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# On input–to–state stability analysis of discrete–time systems via finite–time Lyapunov functions

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**Abstract:**
This paper considers the problem of inherent robustness analysis for nonlinear discrete–time systems using the concept of a finite–time Lyapunov function. The main contribution is to prove that, for sufficiently continuous dynamics and finite–time Lyapunov functions, inherent global input–to–state stability to general disturbances can be established for nominally stable nonlinear systems. Moreover, under mere continuity of the finite–time Lyapunov function and of the dynamics, inherent input–to–state stability on a compact set is obtained.

*Keywords:* Finite–time Lyapunov functions, input–to–state stability, discrete–time systems

## 1. INTRODUCTION

Stability analysis of nonlinear systems is an inherently difficult problem which is usually addressed by constructing Lyapunov functions, see, e.g., (Khalil, 2002) and (Vidyasagar, 2002). However, computing a Lyapunov function for general nonlinear systems is rather difficult.

The finite–time Lyapunov function (FTLF) is a relaxation of the classical Lyapunov function, where the decrease of the Lyapunov function is required in a finite number of steps rather than at each step. A similar relaxation was originally proposed in (Aeyels and Peuteman, 1998), and it was also used in (Böhm et al., 2012) and (Gielen and Lazar, 2012). Recently, it was proven in (Bobiti et al., 2013) that for FTLF, any candidate function can be used for stability analysis. Therein, FTLF were proven to provide non–conservative stability analysis tests for globally exponentially stable nonlinear systems, with a focus on the tractability of such tests for linear systems. Given the freedom of choosing any candidate function, FTLF is an attractive approach for stability analysis of nonlinear systems. Indeed, stability analysis via FTLF is opening new opportunities, see (Lazar et al., 2013a), where scalable and non–conservative FTLF stability tests were developed for switched linear systems.

However, while FTLF are attractive for stability analysis, it is yet unknown if inherent robustness can be guaranteed by FTLF. Since inherent robustness is a major concern for discrete–time, possibly discontinuous systems, see, e.g., (Grimm et al., 2004), (Lazar et al., 2009), (Lazar et al., 2013b), the goal of this paper is to provide a framework for inherent input–to–state stability (ISS) analysis via FTLF.

To this end, let us recall the usual approach to inherent ISS via standard Lyapunov functions. ISS of discrete–time systems was formulated in (Jiang and Wang, 2001) and it was further explored in (Limón et al., 2006), (Magni et al., 2006), (Lazar et al., 2008) and (Lazar and Heemels, 2009) in a Lyapunov functions context. As shown in (Lazar et al., 2009), globally exponentially stable systems may lack ISS even to arbitrarily small inputs, if the nominal dynamics and the Lyapunov function are discontinuous. On the other hand, in (Lazar et al., 2013b) it is shown that if the Lyapunov function is sufficiently continuous, then inherent, even global ISS can be guaranteed.

In this context, the question to be addressed by this paper is to find the conditions under which inherent ISS is guaranteed for nominally stable systems in a FTLF framework. It is illustrated through an example that, unlike the classical Lyapunov functions, see (Lazar et al., 2013b), the existence of a $\mathcal{K}_\infty$–continuous (KIC) FTLF does not necessarily imply inherent ISS. Therefore, a formal proof of inherent ISS is established, under a set of assumptions concerning the nominal system dynamics, the perturbed system dynamics and the class of FTLF candidates. More specifically, the main contribution of this paper is to prove that ISS FTLF implies global ISS under the assumptions that the nominal system dynamics is KIC and the system is uniformly KIC with respect to the disturbance. Moreover, it is proven in this paper that under the same assumptions, a KIC FTLF becomes an ISS FTLF as well. Therefore, under certain conditions, existence of a FTLF guarantees ISS.

Furthermore, the paper introduces inherent ISS results for systems defined on compact sets, with general disturbances taking values from a compact set, by FTLF.

The remainder of this paper is organized as follows. In Section 2, basic notation and definitions are introduced, together with the system class and the definition of a FTLF. Then, Section 3 introduces an example motivating the analysis of inherent robustness and the ISS results for discrete–time systems in the context of FTLF. Section 4 provides conditions for ISS on compact sets. Section 5 concludes the paper.

## 2. PRELIMINARIES

This section introduces basic notation as well as the system class considered in this paper and the concept of a FTLF.

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non–negative reals, the set of integers and the set of non–negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$, define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$. Furthermore, $\mathbb{R}_{\Pi} := \mathbb{R} \cap \Pi$ and $\mathbb{Z}_{\Pi} := \mathbb{Z} \cap \Pi$. Let $\mathbb{S}^h := \mathbb{S} \times \ldots \times \mathbb{S}$ for any $h \in \mathbb{Z}_{\geq 1}$ denote the $h$–times Cartesian–product of $\mathbb{S} \subseteq \mathbb{R}^n$. Denote $\circ$ the operator of maps composition, i.e., for two arbitrary maps $\alpha_1 : \mathbb{D}_1 \to \mathbb{C}_1$, and $\alpha_2 : \mathbb{D}_2 \to \mathbb{C}_2$, with $\mathbb{C}_2 \subseteq \mathbb{D}_1$, $\alpha_1 \circ \alpha_2(x) = \alpha_1(\alpha_2(x)), \forall x \in \mathbb{D}_2$. Let $\alpha^h := \alpha \circ \ldots \circ \alpha$ for any $h \in \mathbb{Z}_{\geq 1}$ denote the $h$–times map composition of $\alpha : \mathbb{C} \to \mathbb{C}$. Let $id : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function, i.e., $id(x) = x, \forall x \in \mathbb{R}^n$. Observe that $id^{-1}(x) = id(x) = x$. Let $\mathcal{B}$ denote the open unit ball in $\mathbb{R}^n$. Let $\overline{\mathcal{O}}$ define the closure of a set $\mathcal{O}$. Also, let $\mathcal{B}$ (or $\overline{\mathcal{B}}$) denote the open (or closed) unit ball in $\mathbb{R}^n$.

