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Feedforward and Feedback Control of Dynamic Systems

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Abstract: Recently the control methods based on disturbance rejection have largely renewed our understanding of automatic control, especially the role of feedforward control. With the renewed understanding, this work presents a feedforward and feedback approach for control of general dynamic systems under a state tracking framework. This is realized by introducing a fictitious system model and employing a proper state and disturbance estimator. The resultant control scheme embodies a close cooperation between feedforward and feedback controls: Feedforward control rejects the general disturbance and embeds a reference state trajectory, whilst feedback control cancels the fictitious dynamics and enforces desired tracking error dynamics. A notable advantage of the approach is that control designs of linear and nonlinear systems can follow a common routine and the closed-loop stability is guaranteed under standard conditions.

Keywords: Linear systems, nonlinear systems, feedforward control, feedback control, disturbance estimation, model-free control, active disturbance rejection control, uncertainty and disturbance estimator, pendulum

1. INTRODUCTION

As two fundamental philosophies, feedback and feedforward, sit in the center of control theory (Åström and Hägglund, 2005). Generally speaking, feedback enforces compensation based on the effects, while feedforward enforces compensation based on the causes. The state-of-the-art control theory has been dominated by feedback control (Åström and Murray, 2010) although feedforward control is useful and highly demanded in practice (Skogestad, 2009; Gao, 2013). The inferior status of feedforward control can owe to two bottlenecks of recognition: a) the limited understanding of what can be used as feedforward signals (Gao, 2013) and b) the limited knowledge of how the feedforward signals can be obtained (Skogestad, 2009). These bottlenecks prevent us fully recognizing the power of feedforward control. But, fortunately they have begun to crack for a renewed understanding of automatic control (Gao, 2013). The new understanding contributes to identifying useful feedforward signals but also the ways of extracting them, which shall together free the hidden power of feedforward. The renewals have been led by the developments of active disturbance rejection control (ADRC) (Han, 1998, 1999; Gao, 2006), uncertainty and disturbance estimator (UDE) based control (Youcef-Toumi and Ito, 1988; Zhong and Rees, 2004; Zhong et al., 2011), and model-free control (MFC) (Fliess and Join, 2009, 2013), which emerge almost independently within their own research domains.

The ADRC was proposed by Han (1995; 1998; 1999). It introduces two nontrivial innovations in philosophy: i) Almost all dynamic systems can be transformed into a canonical form as represented by a cascade of integrators via input-dependent state transformations (Han, 1981); ii) By extracting a nominal model from the canonical form, any unmodeled dynamics including internal and external uncertainties can be lumped as a *total disturbance* and then estimated and compensated online. The innovation i) was later rediscovered (Youcef-Toumi and Ito, 1988; Fliess, 1990) and more completely elaborated by Fliess (1990). As to innovation ii), Han treats the total disturbance as an additional state which is then estimated together with the states by an extended state observer (ESO) (Han, 1995). The estimate is used as a feedforward control to compensate the general disturbance before its affects the system. This largely renews our understanding and digs out the potential of feedforward. A direct consequence is that feedback control can be designed based on a simple (and even standard) plant model and the rest uncertainties are handled by feedforward control. To date, ADRC has been testified by a range of applications and its philosophy as a medium to unifying various disturbance rejection based control methods has become clear (Gao, 2013). However, the theoretical basis of ADRC is still in the infancy. Related analyses have been mainly on the capacity of the disturbance observer (Yang and Huang, 2009; Zheng et al., 2012; Huang and Xue, 2012) and the closed-loop stability for single-input single-output (SISO) systems in Han's canonical form (Zheng et al., 2007). Some stability results (limited to a class of ESOs) are also available for multi-input multi-output (MIMO) systems (Huang and Xue, 2012), but it is unclear how they can be extended to general dynamic systems which are not in or unable to be transformed into a canonical form.

