equation bearing on $\bar{\alpha}$. As it is linear, one obtains straightforwardly the existence of Ψ_1 such that $\bar{\alpha}(x) = \Psi_1(x)\bar{\alpha}(0) = \Psi_1(x)\tilde{\theta}$. Consequently, considering the equation governing $\bar{\beta}$ in (14) and integrating, one obtains the existence of Ψ_2 such that $\bar{\beta}(x) = \Psi_2(x)\bar{\alpha}(0) = \Psi_2(x)\tilde{\theta}$. Finally, we have that $(\bar{u}^i(x), \bar{v}(x)) = T^{-1}(\bar{\alpha}^i(x), \bar{\beta}(x)) = T^{-1}(\Psi_1(x)\tilde{\theta}, \Psi_2(x)\tilde{\theta})$. Consequently, there exists Ψ_3 such that $\bar{v}(x) = \Psi_3(x)\tilde{\theta}$. Now, consider the equation governing $\bar{u}$ in (5), which is a linear ODE with a space-varying source term multiplying $\bar{v}(0)$. Therefore, as $\bar{v}(x) = \Psi_3(x)\tilde{\theta}$, there exists Φ such that $\bar{u}(x) = \Phi(x)\tilde{\theta}$. One can observe that this transition matrix only depends on the constants involved in (1)–(2) because the various transition matrix introduced above only depends on the constants and kernels in (14). As these kernels only depend on the backstepping transformation which are defined according to the constants in (1)–(2), following Di Meglio et al. [2013b], the result follows. Finally, the fact that none of the eigenvalues of this matrix is equal to zero straightforwardly follows from an extension of Cauchy-Lipschitz theorem to linear integro-differential equations. ■

Finally, we define $\check{\alpha}^i = \tilde{\alpha}^i - \bar{\alpha}^i$, $\check{\beta} = \tilde{\beta} - \bar{\beta}$ and a time-varying estimate $\hat{\theta}(t)$ such that the complete system rewrites

$$\begin{cases} \tilde{\alpha}^i_t + \lambda_i \tilde{\alpha}^i_x = \sum_{j=1}^n \sigma_{i,j}\tilde{\alpha}^j + \sum_{j=1}^n \int_0^x g_{i,j}(x,y)\tilde{\alpha}^j(y,t)dy \\ \tilde{\beta}_t - \mu\tilde{\beta}_x = \sum_{j=1}^n \pi_j \tilde{\alpha}^j + \sum_{j=1}^n \int_0^x h_j(x,y)\tilde{\alpha}^j(y,t)dy \\ \check{\alpha}^i_t + \lambda_i \check{\alpha}^i_x = \sum_{j=1}^n \sigma_{i,j}\check{\alpha}^j + \sum_{j=1}^n \int_0^x g_{i,j}(x,y)\check{\alpha}^j(y,t)dy \\ \qquad + \bar{\alpha}^i_{\tilde{\theta}}(x,\tilde{\theta}(t))\dot{\hat{\theta}}(t) \\ \check{\beta}_t - \mu\check{\beta}_x = \sum_{j=1}^n \pi_j \check{\alpha}^j + \sum_{j=1}^n \int_0^x h_j(x,y)\check{\alpha}^j(y,t)dy \\ \qquad + \dot{\hat{\theta}}(t)^T \bar{\beta}_{\tilde{\theta}}(x,\tilde{\theta}(t)) \\ \tilde{\alpha}^i(0,t) = \tilde{\theta}_i\,, \quad \tilde{\beta}(1,t) = \sum_{j=1}^n \rho_j \tilde{\alpha}^j(1,t) \\ \check{\alpha}^i(0,t) = 0\,, \quad \check{\beta}(1,t) = \sum_{j=1}^n \rho_j \check{\alpha}^j(1,t) \end{cases}$$

Lyapunov analysis We now consider the following Lyapunov functional candidate

$$V(t) = \int_0^1 e^{-\delta x}\left[p_1|\tilde{\alpha}(x,t)|^2 + p_2|\check{\alpha}(x,t)|^2\right]dx + \int_0^1 e^{\delta x}\left[\tilde{\beta}(x,t)^2 + p_3\check{\beta}(x,t)^2\right]dx + \frac{p_4}{\gamma}|\tilde{\theta}(t)|^2 \tag{15}$$