For a vector $x \in \mathbb{R}^n$, the symbol $\|x\|$ is used to denote an arbitrary $p$–norm; it will be made clear when a specific norm is considered. For a sequence $\{x_j\}_{j \in \mathbb{Z}_+}$, with $x_j \in \mathbb{R}^n$, $x_{[k]}$ denotes the truncation of $\{x_j\}_{j \in \mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e., $x_{[k]} = \{x_j\}_{j \in \mathbb{Z}_{[0,k]}}$, and $x_{[k_1,k_2]}$ denotes the truncation of $\{x_j\}_{j \in \mathbb{Z}_+}$ at times $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\geq k_1}$, i.e., $x_{[k_1,k_2]} = \{x_j\}_{j \in \mathbb{Z}_{[k_1,k_2]}}$. For a sequence $\{x_j\}_{j \in \mathbb{Z}_+}$, with $x_j \in \mathbb{R}^n$, $\|\{x_j\}_{j \in \mathbb{Z}_+}\| := \sup\{\|x_j\| | j \in \mathbb{Z}_+\}$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, i.e., $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. Furthermore, $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s \to \infty} \alpha(s) = \infty$. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$, i.e., $\beta \in \mathcal{KL}$, if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is decreasing and $\lim_{s \to \infty} \beta(r, s) = 0$.

*Fact 1.*

The following statements are true:

**(i)** If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then $\alpha_1 + \alpha_2 \in \mathcal{K}_\infty$;

**(ii)** If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then $\max(\alpha_1, \alpha_2) \in \mathcal{K}_\infty$;

**(iii)** If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then $\alpha_1 \circ \alpha_2$ belongs also to class $\mathcal{K}_\infty$;

**(iv)** If $\alpha \in \mathcal{K}_\infty$, then $\alpha^{-1} \in \mathcal{K}_\infty$;

**(v)** If $\beta_1 \in \mathcal{KL}$ and $\alpha_1 \in \mathcal{K}$, then $\beta := \beta_1(\alpha_1(s), k)$ is also of class $\mathcal{KL}$;

**(vi)** If a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$, then $\alpha(x_1 + x_2) \leq \alpha(2\max(x_1, x_2)) \leq \alpha(2x_1) + \alpha(2x_2)$, for all $(x_1, x_2) \in \mathbb{R}^2_+$.

Some of the above statements can also be found in (Limón et al., 2006).

### 2.1 System class

Consider the discrete–time perturbed autonomous nonlinear system

$$x_{k+1} = \Phi(x_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $v_k \in \mathbb{R}^{d_v}$ is an unknown disturbance input and $\Phi : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is a nonlinear, possibly discontinuous function. Denote the corresponding non–disturbed system by

$$G(x) := \Phi(x, 0), \quad \forall x \in \mathbb{R}^n \tag{2}$$

with $G : \mathbb{R}^n \to \mathbb{R}^n$ and $G(0) = \Phi(0, 0) = 0$.

Let $\{x_k(\xi)\}_{k \in \mathbb{Z}_+}$ denote the solution of (2) from initial condition $\xi \in \mathbb{R}^n$, i.e., such that $x_0(\xi) := \xi$ and $x_{k+1}(\xi) := G(x_k(\xi))$ for all $k \in \mathbb{Z}_+$. Let $\{x_k(\xi, v_{[k-1]})\}_{k \in \mathbb{Z}_+}$ denote the solution of (1) from initial condition $\xi \in \mathbb{R}^n$, such that $x_0(\xi, 0) := \xi$ and $x_{k+1}(\xi, v_{[k]}) := \Phi(x_k(\xi, v_{[k-1]}), v_k)$ for all $k \in \mathbb{Z}_+$. Denote $x_k := x_k(\xi, v_{[k-1]})$, or $\Phi^k(\xi, v_{[k-1]}) := x_k(\xi, v_{[k-1]})$ for all $k \in \mathbb{Z}_{\geq 1}$ and all $v_{[k-1]} \in (\mathbb{R}^{d_v})^k$.

For any $i \in \mathbb{Z}_{\geq 1}$, let $\overline{G}^i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued map such that

$$\overline{G}^i(\mathbb{X}) := \{G^i(x) | x \in \mathbb{X}\},$$

for any set $\mathbb{X} \subseteq \mathbb{R}^n$. By convention, $\overline{G}^0(\mathbb{X}) := \mathbb{X}$. Similarly, let $\overline{\Phi}^i : \mathbb{R}^n \times (\mathbb{R}^d)^i \rightrightarrows \mathbb{R}^n$ such that

$$\overline{\Phi}^i(\mathbb{X}, \mathbb{D}^i) := \{\Phi^i(x, v_{[i-1]}) | x \in \mathbb{X}, v_{[i-1]} \in \mathbb{D}^i\},$$

for any sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{D} \subseteq \mathbb{R}^{d_v}$. By convention, $\overline{\Phi}^0(\mathbb{X}, \mathbb{D}^0) := \mathbb{X}$.