A related but different idea for rejecting the total disturbance was proposed by Youcef-Toumi and Ito (1988). The authors considered a class of nonlinear systems with unknown nonlinear dynamics and external disturbances and proposed a robust control scheme, called time delay control (TDC), to make it track reference dynamics. By assuming that a continuous signal changes little during a small enough period, TDC uses past observation of the total disturbance as feedforward control to approximately cancel the current one. The closed-loop performance is then governed by state feedback and model reference feedforward controls. Later, Zhong and Rees generalize TDC by replacing the time-delay filter with a general low pass filter, resulting in the so-called UDE-based control (Zhong and Rees, 2004). The generalization avoids several drawbacks inherent in TDC, and allows decoupled designs of the disturbance filter and the reference dynamics (Zhong et al., 2011). The control method has been used to handle both linear and nonlinear systems with state delays, where the uncertainties and disturbances are of general sense as that in ADRC (Kuperman and Zhong, 2011; Stobart et al., 2011). To date, the stability of UDE-based control has been proved mainly for linear time-invariant (LTI) SISO systems. While, its assumption of true states being available also limits its applications.

Another closely related method, MFC, was introduced recently by Fliess and Join (2009; 2013). The method approximates a continuous-time system by a local model within a very short time period. The model is a differential equation with respect to the system's input and output. The output differential consists of two parts, one related to the control input and the other lumps all rest dynamics (i.e., a total disturbance). The total disturbance is estimated and canceled online (which actually enforces feedforward control), by using an algebraic identification technique developed in (Fliess and Sira-Ramírez, 2003, 2008) or its improved version (Hu and Mao, 2014). With the total disturbance being (approximately) canceled, the local model reduces to a cascade of integrators for which the feedback control design becomes straightforward. By specifying a first- or second-order local model, MFC enables PID feedback control to work with feedforward compensation for output tracking, which results in the so-called intelligent PID control (Fliess and Join, 2009, 2013). To date, MFC has been studied mainly for SISO systems though its extension to MIMO systems seems possible (Fliess and Join, 2013). Furthermore, a strict stability analysis of MFC is still missing.

By reviewing the above control methods at a high level, it is not difficult to see that they are essentially different manifestations of the same philosophies of feedforward and feedback controls. The methods differ mainly in how a total disturbance is defined and how it is estimated and compensated. Motivated by the limited theoretical results available for each of these methods, this work is devoted to presenting a feedforward and feedback control framework for general MIMO dynamic systems which are not necessarily transformable into Han's canonical form. The design relies on the concept that disturbance in a control design means the difference between the system model in use and the real system it should be (Gao, 2013), and on the principle that the disturbance is rejected by feedforward control and its effect is attenuated by feedback control. Specific contributions of this work are listed as follows:

- It provides a feedforward and feedback control framework for general MIMO dynamic systems;
- It identifies and discusses two types of total disturbance estimators under a same roof;
- It analyzes the closed-loop stability of the proposed control scheme.

The problems treated by ADRC, UDE and MFC so far can most, if not all, be viewed as special cases of the problem considered in this work. The new theoretical results, however, do not prove the internal stability of the closed-loop system if a low-order (as contrast to full-order) model is used, which is also the case of the state-of-the-art MFC (Fliess and Join, 2013).

2. PROBLEM FORMULATION

Consider a dynamic system described by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}_0(t,\, \mathbf{x},\, \mathbf{u},\, \mathbf{w}_0),\\ \mathbf{y}(t) &= \mathbf{g}_0(t,\, \mathbf{x},\, \mathbf{u},\, \mathbf{v}_0),\end{aligned} \tag{1}$$

for $t \geq t_0$ (an initial time), where $t \in \mathbb{R}$ is the time, and $\mathbf{x},\, \mathbf{w}_0 \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ $(m \leq n)$ and $\mathbf{y},\, \mathbf{v}_0 \in \mathbb{R}^l$ $(l \leq n)$, are vectors of the state, disturbance, control input, measurement (i.e., measured output) and measurement noise, respectively. (The arguments of a variable or function are ignored whenever no ambiguity arises.) As the true state and measurement functions, $\mathbf{f}_0$ and $\mathbf{g}_0$, are unknown in practice, we consider a model of the system instead:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(t,\, \mathbf{x},\, \mathbf{u}) + \mathbf{w}(t,\, \mathbf{x},\, \mathbf{u},\, \mathbf{w}_0),\\ \mathbf{y}(t) &= \mathbf{g}(t,\, \mathbf{x},\, \mathbf{u}) + \mathbf{v}(t,\, \mathbf{x},\, \mathbf{u},\, \mathbf{v}_0),\end{aligned} \tag{2}$$

where $\mathbf{w}$ and $\mathbf{v}$ lump all unmodeled dynamics in the state and measurement functions, respectively. Based on this model is then defined the control problem.