Taking a time-derivative and using suitable integrations by parts, the boundary conditions and the update law, one gets

$$\begin{aligned} \dot{V}(t) = {} & p_1\left[|\lambda^T\tilde{\alpha}(0,t)|^2 - e^{-\delta}|\lambda^T\tilde{\alpha}(1,t)|^2 - \delta\left\|\lambda^T\tilde{\alpha}(t)\right\|^2_{-\delta}\right] \\ & + \mu\left[e^{\delta}\tilde{\beta}(1,t)^2 - \tilde{\beta}(0,t)^2 - \delta\left\|\tilde{\beta}(t)\right\|^2_{\delta}\right] \\ & + p_2\left[|\lambda^T\check{\alpha}(0,t)|^2 - e^{-\delta}|\lambda^T\check{\alpha}(1,t)|^2 - \delta\left\|\lambda^T\check{\alpha}(t)\right\|^2_{-\delta}\right] \\ & + \mu p_3\left[e^{\delta}\check{\beta}(1,t)^2 - \check{\beta}(0,t)^2 - \delta\left\|\check{\beta}(t)\right\|^2_{\delta}\right] \\ & + \int_0^1 e^{-\delta x}\left[p_1\tilde{\alpha}(x,t)^T\Lambda_1(\tilde{\alpha}) + p_2\check{\alpha}(x,t)^T\Lambda_1(\check{\alpha})\right]dx \end{aligned}$$

$$+\int_0^1 e^{\delta x}\left[\tilde{\beta}(x,t)\Lambda_2(\tilde{\alpha})+p_3\check{\beta}(x,t)\Lambda_2(\check{\alpha})\right]dx$$
$$+p_2\int_0^1 e^{-\delta x}\check{\alpha}(x,t)^T\bar{\alpha}_{\tilde{\theta}}(x,\tilde{\theta}(t))\dot{\hat{\theta}}(t)dx$$
$$+p_3\int_0^1 e^{\delta x}\check{\beta}(x,t)\dot{\hat{\theta}}(t)^T\bar{\beta}_{\tilde{\theta}}(x,\tilde{\theta}(t))dx$$
$$-p_4\tilde{\theta}^T\Phi(1)^T\left[\bar{u}(1,t)+\check{\alpha}(1,t)+\int_0^1 m(1,x)\check{\beta}(x,t)dx\right] \tag{16}$$

with

$$\Lambda_1(\tilde{\alpha})=\begin{pmatrix}\sum_{j=1}^{n}\left[\sigma_{1,j}\tilde{\alpha}^j(x,t)+\int_0^x g_{1,j}(x,y)\tilde{\alpha}^j(y,t)dy\right]\\ \vdots\end{pmatrix} \tag{17}$$

$$\Lambda_2(\tilde{\alpha})=\sum_{j=1}^{n}\left[\pi_j\tilde{\alpha}^j(x,t)+\int_0^x h_j(x,y)\tilde{\alpha}^j(y,t)dy\right] \tag{18}$$

Using Young's and Cauchy-Schwartz's inequality , the fact that $(\hat{\theta},\theta)\in[\underline{\theta},\overline{\theta}]^2$ and the continuity of the variables at stake, one obtains the existence of constants M and $\tilde{M}$ independent of δ and γ such that