*Definition 2.* The system (2) is called globally $\mathcal{KL}$–stable if there exists a $\mathcal{KL}$ function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|x_k(\xi)\| \leq \beta(\|\xi\|, k)$ for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_+$.

*Definition 3.* (Jiang and Wang, 2001) The perturbed system (1) is globally input–to–state stable (globally ISS) if there exists a $\mathcal{KL}$ function $\beta$ and a $\mathcal{K}$ function $\gamma$ such that the corresponding state trajectory satisfies

$$\|x_k(\xi, v_{[k-1]})\| \leq \beta(\|\xi\|, k) + \gamma(\|v_{[k-1]}\|), \tag{3}$$

for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_+$ and all $\{v_j\}_{j \in \mathbb{Z}_+}$ with $v_j \in \mathbb{R}^{d_v}$ for all $j \in \mathbb{Z}_+$.

*Definition 4.* A system is called zero–robust if it is not ISS for any, arbitrarily small non–zero disturbances.

*Definition 5.* A real valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called $\mathcal{K}$–infinity continuous (KIC) if there exists a function $\sigma_V \in \mathcal{K}_\infty$ such that

$$|V(x) - V(y)| \leq \sigma_V(\|x - y\|), \tag{4}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

*Definition 6.* A map $G : \mathbb{R}^n \to \mathbb{R}^n$ is called KIC if there exists a function $\sigma_x \in \mathcal{K}_\infty$ such that

$$\|G(x) - G(y)\| \leq \sigma_x(\|x - y\|), \tag{5}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

*Definition 7.* Let $\mathbb{X} \subseteq \mathbb{R}^n$. Then, a map $G : \mathbb{X} \to \mathbb{X}$ is called Lipshitz continuous in $\mathbb{X}$ if there exists an $a \in \mathbb{R}_+$ such that

$$\|G(x) - G(y)\| \leq a\|x - y\|, \tag{6}$$

for all $(x, y) \in \mathbb{X} \times \mathbb{X}$.

*Definition 8.* A map $G : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathcal{K}_\infty$–bounded if there exists a function $\alpha \in \mathcal{K}_\infty$ such that

$$\|G(x)\| \leq \alpha(\|x\|), \tag{7}$$

for all $x \in \mathbb{R}^n$.

Observe that the condition of $\mathcal{K}_\infty$–boundedness on a system which is globally $\mathcal{KL}$–stable is not restrictive, since $\mathcal{K}_\infty$–boundedness is derived from $\mathcal{KL}$–stability when $k = 1$. Moreover, if $G$ is KIC, then $G$ is also $\mathcal{K}_\infty$–bounded. Nevertheless, the converse does not hold.

*Definition 9.* A map $\Phi : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^m$ is called KIC uniformly in $x$ if there exists a function $\sigma_d \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$,

$$\|\Phi(x, v) - \Phi(x, w)\| \leq \sigma_d(\|v - w\|), \tag{8}$$

for all $(v, w) \in \mathbb{R}^{d_v} \times \mathbb{R}^{d_v}$.

### 2.2 Definition of a FTLF

Let us state the definition of a FTLF, as introduced in (Bobiti et al., 2013) and (Lazar et al., 2013a).

*Proposition 10.* Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Suppose that the function $G : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to the dynamics (2) is $\mathcal{K}_\infty$–bounded and there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{R}^n, \tag{9a}$$

and that there exists an $M \in \mathbb{Z}_{\geq 1}$ and corresponding $\rho \in \mathbb{R}_{[0,1)}$ such that

$$V(x_M(\xi)) \leq \rho V(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{9b}$$

Then, system (2) is globally $\mathcal{KL}$-stable.

*Definition 11.* The real valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ which satisfies the conditions of Proposition 10 is called a global *FTLF*.

The proof of Proposition 10 is a particular case of the proof of Theorem 13 on ISS, which corresponds to zero input, and is omitted for brevity.

The idea that is employed in Proposition 10, i.e., to relax the classical Lyapunov conditions such that the corresponding function is decreasing after a finite time rather than at each time instance, was inspired by the asymptotic stability criterion by (Aeyels and Peuteman, 1998) for *time–variant* dynamical systems. Therein, conditions for asymptotic stability of both differential and difference time–varying equations were obtained in a similar fashion.

## 3. SUFFICIENT ISS THEOREMS BASED ON FTLF

### 3.1 Motivating example

It was indicated in (Lazar et al., 2013b) that the existence of a KIC Lyapunov function is sufficient for inherent global ISS. It is of interest to verify whether inherent ISS is also guaranteed for a KIC FTLF.

An example of a system which admits a KIC FTLF and is not ISS is inspired by Example 2 in (Lazar et al., 2009).

*Example 1.* Let the system

$$x_{k+1} = G(x_k) = A_i x_k + f_i, \quad \text{if} \quad x_k \in \Omega_i \tag{10}$$

with $i \in \{1, 2\}$, $A_1 = A_2 = 0$, $f_1 = 0$, $f_2 = 1$ and a partition given by $\Omega_1 = \{x \in \mathbb{R} | x \leq 1\}$, $\Omega_2 = \{x \in \mathbb{R} | x > 1\}$.