The system needs to track a reference state trajectory, as generated by

$$\dot{\mathbf{x}}_r(t) = \mathbf{f}_r(t,\, \mathbf{x}_r,\, \mathbf{u}_r), \tag{3}$$

for $t \geq t_0$, where $\mathbf{x}_r \in \mathbb{R}^n$ is the reference state trajectory and $\mathbf{u}_r \in \mathbb{R}^m$ is the input used to excite the reference system. To meet design specifications, the desired tracking error dynamics is imposed as

$$\dot{\mathbf{e}} = \mathbf{h}(t,\, \mathbf{e}), \tag{4}$$

where $\mathbf{e} := \mathbf{x}_r - \mathbf{x}$ which defines the error. With (2) and (3), it follows that $\mathbf{f}_r(t,\, \mathbf{x}_r,\, \mathbf{u}_r) - \mathbf{f}(t,\, \mathbf{x},\, \mathbf{u}) - \mathbf{w} = \mathbf{h}(t,\, \mathbf{e})$, from which an ideal control $\mathbf{u}$ is solved. Since the true state and disturbance are unavailable in practice, they have to be estimated from the measurement $\mathbf{y}$. Let the estimates of $\mathbf{x}$ and $\mathbf{w}$ be $\hat{\mathbf{x}}$ and $\hat{\mathbf{w}}$, respectively, and let $\hat{\mathbf{e}} := \mathbf{x}_r - \hat{\mathbf{x}}$. Then the equation becomes

$$\mathbf{f}_r(t,\, \mathbf{x}_r,\, \mathbf{u}_r) - \hat{\mathbf{w}} - \mathbf{f}(t,\, \hat{\mathbf{x}},\, \mathbf{u}) - \mathbf{h}(t,\, \hat{\mathbf{e}}) = \delta_{\mathbf{f}} + \delta_{\mathbf{w}} + \delta_{\mathbf{h}}, \tag{5}$$

where $\delta_{\mathbf{f}} := \mathbf{f}(t,\, \mathbf{x},\, \mathbf{u}) - \mathbf{f}(t,\, \hat{\mathbf{x}},\, \mathbf{u})$, $\delta_{\mathbf{w}} := \mathbf{w} - \hat{\mathbf{w}}$ and $\delta_{\mathbf{h}} := \mathbf{h}(t,\, \mathbf{e}) - \mathbf{h}(t,\, \hat{\mathbf{e}})$, which are errors caused by inexact estimation. As a consequence, the target control vector $\mathbf{u}$ has to be estimated from (5) subject to (2).

The estimation is affected by the system model in use. Depending on the information available, two cases can be considered: i) *Neither a state nor a measurement model is available*; ii) *A state or measurement model, accurate or not, is available.* Case i) occurs when modeling of the system dynamics is difficult or costly or even unnecessary, which can be viewed as an extreme scenario of case ii). Case ii) occurs when a good approximation of the system dynamics is possible. The next section presents a feedforward and feedback control approach to tackling the more challenging case i); while, case ii) can be handled alike.

3. THE CONTROLLER DESIGN

3.1 Feedforward and feedback control form

Consider the most challenging case where no system model is available at all, i.e., the vector functions $\mathbf{f}$ and $\mathbf{g}$ are null and $\mathbf{w}$ and $\mathbf{v}$ contain all unmodeled dynamics. In this case, it is normally impossible to estimate the control $\mathbf{u}$ from (2) and (5) for lack of an explicit relation with the available information. In order to establish an explicit relation, let us introduce a fictitious system model. That is, $\mathbf{f}$ and $\mathbf{g}$ are both artificial vector functions of t, $\mathbf{u}$ and $\mathbf{x}$, instead of being constantly zero, and consequently $\mathbf{w}$ and $\mathbf{v}$ are different total disturbance and noise, respectively. The control $\mathbf{u}$ is then estimated from (5) subject to (2).