$$\left|2\int_0^1 e^{-\delta x}\tilde{\alpha}(x,t)^T\Lambda_1(\tilde{\alpha})dx\right|\le M\|\tilde{\alpha}(t)\|_{-\delta}$$
$$\left|2\int_0^1 e^{\delta x}\tilde{\beta}(x,t)\Lambda_2(\tilde{\alpha})dx\right|\le M(e^{2\delta}\|\tilde{\alpha}(t)\|_{-\delta}+\left\|\tilde{\beta}(t)\right\|_{\delta}^2)$$
$$\left|2\int_0^1 e^{-\delta x}\check{\alpha}(x,t)^T\Lambda_1(\check{\alpha})dx\right|\le M\|\check{\alpha}(t)\|_{-\delta}$$
$$\left|2\int_0^1 e^{\delta x}\check{\beta}(x,t)\Lambda_2(\check{\alpha})dx\right|\le M(e^{2\delta}\|\check{\alpha}(t)\|_{-\delta}+\left\|\check{\beta}(t)\right\|_{\delta}^2)$$
$$\left|2\int_0^1 e^{-\delta x}\check{\alpha}(x,t)^T\bar{\alpha}_{\tilde{\theta}}(x,\tilde{\theta}(t))\dot{\hat{\theta}}(t)dx\right|$$
$$\le M(\|\check{\alpha}(t)\|_{-\delta}^2+\gamma^2\tilde{\alpha}(1,t)^2+\gamma^2\left\|\tilde{\beta}(t)\right\|_{\delta}^2)$$
$$\left|2\int_0^1 e^{\delta x}\check{\beta}(x,t)\dot{\hat{\theta}}(t)^T\bar{\beta}_{\tilde{\theta}}(x,\tilde{\theta}(t))dx\right|$$
$$\le M(\left\|\check{\beta}(t)\right\|_{\delta}^2+\gamma^2\tilde{\alpha}(1,t)^2+\gamma^2\left\|\tilde{\beta}(t)\right\|_{\delta}^2)$$
$$2\tilde{\theta}^T\Phi(1)^T\left[\check{\alpha}(1,t)+\int_0^1 m(1,x)\check{\beta}(x,t)dx\right]$$
$$\le\lambda_m(\Phi(1)^T\Phi(1))|\tilde{\theta}(t)|^2+\tilde{M}(|\check{\alpha}(1,t)|^2+\left\|\check{\beta}(t)\right\|_{\delta}^2)$$

This yields the following upper bound

$$\dot{V}(t)\le-\left(p_2e^{-\delta}\min\lambda_i-p_3\mu e^{\delta}n|\rho|^2-p_4\tilde{M}\right)|\check{\alpha}(1,t)|^2$$
$$-\left(p_1e^{-\delta}\min\lambda_i-\mu e^{\delta}n|\rho|^2-(p_2+p_3)\gamma^2M\right)|\tilde{\alpha}(1,t)|^2$$
$$-\left(p_1(\delta\min\lambda_i-M)-e^{2\delta}M\right)\|\tilde{\alpha}(t)\|_{-\delta}^2$$
$$-\left(p_2(\delta\min\lambda_i-2M)-p_3e^{2\delta}M\right)\|\check{\alpha}\|_{-\delta}^2$$
$$-\left((\mu\delta-M)-p_2\gamma^2M\right)\left\|\tilde{\beta}(t)\right\|_{\delta}^2$$
$$-\left(p_3(\delta\mu-2M)-p_4\tilde{M}\right)\|\check{\beta}(t)\|_{-\delta}^2$$
$$-\left(p_4\lambda_m\left(\Phi(1)^T\Phi(1)\right)-p_1|\lambda|\right)|\tilde{\theta}(t)|^2$$

in which $\lambda_m(\Phi(1)^T\Phi(1))>0$ as $\Phi(1)^T\Phi(1)$ is symmetric positive and also definite following Lemma 2. Therefore, choosing

$$\delta>\max\left\{\frac{2M}{\mu},\frac{2M}{\min\lambda_i}\right\} \tag{19}$$
$$p_1>\max\left\{\frac{\mu e^{2\delta}n|\rho|^2}{\min\lambda_i},\frac{e^{2\delta}M}{\delta\min\lambda_i-M}\right\} \tag{20}$$
$$p_4>\frac{p_1|\lambda|}{\lambda_m(\Phi(1)^T\Phi(1))} \tag{21}$$
$$p_3>\frac{p_4\tilde{M}}{\delta\mu-2M} \tag{22}$$
$$p_2>\max\left\{\frac{e^{\delta}(p_3\mu e^{\delta}n|\rho|^2+p_4\tilde{M}))}{\min\lambda_i},\frac{p_3e^{2\delta}M}{\delta\min\lambda_i-2M}\right\} \tag{23}$$
$$\gamma^2<\frac{p_1e^{-\delta}\min\lambda_i-\mu e^{\delta}n|\rho|^2}{(p_2+p_3)M}=(\gamma^\star)^2 \tag{24}$$