System (10) admits a FTLF $V(x) = \|x\|$, for all $x \in \mathbb{R}$, with $M = 2$. Because the norm is KIC, it follows that system (10) admits a KIC FTLF. However, as shown in (Lazar et al., 2009), system (10) is not ISS, not even for arbitrarily small inputs. □

Example 1 shows a system that is GES, but it has zero–robustness and it illustrates the fact that the existence of a KIC FTLF does not guarantee ISS. This is in contrast with the results in (Lazar et al., 2013b) on KIC Lyapunov functions, which are proven to grant inherent ISS. This observation motivates finding the conditions under which a KIC FTLF implies inherent ISS.

### 3.2 ISS from an ISS FTLF

The following assumptions are of use for the ISS analysis.

*Assumption 1.* The map $G$ of system (2) is $\mathcal{K}_\infty$–bounded, i.e., (7) holds for all $x \in \mathbb{R}^n$.

*Assumption 2.* The map $G$ of system (2) is KIC, i.e., (6) holds for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Observe that Assumption 2 implies Assumption 1, while the converse does not necessarily hold.

*Assumption 3.* The map $\Phi$ underlying the perturbed system (1) is KIC uniformly in $x$, i.e., for all $x \in \mathbb{R}^n$ (8) holds for all $(v, w) \in \mathbb{R}^{d_v} \times \mathbb{R}^{d_v}$.

*Lemma 12.* Suppose Assumption 1 and Assumption 3 hold. Then, there exist functions $\omega, \eta \in \mathcal{K}_\infty$ such that, for all $j \in \mathbb{Z}_{[1,M-1]}$ and all $v_{[j-1]} \in \left(\mathbb{R}^{d_v}\right)^j$,

$$\|x_j\| \leq \omega(\|x_0\|) + \eta(\|v_{[j-1]}\|), \quad \forall x_0 \in \mathbb{R}^n. \tag{11}$$

**Proof.** Let us first prove by induction that $\forall j \in \mathbb{Z}_{[1,M-1]}$, $\exists \omega_j, \eta_j \in \mathcal{K}_\infty$ such that

$$\|x_j\| \leq \omega_j(\|x_0\|) + \eta_j(\|v_{[j-1]}\|). \tag{12}$$

By Assumption 1 we get:

$$\|G(x_0)\| \leq \alpha(\|x_0\|), \tag{13}$$

for all $x_0 \in \mathbb{R}^n$. Moreover, from Assumption 3, with $w = 0$, and using the triangle inequality and the inequality (13), it follows:

$$\begin{aligned} \|x_1\| = \|\Phi(x_0, v_0)\| &\leq \|\Phi(x_0, 0)\| + \sigma_d(\|v_0\|) \\ &\leq \alpha(\|x_0\|) + \sigma_d(\|v_0\|), \end{aligned} \tag{14}$$

for all $(x_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^{d_v}$, which means that (12) holds for $j = 1$, with $\omega_1 := \alpha$ and $\eta_1 := \sigma_d$. Suppose that (12) holds for some $j \in \mathbb{Z}_{\geq 1}$. Following the same reasoning as in (14), it follows that

$$\|x_{j+1}\| = \|\Phi(x_j, v_j)\| \leq \alpha(\|x_j\|) + \sigma_d(\|v_j\|). \tag{15}$$

Using (12) and Fact 1-(vi) further yields:

$$\begin{aligned} \|x_{j+1}\| &\leq \alpha(\omega_j(\|x_0\|) + \eta_j(\|v_{[j-1]}\|)) + \sigma_d(\|v_j\|) \\ &\leq \alpha \circ 2id \circ \omega_j(\|x_0\|) \\ &\quad + (\alpha \circ 2id \circ \eta_j + \sigma_d)(\|v_{[j]}\|). \end{aligned} \tag{16}$$

Next, letting $\omega_{j+1} := \alpha \circ 2id \circ \omega_j \in \mathcal{K}_\infty$ and $\eta_{j+1} := \alpha \circ 2id \circ \eta_j + \sigma_d \in \mathcal{K}_\infty$, the inequality in (16) recovers inequality (12) for $j + 1$. Then, letting

$$\omega := \max_{j \in \mathbb{Z}_{[1,M-1]}} \omega_j \in \mathcal{K}_\infty, \tag{17}$$

and

$$\eta := \max_{j \in \mathbb{Z}_{[1,M-1]}} \eta_j \in \mathcal{K}_\infty, \tag{18}$$

where Fact 1-(ii) was used, yields that (7) holds. □

Let us define the main result which illustrates conditions under which a system is globally ISS.

*Theorem 13.* Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Suppose Assumption 1 and Assumption 3 hold and there exists a real valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{R}^n, \tag{19a}$$

and that there exists an $M \in \mathbb{Z}_{\geq 1}$, a corresponding $\rho \in \mathbb{R}_{[0,1)}$ and a $\sigma \in \mathcal{K}$ such that

$$V(x_M(\xi, v_{[M-1]})) \leq \rho V(\xi) + \sigma(\|v_{[M-1]}\|) \tag{19b}$$

holds $\forall \xi \in \mathbb{R}^n, \forall v_{[M-1]} \in \left(\mathbb{R}^{d_v}\right)^M$. Then, system (1) is globally ISS.