While it is feasible to have a numerical estimate, an analytical estimate of $\mathbf{u}$ is preferred. This is possible if the fictitious state and measurement functions are specified in linear forms as follows:

$$\begin{aligned} \mathbf{f}(t,\,\mathbf{x},\,\mathbf{u}) &:= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{g}(t,\,\mathbf{x},\,\mathbf{u}) &:= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \end{aligned} \tag{6}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are constant matrices of compatible dimensions, satisfying that $(\mathbf{A},\,\mathbf{B})$ is controllable and $(\mathbf{A},\,\mathbf{C})$ is observable. Then the key equation (5) becomes

$$\underbrace{\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r) - \hat{\mathbf{w}}}_{\text{feedforward signals}} \underbrace{-\mathbf{A}\hat{\mathbf{x}} - \mathbf{h}(t,\,\hat{\mathbf{e}})}_{\text{feedback signals}} - \mathbf{B}\mathbf{u} = \mathbf{A}\delta_{\mathbf{x}} + \delta_{\mathbf{w}} + \delta_{\mathbf{h}}. \tag{7}$$

The equation embodies feedforward signals, $\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r)$, which embeds a reference state trajectory, and $\hat{\mathbf{w}}$, which rejects the total disturbance, and feedback signals, $\mathbf{A}\hat{\mathbf{x}}$, which cancels the fictitious state dynamics, and $\mathbf{h}(t,\,\hat{\mathbf{e}})$, which enforces desired tracking error dynamics. The control estimate will be coded by these two kinds of signals.

Depending on forms of the estimates $\hat{\mathbf{x}}$ and $\hat{\mathbf{w}}$, different control estimates can be obtained from (7). If either $\hat{\mathbf{x}}$ or $\hat{\mathbf{w}}$ has an explicit relation with $\mathbf{u}$, then it will be better to substitute the relation into the above equation before estimating $\mathbf{u}$: This makes the estimate $\hat{\mathbf{x}}$ or $\hat{\mathbf{w}}$ be transparent in the implementation, i.e., it is merely used in the deduction but not computed in practice. Otherwise, it is straightforward to estimate $\mathbf{u}$ as

$$\hat{\mathbf{u}} = \mathbf{B}^{\dagger}\left(\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r) - \hat{\mathbf{w}} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{h}(t,\,\hat{\mathbf{e}})\right), \tag{8}$$

which minimizes the least-square (LS) equation error of (7) (where $\mathbf{B}^{\dagger} := (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$). The feedforward and feedback components of the control are clear as in (7). Note that this control estimate may introduce a bias even if $\hat{\mathbf{x}}$ and $\hat{\mathbf{w}}$ are equal to the true values. This is seen by replacing $\mathbf{u}$ in (7) with $\hat{\mathbf{u}}$, resulting in a bias

$$\delta_{\mathbf{u}} := (\mathbf{I} - \mathbf{B}\mathbf{B}^{\dagger})\left(\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r) - \hat{\mathbf{w}} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{h}(t,\,\hat{\mathbf{e}})\right). \tag{9}$$

Though it will be compensated by the next update of control once it is incorporated in the estimate of a renewed disturbance $\mathbf{w}$, it is preferable if the bias does not appear at all. The following lemma gives a sufficient condition under which this actually happens.

Lemma 1. (Zero control-estimate-induced bias) If there is a nonsingular matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ such that $\mathbf{P}\mathbf{B} = \begin{bmatrix} \mathbf{0}_{(n-m)\times m} \\ *_1 \end{bmatrix}$ and $\mathbf{P}\left(\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r) - \hat{\mathbf{w}} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{h}(t,\,\hat{\mathbf{e}})\right) = \begin{bmatrix} \mathbf{0}_{(n-m)\times 1} \\ *_2 \end{bmatrix}$, where $*_1$ denotes a nonsingular $m \times m$ matrix and $*_2$ an $m \times 1$ vector, then the induced bias $\delta_{\mathbf{u}}$ is equal to zero.