one obtains the existence of $\eta_0>0$ such that

$$\dot{V}(t)\le-\eta_0V(t) \tag{25}$$

Now, using the backstepping transformations (11)–(12) and their inverse, which are given in Di Meglio et al. [2013a], and applying Young's and Cauchy-Schwartz's inequalities, one obtains the existence of positive constants r_1,r_2,r_3,s_1,s_2 and s_3 such that

$$\|\tilde{\alpha}(t)\|_{-\delta}^2\le r_1\|\tilde{u}(t)\|^2+r_2\|\tilde{v}(t)\|^2\ ,\quad \left\|\tilde{\beta}(t)\right\|_{\delta}^2\le r_3\|\tilde{v}(t)\|^2$$
$$\|\tilde{u}(t)\|^2\le s_1\|\tilde{\alpha}(t)\|_{-\delta}^2+s_2\left\|\tilde{\beta}(t)\right\|_{\delta}^2\ ,\ \|\tilde{v}(t)\|^2\le s_3\left\|\tilde{\beta}(t)\right\|_{\delta}^2$$

and such that similar equations hold for $(\bar{\alpha},\bar{\beta},\bar{u},\bar{v})$ instead of $(\tilde{\alpha},\tilde{\beta},\tilde{u},\tilde{v})$. Consequently, considering (7) and (15), one obtains the existence of positive constants ρ_1 and ρ_2 such that

$$\rho_1\Gamma(t)\le V(t)\le\rho_2\Gamma(t)$$

Therefore, integrating (25), one gets

$$\Gamma(t)\le\frac{V(t)}{\rho_1}\le\frac{V(0)}{\rho_1}e^{-\eta_0t}\le\frac{\rho_2}{\rho_1}\Gamma(0)e^{-\eta_0t}$$

and the result follows defining $R=\rho_2/\rho_1$ and $\tau=1/\eta_0$.

We aim at applying the method proposed in Theorem 1 to reservoir pressure estimation during UBD. Before doing so, we describe in the next section the process and its modelling.

3. UBD: MODELLING AND PROCESS DESCRIPTION

Consider the drilling system schematically depicted in Fig. 1. It consists of a circulation system: The drilling fluid is pumped into the top of the drill string, and circulated out at the bottom at the drilling bit. The drilling fluid flows up trough the annular section surrounding the drill string transporting the formation particles, referred as cutting and cavings, out of the well. The cuttings and any produced fluids are then separated from the drilling fluid before it is injected into the drill string again.

In Under-Balanced Drilling operations the downhole pressure is deliberately kept below the reservoir pore pres-

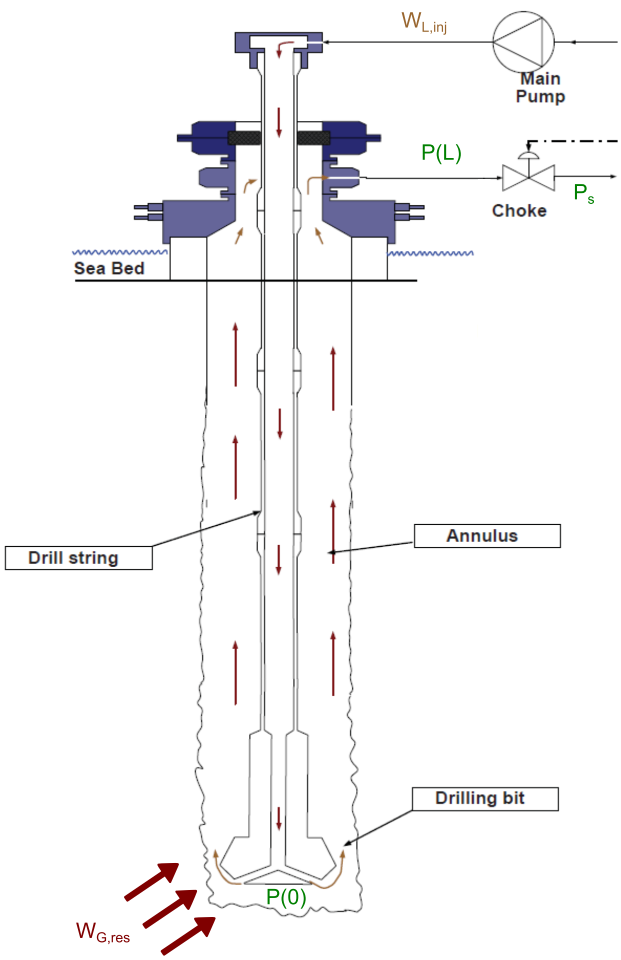

Fig. 1. Drilling process schematic for UBD.