**Proof.** Let $k = MN + j$, where $N \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_{[M-1]}$. From (19b) it follows:

$$\begin{aligned} V(x_k(\xi, v_{[k-1]})) &= V(x_M(x_{k-M}, v_{[k-M,k-1]})) \\ &\leq \rho V(x_{k-M}) + \sigma(\|v_{[k-M,k-1]}\|), \end{aligned} \tag{20}$$

for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_{\geq 1}$. Applying recursively the inequality from (20) it follows that:

$$V(x_k) \leq \rho^N V(x_j) + \sum_{i=0}^{N-1} \rho^i \sigma(\|v_{[k-(1+i)M,k-iM-1]}\|), \quad (21)$$

for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_{\geq 1}$.

Moreover, the inequality $\|v_{[k-(1+i)M,k-iM-1]}\| \leq \|v_{[k-1]}\|$ holds for any $i \in \mathbb{Z}_{[N-1]}$. Replacing this in (21), using (19a) and the facts that $N = \frac{k-j}{M}$ and $\sigma \in \mathcal{K}$ further yields:

$$\begin{aligned}\alpha_1(\|x_k\|) \leq V(x_k) &\leq \rho^{\frac{k-j}{M}} V(x_j) + \sum_{i=0}^{N-1} \rho^i \sigma(\|v_{[j,k-1]}\|) \\ &\leq \rho^{\frac{k-j}{M}} \alpha_2(\|x_j\|) + \sigma(\|v_{[k-1]}\|)\frac{1}{1-\rho}, \quad (22)\end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 1}$. Taking into account that $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$, the inequality in (22) can be rewritten as:

$$\begin{aligned}\|x_k\| &\leq \alpha_1^{-1}\left(\rho^{\frac{k-j}{M}} \alpha_2(\|x_j\|) + \sigma(\|v_{[k-1]}\|)\frac{1}{1-\rho}\right) \\ &\leq \alpha_1^{-1}\left(2\rho^{\frac{k-j}{M}} \alpha_2(\|x_j\|)\right) + \\ &\quad + \alpha_1^{-1}\left(2\sigma(\|v_{[k-1]}\|)\frac{1}{1-\rho}\right), \quad (23)\end{aligned}$$

for all $k \in \mathbb{Z}_{\geq 1}$. Following Lemma 12 and considering that $\alpha_1^{-1}, \alpha_2 \in \mathcal{K}_\infty$ and $\xi := x(0)$, inequality (23) becomes:

$$\begin{aligned}\|x_k\| \leq &\alpha_1^{-1}\left(2\rho^{\frac{k-j}{M}} \alpha_2(\omega(\|\xi\|) + \eta(\|v_{[j-1]}\|))\right) \\ &+ \alpha_1^{-1}\left(2\sigma(\|v_{[k-1]}\|)\frac{1}{1-\rho}\right), \quad (24)\end{aligned}$$

for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_{\geq 1}$. Denote

$$\gamma_1 := \alpha_1^{-1} \circ 2id \circ \sigma \circ \frac{1}{1-\rho} id, \quad (25)$$

which is a $\mathcal{K}$–class function, because of Fact 1–(iii)-(iv). Observe that

$$\rho^{\frac{k-j}{M}} \leq \rho^{\frac{k-M+1}{M}}. \quad (26)$$

Moreover, by definition, it holds that $\|v_{[j-1]}\| \leq \|v_{[M-1]}\|$, for all $j \in \mathbb{Z}_{[1,M]}$. With these considerations, together with the notation in (25) and with observation (26), the inequality in (24) becomes:

$$\begin{aligned}\|x_k\| \leq &\alpha_1^{-1}\left(2\rho^{\frac{k-M+1}{M}} \alpha_2(\omega(\|\xi\|) + \eta(\|v_{[M-1]}\|))\right) \\ &+ \gamma_1(\|v_{[k-1]}\|), \quad (27)\end{aligned}$$

for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_{\geq 1}$. Denote $c := 2\rho^{\frac{-M+1}{M}} > 0$, $\bar{\rho} := \rho^{\frac{1}{M}} \in \mathbb{R}_{[0,1)}$ and $\bar{\beta}(s,k) := \alpha_1^{-1} \circ c\bar{\rho}^k id \circ \alpha_2(s), \forall k \in \mathbb{Z}_{\geq 1}$. From Fact 1–(iii)-(iv) it follows that $\bar{\beta} \in \mathcal{KL}$. This means that:

$$\begin{aligned}&\bar{\beta}(\omega(\|\xi\|) + \eta(\|v_{[M-1]}\|), k) \leq \\ &\quad \leq \bar{\beta}(2\omega(\|\xi\|), k) + \bar{\beta}(2\eta(\|v_{[M-1]}\|), k) \\ &\quad \leq \bar{\beta}(2\omega(\|\xi\|), k) + \bar{\beta}(2\eta(\|v_{[M-1]}\|), 0), \quad (28)\end{aligned}$$

for all $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}_{\geq 1}$. Define:

$$\beta(s,k) := \bar{\beta}(2id \circ \omega(s), k), \quad (29)$$

for all $(s,k) \in \mathbb{R}_+ \times \mathbb{Z}_{\geq 1}$, which is a class $\mathcal{KL}$ function, by Fact 1-(v). Moreover, because

$$\begin{aligned}\bar{\beta}(2\eta(\|v_{[M-1]}\|), 0) &= (\alpha_1^{-1} \circ cid \circ \alpha_2 \circ 2id \circ \eta)(\|v_{[M-1]}\|) \\ &\leq (\alpha_1^{-1} \circ cid \circ \alpha_2 \circ 2id \circ \eta)(\|v_{[k-1]}\|) \\ &=: \gamma_2(\|v_{[k-1]}\|) \quad (30)\end{aligned}$$

is a $\mathcal{K}$ function, then

$$\gamma(\|v_{[k-1]}\|) := (\gamma_1 + \gamma_2)(\|v_{[k-1]}\|) \quad (31)$$

is a $\mathcal{K}$ function. Replacing the inequality (28) in (27) and given the notation in (29) and (31), with $\gamma_1$ and $\gamma_2$ defined as in (25) and (30), respectively, it follows that (3) holds. Therefore, system (1) is globally ISS. □

*Definition 14.* The real valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ which satisfies the conditions of Theorem 13 is called an *ISS FTLF*.

Observe that indeed, the proof of Proposition 10 follows directly from the proof of Theorem 13, in the particular case when $v_k = 0, \forall k \in \mathbb{Z}_+$.

Theorem 13 indicates that one way of proving ISS is by finding an ISS FTLF. The next result illustrates an alternative way of proving ISS directly from an existing KIC FTLF.

### 3.3 ISS FTLF from a FTLF

*Theorem 15.* Suppose Assumption 2 and Assumption 3 hold and system (2) admits a FTLF, $V$, which is KIC, i.e., it satisfies inequality (4). Then $V$ is a ISS FTLF for system (1).

**Proof.** Let us first prove that

$$V\left(x_k(\xi, v_{[k-1]})\right) \leq V\left(\Phi^k(\xi, 0)\right) + \sigma_k(\|v_{[k-1]}\|) \quad (32)$$

holds for all $k \in \mathbb{Z}_{\geq 1}$, all $\xi \in \mathbb{R}^n$ and all $v_{[k-1]} \in \left(\mathbb{R}^{d_v}\right)^k$, with $\sigma_k \in \mathcal{K}$.

To proceed with this proof we use mathematical induction.

For $k = 1$, inequality (32) follows directly by using, in order, $w = 0$, the triangle inequality, the KIC property of $V$ and Assumption 3:

$$\begin{aligned}V\left(\Phi(\xi, v_0)\right) - V\left(\Phi(\xi, 0)\right) &\leq |V\left(\Phi(\xi, v_0)\right) - V\left(\Phi(\xi, 0)\right)| \\ &\leq \sigma_V(\sigma_d(\|v_0\|)). \quad (33)\end{aligned}$$

Denote $\sigma_1 = \sigma_v \circ \sigma_d \in \mathcal{K}$ in (33) and the inequality in (32) is recovered for $k = 1$.

Suppose inequality (32) holds for $k-1$:

$$V\left(x_{k-1}\right) \leq V\left(\Phi^{k-1}(\xi, 0)\right) + \sigma_{k-1}(\|v_{[k-2]}\|). \quad (34)$$

Then, for $k$ we can write:

$$V\left(x_k\right) = V\left(\Phi^{k-1}(\Phi(\xi, v_0), v_{[1,k-1]})\right). \quad (35)$$

Using (34) in (35) followed by addition and subtraction of a $V\left(G^k(\xi)\right)$ term together with the triangle inequality we obtain:

$$\begin{aligned}V\left(x_k\right) \leq &|V\left(G^{k-1}(\Phi(\xi, v_0))\right) - V\left(G^{k-1}(\Phi(\xi, 0))\right)| \\ &+ V\left(G^k(\xi)\right) + \sigma_{k-1}(\|v_{[1,k-1]}\|). \quad (36)\end{aligned}$$

By considering the KIC property of $V$ and repeatedly using Assumption 2 in (36) and then Assumption 3, together with the notation $\sigma_k := \sigma_V \circ \sigma_x^{k-1} \circ \sigma_d + \sigma_{k-1}$, we obtain

$$\begin{aligned}V\left(x_k\right) \leq &\sigma_V(\|G^{k-1}(\Phi(\xi, v_0)) - G^{k-1}(\Phi(\xi, 0))\|) \\ &+ V\left(G^k(\xi)\right) + \sigma_{k-1}(\|v_{[k-1]}\|) \\ \leq &V\left(G^k(\xi)\right) + \sigma_k(\|v_{[k-1]}\|), \quad (37)\end{aligned}$$

which means that (32) holds for all $k \in \mathbb{Z}_{\geq 1}$. If $V$ is a FTLF, then there exists an $M \in \mathbb{Z}_{\geq 1}$ and corresponding $\rho \in \mathbb{R}_{[0,1)}$ such that

$$V(G^M(\xi)) \leq \rho V(\xi), \quad \forall \xi \in \mathbb{R}^n. \quad (38)$$

Moreover, let

$$\sigma := \sigma_M = \sum_{j=1}^{M} \sigma_V \circ \sigma_x^{j-1} \circ \sigma_d. \quad (39)$$