Proof. Let $\mathbf{Q} = [\mathbf{Q}_1\ \mathbf{Q}_2] := \mathbf{P}^{-1}$, where $\mathbf{Q}_1 \in \mathbb{R}^{n\times(n-m)}$ and $\mathbf{Q}_1 \in \mathbb{R}^{n\times m}$. Then $\mathbf{B} = \mathbf{Q}\mathbf{P}\mathbf{B} = \mathbf{Q}_2 *_1$ and similarly $\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r) - \hat{\mathbf{w}} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{h}(t,\,\hat{\mathbf{e}}) = \mathbf{Q}_2 *_2$. Hence

$$\begin{aligned} \delta_{\mathbf{u}} &= (\mathbf{I} - \mathbf{B}\mathbf{B}^{\dagger})\left(\mathbf{f}_r(t,\,\mathbf{x}_r,\,\mathbf{u}_r) - \hat{\mathbf{w}} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{h}(t,\,\hat{\mathbf{e}})\right) \\ &= \left(\mathbf{I} - \mathbf{Q}_2 *_1 (*_1^T \mathbf{Q}_2^T \mathbf{Q}_2 *_1)^{-1} *_1^T \mathbf{Q}_2^T\right)\mathbf{Q}_2 *_2 \\ &= \left(\mathbf{I} - \mathbf{Q}_2(\mathbf{Q}_2^T\mathbf{Q}_2)^{-1}\mathbf{Q}_2^T\right)\mathbf{Q}_2 *_2 = 0, \end{aligned}$$

which completes the proof.

A specific case when the above condition is satisfied is that the true system is in Han's canonical form (Han, 1981; Youcef-Toumi and Ito, 1988; Fliess, 1990) and so is the fictitious system model, where $\mathbf{P}$ is an identify matrix.

3.2 State and disturbance estimators

Given the fictitious state model in (6), the actual complete system dynamics has the form of

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\hat{\mathbf{u}}(t) + \mathbf{w}(t,\,\mathbf{x}(t),\,\hat{\mathbf{u}}(t)), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\hat{\mathbf{u}}(t) + \mathbf{v}(t,\,\mathbf{x}(t),\,\hat{\mathbf{u}}(t)), \end{aligned} \tag{10}$$

which is an LTI system subject to general disturbances and measurement noises. To compute the control $\hat{\mathbf{u}}$ from (7) or implement it per (8), the state $\mathbf{x}$ and the disturbance $\mathbf{w}$ need to be estimated (implicitly or explicitly) from the measurement $\mathbf{y}$. Depending on the invertibility of the measurement matrix $\mathbf{C}$, two kinds of estimators can be used for this purpose. If $\mathbf{C}$ is square and invertible, then it is feasible to estimate $\mathbf{x}$ by direct filtering of $\mathbf{y}$ and subsequently estimate $\mathbf{w}$ by a different filtering, which results in a Type-I estimator. Otherwise, an ESO can be used to estimate $\mathbf{x}$ and $\mathbf{w}$ simultaneously, resulting in a Type-II estimator. Type-II estimator is also applicable when $\mathbf{C}$ is invertible, though its implementation is more complicated as compared to Type-I estimator. In this case, Type-I estimator has another advantage that the disturbance is only estimated implicitly without being really computed.

Type-I estimator (if $\mathbf{C}$ *is invertible).* The state $\mathbf{x}$ is estimated by filtering of $\mathbf{y}$ as

$$\hat{\mathbf{x}}(t) = \mathbf{C}^{-1}\left(\mathbf{f}_{\mathbf{y}}(t).\star\mathbf{y}(t)\right), \tag{11}$$

where $\mathbf{f}_{\mathbf{y}}(t) \in \mathbb{R}^l$ is a vector of impulse responses of filters that suppress the measurement noises, and $.\star$ denotes the convolution operator as applied to corresponding elements of the two vectors.

With the state estimate, the disturbance $\mathbf{w}$ is then estimated by filtering on the state equation, yielding $\hat{\mathbf{w}}(t) =$