sure[3], causing continuous inflow of produced fluid from the reservoir. The reservoir inflow is related to the down-hole pressure by the Production Index (PI) and pore pressure. It is highly desirable to do real-time estimation of the Production Index and pore pressure of the reservoir for several reasons:

- the lower limit on the allowable downhole pressure is sometimes given by a constraint on the rate of produced gas. Hence better estimates of these coefficients enable a less conservative lower limit or reduce the risk of breaking this constraint;
- the upper limit on the allowable downhole pressure is given by the reservoir pressure;
- ultimately, the goal of drilling a production well is to enable production. Hence good estimates of these parameters give the necessary information w.r.t. deciding whether to keep drilling or declaring total depth;
- the PI significantly affects the dynamics of the system.

3.1 Modelling

In the literature, the most used model of multiphase flow in drilling is the Drift Flux Model (DFM). To keep the complexity of the model manageable, the fluids in the well are typically lumped into a gas and a liquid phase and the energy equation ignored thereby reducing the number of distributed states to 3 (see Fjelde and Rommetveit [2003],Lage and Fjelde [2000]). We stress that this model requires tuning to actual data or a high fidelity model before use.

[3] I.e. the pressure in the fluids in the reservoir surrounding the borehole.

Description	Symbol
Area of flow	A
Velocity of sound, liquid	a_L^2
Slip parameter	C_0
Choke constant	C_v
Hydraulic diameter	D
Gravity constant	g
Friction factor	f
Gas inflow parameter	k_G
Liquid inflow parameter	k_L
Reservoir pressure	P_{res}
Separator pressure	P_s
Reference liquid pressure	p_0
Specific gas constant	R_G
Temperature	$T(s)$
Slip parameter	v_∞
Choke correction factor	Y
Mass fraction, liquid, gas	$x_{L,G}$
Gas compression factor	Z_G
Liquid density	ρ_L
Reference liquid density	$\rho_{L,0}$
Inclination	$\phi(s)$

Table 1. List of parameters

3.2 The drift flux model:

The model consists of expressing the mass conservation law for the gas and the liquid separately, and a combined momentum equation. The drilling fluid, oil and water are lumped into one single liquid phase. For $k = L, G, m$ denoting liquid, gas or mixture, we denote α_k the volume fractions, ρ_k the densities, v_k the superficial velocities, and P the pressure. All of these variables are functions of time and space. We denote $t \geq 0$ the time variable, and $s \in [0, L]$ the space variable, corresponding to a curvilinear abscissa with s = 0 corresponding to the bottom hole and x = L to the outlet choke position (see Fig. 1).

The dynamics equations are as follows,

$$\frac{\partial \alpha_L \rho_L}{\partial t} + \frac{\partial \alpha_L \rho_L v_L}{\partial x} = 0, \tag{26}$$

$$\frac{\partial \alpha_G \rho_G}{\partial t} + \frac{\partial \alpha_G \rho_G v_G}{\partial x} = 0, \tag{27}$$

$$\frac{\partial \alpha_L \rho_L v_L + \alpha_G \rho_G v_G}{\partial t} + \frac{\partial P + \alpha_G \rho_G v_G^2 + \alpha_L \rho_L v_L^2}{\partial x} = -\rho_m g \sin\phi(s) - \frac{2 f \rho_m v_m |v_m|}{D}. \tag{28}$$

In the momentum equation (28), the term $\rho_m g \sin\theta$ represents the gravitational source term, while $-\frac{2f\rho_m v_m |v_m|}{D}$ accounts for frictional losses. The parameters are detailed in Table 1. The mixtures are given as

$$\rho_m = \alpha_G \rho_G + \alpha_L \rho_L, \qquad v_m = \alpha_G v_G + \alpha_L v_L. \tag{29}$$