Then, by inequality (32) it holds that:

$$V\left(x_M(\xi, v_{[M-1]})\right) \le V\left(\Phi^M(\xi, 0)\right) + \sigma(\|v_{[M-1]}\|). \quad (40)$$

Therefore, from (38) and (40) we obtain that:

$$V(x_M(\xi, v_{[M-1]})) \le \rho V(\xi) + \sigma(\|v_{[M-1]}\|), \quad (41)$$

for all $v_i \in \mathbb{R}^{d_v}, i \in \mathbb{Z}_{[0,M-1]}$. Hence, $V$ is an ISS FTLF for system (1), which completes the proof. □

*Corollary 16.* Suppose Assumption 2 and Assumption 3 hold and system (2) admits a FTLF, $V$, which is KIC, i.e., it satisfies inequality (4). Then system (1) is globally ISS.

The proof of Corollary 16 follows directly from Theorem 15 and Theorem 13.

### 3.4 Remarks on ISS by FTLF

With respect to the results derived so far the following remarks are of interest.

*Remark 1.* Coming back to Example 1, observe that system (10) is not KIC. The conclusion that system (10) is not KIC can be deducted by observing that the system is not continuous, which is a necessary condition for a system to be KIC, see (Lazar et al., 2013b). Therefore, even though the system admits a KIC FTLF, as the system dynamics $G$ is not KIC, Corrolary 16 does not apply. □

The conditions derived above are sufficient, but not necessary for ISS. The following example shows a discontinuous piecewise linear system which is ISS.

*Example 2.* Define the system

$$x_{k+1} = G(x_k) = A_i x_k, \quad \text{if } x_k \in \Omega_i \quad (42)$$

with $i \in \{1, 2\}$, $A_1 = 0.3$, $A_2 = 0.5$ and a partition given by $\Omega_1 = \{x \in \mathbb{R} | x > 1\}$, $\Omega_2 = \{x \in \mathbb{R} | x \le 1\}$, see Fig. 1 for a graphical illustration.

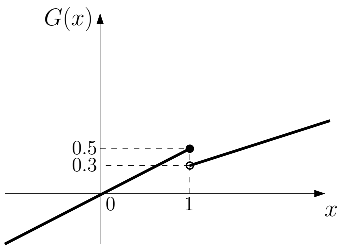


Fig. 1. Globally ISS discontinuous system.

Let us consider a Lyapunov function of the form $V(x) = \|x\|$. Then $\|G(x)\| \le 0.5\|x\|$ holds, which makes $V(x)$ a KIC Lyapunov function for system (42), and therefore, according to Theorem IV.8 of (Lazar et al., 2013b), system (42) with additive disturbance is globally ISS. However, as $G$ is not KIC, Corollary 16 can not be of use. □

Global results are difficult to verify in general, for nonlinear systems. That is why in what follows we will focus on ISS results for compact subsets of $\mathbb{R}^n$.

## 4. INHERENT ISS ON COMPACT SETS

The setting of compact sets in ISS analysis allows the relaxation of all KIC assumptions to $\mathcal{K}$–continuity, which is equivalent to mere continuity on compact sets, see (Lazar et al., 2013b).

Furthermore, it will be shown that compared to the standard notion of a robustly positive invariant set, as used in the standard Lyapunov approach in (Lazar et al., 2013b), a relaxed notion of an $k$–periodically robustly invariant set suffices in the FTLF approach, where $k \in \mathbb{Z}_{\ge 1}$. For $k = 1$, the standard notion of a robustly invariant set is recovered.

To this end, let us first recall the definition of ISS with respect to $\mathbb{X}$ and $\mathbb{D}$, where $\mathbb{X}$ and $\mathbb{D}$ are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^{d_v}$, respectively, with the origin in their interior.

*Definition 17.* The perturbed system (1) is ISS on $\mathbb{X}$ with respect to disturbances in $\mathbb{D}$ if there exists a $\mathcal{KL}$ function $\beta$ and a $\mathcal{K}$ function $\gamma$ such that the corresponding state trajectory satisfies

$$\|x_k(\xi, v_{[k-1]})\| \le \beta(\|\xi\|, k) + \gamma(\|v_{[k-1]}\|), \quad (43)$$

for all $(\xi, k) \in \mathbb{X} \times \mathbb{Z}_+$ and all $\{v_j\}_{j \in \mathbb{Z}_+}$ with $v_j \in \mathbb{D}$ for all $j \in \mathbb{Z}_+$.

*Definition 18.* Let $k \in \mathbb{Z}_{\ge 1}$. The set $\mathbb{X}$ is called $k$–periodically invariant with respect to the map $G$ of system (2), if for all $x \in \mathbb{X}$, then $G^k(x) \in \mathbb{X}$.

*Definition 19.* Let $k \in \mathbb{Z}_{\ge 1}$. The set $\mathbb{X}$ is called $k$–periodically robustly invariant with respect to the map $\Phi$ of system (1) and the set $\mathbb{D}$, if for all $x \in \mathbb{X}$, then $\Phi^k(x, v_{[k-1]}) \in \mathbb{X}$, $\forall v_{[k-1]} \in \mathbb{D}^k$.