$\mathbf{f_w}(t).\star\left(\dot{\hat{\mathbf{x}}}(t)-\mathbf{A}\hat{\mathbf{x}}(t)-\mathbf{B}\hat{\mathbf{u}}(t)\right)$, where $\mathbf{f_w}(t)\in\mathbb{R}^n$ is a vector of impulse responses of proper filters, satisfying $\mathbf{f_w}(t).\star\dot{\hat{\mathbf{x}}}(t)$ be realizable. (A more general filtering is that different filters are applied to the three terms in the bracket.) Since it is explicitly dependent on $\hat{\mathbf{u}}$, it is straightforward to substitute $\hat{\mathbf{w}}$ into the key equation (7), from which an LS estimate of the desired control is obtained as follows:

$$\hat{\mathbf{u}}(t)=-\mathbf{B}^\dagger\left(\mathbf{A}\hat{\mathbf{x}}(t)+\mathcal{L}^{-1}\begin{pmatrix}(\mathbf{I}_n-\mathbf{F_w}(s))^{-1}\mathbf{F_w}(s)s\hat{\mathbf{X}}(s)\\-(\mathbf{I}_n-\mathbf{F_w}(s))^{-1}s\mathbf{X}_r(s)\\+(\mathbf{I}_n-\mathbf{F_w}(s))^{-1}\mathbf{H}_{\hat{\mathbf{e}}}(s)\end{pmatrix}\right),\tag{12}$$

where $\mathbf{F_w}(s)$ is a diagonal matrix in Laplace domain whose diagonal elements are the Laplacian transforms of vector $\mathbf{f_w}(t)$, and $\hat{\mathbf{X}}(s)$, $\mathbf{X}_r(s)$ and $\mathbf{H}_{\hat{\mathbf{e}}}(s)$ are the Laplacian transforms of $\hat{\mathbf{x}}(t)$, $\mathbf{x}_r(t)$ and $\mathbf{h}(t,\hat{\mathbf{e}})$, respectively, and $\mathcal{L}^{-1}$ denotes the inverse Laplacian transform. The control input in (12) is ready to implement directly, without explicit estimation of the disturbance.

The estimation errors depend on the filters $\mathbf{f_y}$ and $\mathbf{f_w}$ applied. Designing $\mathbf{f_y}$ properly needs to have prior knowledge about the noise $\mathbf{v}$ which is usually accessible in applications. In contrast, designing $\mathbf{f_w}$ properly is non-trivial and deserves particular investigations. This is because the general disturbance $\mathbf{w}$ contains all unmodeled dynamics and is even dependent on the control input, which would make it difficult to have accurate knowledge of its bandwidth beforehand. Some related results for linear-time varying systems can be found in (Zhong et al., 2011).

Type-II estimator (when $\mathbf{C}$ *is either invertible or not).* An ESO is used to estimate the state $\mathbf{x}$ and the disturbance $\mathbf{w}$, simultaneously. Because the ESO has the form of a conventional linear state observer, a Type-II estimator is rather standard except that the state means an extended state including the total disturbance (Han, 1995).

Extend the state vector as $\bar{\mathbf{x}}=[\mathbf{x}^T\ \mathbf{w}^T]^T$. Then the model equations in (10) are rewritten as

$$\begin{aligned}\dot{\bar{\mathbf{x}}}(t)&=\bar{\mathbf{A}}\bar{\mathbf{x}}(t)+\bar{\mathbf{B}}\hat{\mathbf{u}}(t)+\mathbf{E}\tilde{\mathbf{w}}(t,\mathbf{x}(t),\hat{\mathbf{u}}(t)),\\ \mathbf{y}(t)&=\bar{\mathbf{C}}\bar{\mathbf{x}}(t)+\bar{\mathbf{D}}\hat{\mathbf{u}}(t)+\mathbf{v}(t,\mathbf{x}(t),\hat{\mathbf{u}}(t)),\end{aligned}\tag{13}$$

where

$$\bar{\mathbf{A}}:=\begin{bmatrix}\mathbf{A}&\mathbf{I}_n\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \bar{\mathbf{B}}:=\begin{bmatrix}\mathbf{B}\\\mathbf{0}\end{bmatrix},\ \mathbf{E}:=\begin{bmatrix}\mathbf{0}\\\mathbf{I}_n\end{bmatrix},$$
$$\bar{\mathbf{C}}:=[\,\mathbf{C}\ \mathbf{0}\,],\ \bar{\mathbf{D}}:=\mathbf{D},\ \tilde{\mathbf{w}}(t,\mathbf{x},\hat{\mathbf{u}}):=\dot{\mathbf{w}}(t,\mathbf{x},\hat{\mathbf{u}}).$$