Along with these distributed equations, algebraic relations are needed to close the system.

$$\alpha_L + \alpha_G = 1 \qquad\qquad v_G = C_0 v_m + v_\infty, \tag{30}$$
$$\rho_G = Z_G R_G T P \qquad\qquad \rho_L = \text{const.} \tag{31}$$

where Z_G, R_G, T are the gas compression factor, specific gas constant and temperature respectively, and C_0, v_∞ are parameters giving the slip between the velocity of the gas and liquid phase. For relatively homogeneous operating conditions, setting C_0, v_∞ constant should be satisfactory.

Boundary Conditions Boundary conditions on the left (downhole) boundary are given by the mass-rates of liquid injected from the drill string on the one hand, and gas flowing in from the reservoir on the other hand. The influx of gas depends on the pressure difference between the reservoir and the bottom of the well.

$$A\alpha_L(0)\rho_L(0)v_L(0) = W_{L,inj}, \tag{32}$$
$$A\alpha_G(0)\rho_G(0)v_G(0) = k_G \max(P(0) - P_{res}, 0). \tag{33}$$

Here P_{res} is the reservoir pore pressure and k_G is the production index (PI).

The topside boundary condition is given by a choke equation relating topside pressure to mass flow rates Murdock [1962]

$$A\alpha_L(L)\rho_L(L)v_L(L) + A\alpha_G(L)\rho_G(L)v_G(L) = C_v(Z)\frac{\sqrt{P(L) - P_s}}{\frac{x_L}{\sqrt{\rho_L}} + \frac{x_G}{Y^2\sqrt{\rho_G}}}, \tag{34}$$

where $x_{L,G}$ denotes the mass fraction of liquid and gas, C_v the choke opening given by the manipulated variable Z and Y is a correction factor for gas flow. Changing the choke opening is the primary control actuation for the drilling system.

3.3 Reformulation under a quasilinear form and linearization

Using an appropriate set of variables, e.g. $\xi = (\alpha_G, \rho_G, v_G)$, the system can be put in quasilinear form

$$\frac{\partial \xi}{\partial t} + A(\xi)\frac{\partial \xi}{\partial x} = G(\xi) \tag{35}$$

Considering small perturbations $\delta\xi$ around an equilibrium profile $\bar{\xi}$, the linearized system can be rewritten as (1),(2) where (u, v) is obtained from $\delta\xi$ using a linear change of coordinates. Similarly, the nonlinear boundary conditions (32)–(34) can be rewritten as

$$h_l(\xi(0,t), P_{res}) = 0, \qquad h_r(\xi(L,t)) = 0 \tag{36}$$

Again, considering small perturbations $\delta\xi$, δP_{res} around reference values $\bar{\xi}$, $\bar{P}_{res}$ yields

$$\frac{\partial h_l}{\partial \xi}\left(\bar{\xi}(0), \bar{P}_{res}\right)\delta\xi + \frac{\partial h_l}{\partial P_{res}}\left(\bar{\xi}(0), \bar{P}_{res}\right)\delta P_{res} = 0 \tag{37}$$

which yields, after an appropriate coordinate transformation, boundary conditions of the form (3) in which $\delta P_{res} = \theta$ is uncertain. For details on the linearization of (35),(36), the interested reader is referred to Di Meglio et al. [2012].

We stress that, in all rigor, the obtained linearized PDE system presents space-varying coefficients, which is not the case of (1)–(2). However, we claim that Theorem 1 can be straightforwardly extended to this case. This is another direction of future work.