In this section, the following notation is of use. $\mathbb{E}$ denotes a compact set containing all the trajectories starting from $\mathbb{X}$, i.e.,

$$\bigcup_{i \in \mathbb{Z}_+} \overline{G}^i(\mathbb{X}) \subseteq \mathbb{E},$$

and $\mathbb{V}$ denotes a compact set containing all the trajectories starting from $\mathbb{X}$ with disturbances taking values in the compact set $\mathbb{D}$,

$$\bigcup_{i \in \mathbb{Z}_+} \overline{\Phi}^i(\mathbb{X}, \mathbb{D}^i) \subseteq \mathbb{V}.$$

Let us provide a version of Proposition 10 in terms of compact sets, as in (Lazar et al., 2013a).

*Proposition 20.* Let $\mathbb{X}$ be a compact set. Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Suppose that the function $G : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to the dynamics (2) is $\mathcal{K}$–bounded for all $x \in \mathbb{E}$, $\mathbb{X}$ is $M$–periodically invariant with respect to the map $G$ and there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\alpha_1(\|\xi\|) \le V(\xi) \le \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{E}, \quad (44a)$$

and that there exists an $M \in \mathbb{Z}_{\ge 1}$ and corresponding $\rho \in \mathbb{R}_{[0,1)}$ such that

$$V(x_M(\xi)) \le \rho V(\xi), \quad \forall \xi \in \mathbb{X}. \quad (44b)$$

Then, system (2) is $\mathcal{KL}$-stable in $\mathbb{X}$. □

The instrumental assumptions for global ISS and Theorem 13 are reformulated in the context of compact sets as follows.

*Assumption 4.* The map $G$ of system (2) is continuous on $\mathbb{E}$.

*Assumption 5.* The map $\Phi$ underlying the perturbed system (1) is continuous uniformly in $x$ for all $(x, v) \in \mathbb{V} \times \mathbb{D}$.

*Theorem 21.* Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, let $\sigma \in \mathcal{K}$, $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{D} \subset \mathbb{R}^{d_v}$ be compact sets with the origin in their interior and let $\mathbb{X}$ be $M$–periodically robustly invariant for system (2) with respect to $\mathbb{D}$. Suppose Assumption 4 and Assumption 5 hold and there exists a real valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ with $V(0) = 0$ such that

$$\alpha_1(\|\xi\|) \le V(\xi) \le \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{V}, \quad (45a)$$

and that there exists an $M \in \mathbb{Z}_{\geq 1}$, a corresponding $\rho \in \mathbb{R}_{[0,1)}$ such that

$$V(x_M(\xi, v_{[M-1]})) \leq \rho V(\xi) + \sigma(\|v_{[M-1]}\|) \tag{45b}$$

holds $\forall \xi \in \mathbb{X}, \forall v_{[M-1]} \in \mathbb{D}^M$. Then, system (1) is ISS with respect to $\mathbb{X}$ and $\mathbb{D}$.

**Proof.** Let $k = MN + j$, where $N \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_{[0,M-1]}$. From Lemma 12 it follows:

$$\|x_{MN+j}\| \leq \omega(\|x_{MN}\|) + \eta(\|v_{[k-1]}\|). \tag{46}$$

Similarly to (23), and using inequality (26), then

$$\|x_{MN}\| \leq \alpha_1^{-1} \circ 2\rho^{\frac{k-M+1}{M}} id \circ \alpha_2(\|x_0\|) + \\ + \alpha_1^{-1} \circ 2id \circ \sigma \circ \frac{1}{1-\rho} id(\|v_{[k-1]}\|), \tag{47}$$

for all $k \in \mathbb{Z}_{[MN,MN+M-1]}$. Introduce now $\|x_{MN}\|$ from (47) in (46), and following a similar reasoning as in (27)–(31) from the proof of Theorem 13, it follows that system (1) is ISS with respect to $\mathbb{X}$ and $\mathbb{D}$. □

A function $V$ that satisfies Theorem 21 will be referred to as an ISS FTLF for system (1) with respect to $\mathbb{X}$ and $\mathbb{D}$.

The proof of Proposition 20 can be recovered as a particular case of the proof of Theorem 21, which corresponds to zero disturbance.

Theorem 15 is reformulated in the context of compact sets as follows.

*Theorem 22.* Let $\mathbb{X}, \mathbb{D}$ be compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^{d_v}$ respectively, with the origin in their interior. Suppose that $\mathbb{X}$ is $M$–periodically robustly invariant for system (2) with respect to $\mathbb{D}$. Moreover, suppose Assumption 4 and Assumption 5 hold and system (2) with $x \in \mathbb{X}$ admits a FTLF, $V$, which is continuous on $\mathbb{V}$. Then $V$ is an ISS FTLF for system (1) with respect to $\mathbb{X}$ and $\mathbb{D}$, and hence, system (1) is ISS with respect to $\mathbb{X}$ and $\mathbb{D}$. □

The proof for Theorem 22 is similar with the proof of Theorem 15, and it is omitted for brevity.

## 5. CONCLUSIONS

This paper approached the problem of inherent ISS analysis for nonlinear discrete–time systems using the concept of a finite–time Lyapunov function. It was proven that, for sufficiently continuous dynamics and finite–time Lyapunov functions, inherent global input–to–state stability to general disturbances can be established for nominally stable nonlinear systems.

Moreover, inherent input–to–state stability on a compact set was obtained under simple continuity of the finite–time Lyapunov function and of the dynamics.

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