The derivative $\dot{\mathbf{w}}$ acts as uncertainty instead of $\mathbf{w}$. A linear observer is applicable to estimate $\bar{\mathbf{x}}$, as follows:

$$\begin{aligned}\dot{\hat{\bar{\mathbf{x}}}}(t)&=\bar{\mathbf{A}}\hat{\bar{\mathbf{x}}}(t)+\bar{\mathbf{B}}\hat{\mathbf{u}}(t)+\bar{\mathbf{L}}(\mathbf{y}-\hat{\mathbf{y}}),\\ \hat{\mathbf{y}}(t)&=\bar{\mathbf{C}}\hat{\bar{\mathbf{x}}}(t)+\bar{\mathbf{D}}\hat{\mathbf{u}}(t),\end{aligned}\tag{14}$$

where $\bar{\mathbf{L}}\in\mathbb{R}^{2n\times l}$ is selected such that $\bar{\mathbf{A}}-\bar{\mathbf{L}}\bar{\mathbf{C}}$ is Hurwitz. The existence of such a matrix depends on observability of the extended system model (13), which is guaranteed under conditions summarized in the next lemma.

Lemma 2. (Observability of the extended fictitious model) The extended fictitious LTI system model given in (13) is observable if and only if *one* of the conditions is satisfied:

1) rank $\left([\mathcal{O}\ \tilde{\mathbf{I}}_{nl}\mathcal{O}]\right)=2n$ (full column rank);

2) rank$(\mathcal{O})=n$ and rank $\left((\mathbf{I}_{nl}-\mathcal{O}\mathcal{O}^\dagger)\tilde{\mathbf{I}}_{nl}\mathcal{O}\right)=n$;

where $\mathcal{O}\in\mathbb{R}^{nl\times n}$ is the observability matrix of $(\mathbf{A},\mathbf{C})$ and $\tilde{\mathbf{I}}_{nl}\in\mathbb{R}^{nl\times nl}$ is a downward-row-shifted identity matrix such that $\tilde{\mathbf{I}}_{nl}\mathcal{O}=[\mathbf{0}_{l\times n}^T\ \mathbf{C}^T\ \mathbf{A}^T\mathbf{C}^T\ (\mathbf{A}^T)^2\mathbf{C}^T\ \ldots\ (\mathbf{A}^T)^{n-2}\mathbf{C}^T]^T$.

Proof. To prove 1), it is sufficient to show that the observability matrix for $(\bar{\mathbf{A}},\bar{\mathbf{C}})$ is equal to $[\mathcal{O}\ \tilde{\mathbf{I}}_{nl}\mathcal{O}]$ which is very straightforward. To prove 2), by applying Theorem 5 in (Puntanen et al., 2011) (on pp. 121), it follows that rank $\left([\mathcal{O}\ \tilde{\mathbf{I}}_{nl}\mathcal{O}]\right)$ = rank $(\mathcal{O})$ + Rank $\left((\mathbf{I}_{nl}-\mathcal{O}\mathcal{O}^\dagger)\tilde{\mathbf{I}}_{nl}\mathcal{O}\right)$ and so establishes its equivalence to 1).

Define the estimation error as $\delta_{\bar{\mathbf{x}}}=\bar{\mathbf{x}}-\hat{\bar{\mathbf{x}}}$, and let $\tilde{\mathbf{A}}:=\bar{\mathbf{A}}-\bar{\mathbf{L}}\bar{\mathbf{C}}$. Then the error is bounded if both $\dot{\mathbf{w}}$ and $\mathbf{v}$ are bounded, as explained by the lemma below.

Lemma 3. (Bounded estimation error) If $\|\dot{\mathbf{w}}\|\le c_{\dot{\mathbf{w}}}$, $\|\mathbf{v}\|\le c_{\mathbf{v}}$ and $\tilde{\mathbf{A}}$ is Hurwitz, then there exists a finite time T_1 $(\ge t_0)$ such that $\|\delta_{\bar{\mathbf{x}}}\|\le 2\gamma_{\tilde{\mathbf{A}}}(c_{\dot{\mathbf{w}}}+c_{\mathbf{v}}\|\bar{\mathbf{L}}\|)$ for all $t\ge T_1$, where $\gamma_{\tilde{\mathbf{A}}}=\left\|\int_{t_0}^{\infty}e^{\tilde{\mathbf{A}}^T\tau}e^{\tilde{\mathbf{A}}\tau}d\tau\right\|$.