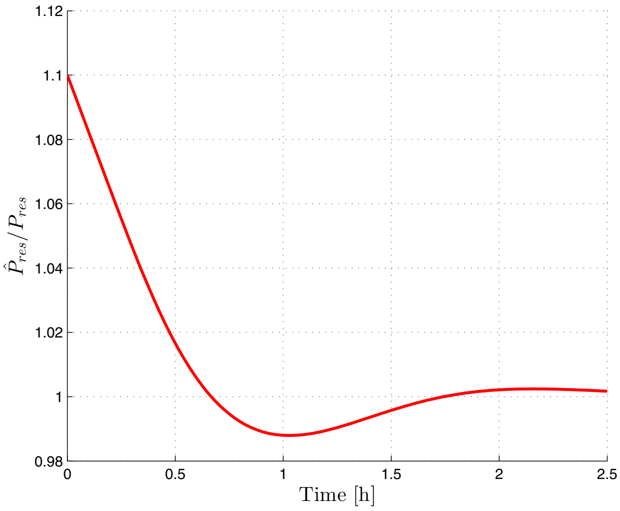

Fig. 2. Evolution of the normalized estimate of the reservoir pressure $\hat{P}_{res}$

4. RESERVOIR PRESSURE ESTIMATION IN UBD

In this section, we aim at illustrating Theorem 1 and the merits of the proposed results by using the adaptive law (6) to estimate reservoir pressure during UBD. We briefly present an extension of the observer design to the considered nonlinear system. Although Theorem 1 only deals with a linear coupled transport PDE, one can expect that the obtained result will also hold locally in all likelihood.

4.1 Adaptive observer for the nonlinear system

We consider a 2200 meter long well, with a reservoir pressure $P_{res} = 225$ bar. At the left (bottom) boundary, only the pressure is being measured. At the right boundary, we assume full measurement of all the states. The observer is a copy of the original model with linear output error injection terms, i.e.

$$\frac{\partial \hat{u}}{\partial t} + A(\hat{u})\frac{\partial \hat{u}}{\partial x} = G(\hat{u}) - L(x)(\hat{P}(0) - P(0)) \tag{38}$$

and the observer boundary condition is designed by directly injecting the measurement $P(0)$ into Equation (33). The observer gains $L(\cdot)$ are chosen such that, when linearized, Equation (38) yields (4). Finally, the update law (6) is used to estimate P_{ref}. Although no proof of convergence has been derived for the nonlinear system, one can reasonably expect the observer to perform well when the system is close to steady-state and during smooth transients. The following simulations give promising results in that regard.

4.2 Simulations

The observer is initialized with a 10% error on the reservoir pressure, i.e. $\frac{\hat{P}_{res} - P_{res}}{P_{res}} = 0.1$. During the simulation, the outlet valve opening Z, defined in Equation (34) is gradually opened to generate a transient behavior. Figure 2 shows the normalized evolution of the estimated reservoir pressure, which converges as expected. Figure 3 pictures the evolution of various states of interest of the model at the outlet.

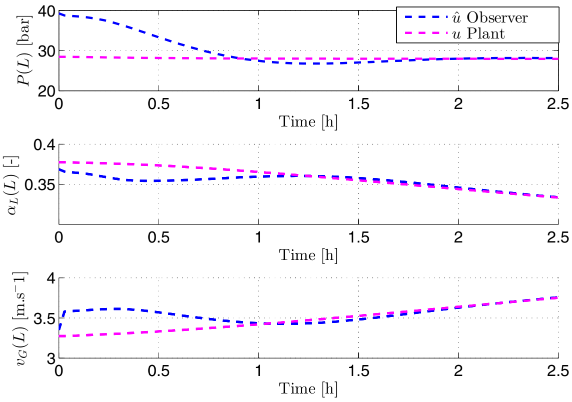

Fig. 3. Comparative evolution of the true states and estimates (topside pressure, liquid holdup and gas velocity) at the right boundary.

5. PERSPECTIVES

The proposed adaptive observer is, to the best of our knowledge, the first analytical result on combined state and parameter estimation for UnderBalanced Drilling operations. The presented simulations are relatively promising as the method, while designed for linear system, seems to apply to the nonlinear flow model. However, it is crucial to notice that the reservoir pressure was, in the presented simulations, the only source of uncertainty. The method needs to be tested against, e.g. high-fidelity simulators, from which our simulation model will significantly differ. This will induce other non-modeled sources of uncertainty (e.g. on in-domain parameters), for which the adaptation law will try to compensate, yielding false estimate of the parameters. Further analysis is required to quantify the impact of other modelling errors and the robustness of the proposed approach.

Other direction of future work include extension of the adaptive observer for linear coupled transport equations to space-varying coefficients and proof of pointwise convergence.

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