Proof. The estimation error dynamics is obtained as $\dot{\delta}_{\bar{\mathbf{x}}}=\tilde{\mathbf{A}}\delta_{\bar{\mathbf{x}}}+\mathbf{d}$, where $\mathbf{d}:=\begin{bmatrix}\mathbf{0}\\\dot{\mathbf{w}}\end{bmatrix}-\bar{\mathbf{L}}\mathbf{v}$. It is easy to show that there exists a finite time T_1 such that $\|\delta_{\bar{\mathbf{x}}}\|\le 2\|\mathbf{Pd}\|\le 2\|\mathbf{P}\|(\|\dot{\mathbf{w}}\|+\|\bar{\mathbf{L}}\|\|\mathbf{v}\|)$ for all $t\ge T_1$ (readers are referred to the proof of Theorem 1 in (Gao, 2006) for this step), where $\mathbf{P}$ is a positive definite matrix satisfying the Lyapunov equation $\tilde{\mathbf{A}}^T\mathbf{P}+\mathbf{P}\tilde{\mathbf{A}}=-\mathbf{I}_{2n}$. Specifically, $\mathbf{P}$ is solved explicitly as $\mathbf{P}=\int_{t_0}^{\infty}e^{\tilde{\mathbf{A}}^T\tau}e^{\tilde{\mathbf{A}}\tau}d\tau$ (Chen, 1999). Inserting this solution and the given bounds on disturbance and noise into the proceeding inequality proves the lemma.

Lemma 3 indicates that the magnitude of the estimation error depends on the speed of the varying disturbance $\mathbf{w}$, and the magnitude of measurement noise $\mathbf{v}$, and also the observer gain $\bar{\mathbf{L}}$ (which determines the value of $\gamma_{\tilde{\mathbf{A}}}$). If the estimation error is large mainly for a rapidly changing $\mathbf{w}$, then it may be remedied by placing the poles of $\tilde{\mathbf{A}}$ further away from the imaginary axis (but limited by simultaneous inflation of the measurement noise via $\|\bar{\mathbf{L}}\|$), or by using a higher-order ESO that treats the derivative(s) of $\mathbf{w}$ as additional state(s) (subject to a renewed observability condition) (Miklosovic et al., 2006; Madonski and Herman, 2013). To deal with measurement noise, the measurements can be filtered before use if the cost incurred is mild compared to the benefit.

3.3 Stability analysis

It is sufficient to analyze the closed-loop stability via the state tracking error dynamics. With the system dynamics described in (10) and the control given in (8), the tracking error dynamics is deduced as follows:

$$\begin{aligned}\dot{\mathbf{e}}&=\dot{\mathbf{x}}_r-\dot{\mathbf{x}}=\mathbf{f}_r-\hat{\mathbf{w}}-\mathbf{A}\hat{\mathbf{x}}-\mathbf{B}\hat{\mathbf{u}}-\mathbf{A}\delta_{\mathbf{x}}-\delta_{\mathbf{w}}\\&=\mathbf{h}(t,\hat{\mathbf{e}})+\begin{pmatrix}\mathbf{f}_r-\hat{\mathbf{w}}-\mathbf{A}\hat{\mathbf{x}}-\mathbf{h}(t,\hat{\mathbf{e}})\\-\mathbf{B}\mathbf{B}^\dagger(\mathbf{f}_r-\hat{\mathbf{w}}-\mathbf{A}\hat{\mathbf{x}}-\mathbf{h}(t,\hat{\mathbf{e}}))-\mathbf{A}\delta_{\mathbf{x}}-\delta_{\mathbf{w}}\end{pmatrix}\\&=\mathbf{h}(t,\hat{\mathbf{e}})+(\mathbf{I}-\mathbf{B}\mathbf{B}^\dagger)(\mathbf{f}_r-\hat{\mathbf{w}}-\mathbf{A}\hat{\mathbf{x}}-\mathbf{h}(t,\hat{\mathbf{e}}))-\mathbf{A}\delta_{\mathbf{x}}-\delta_{\mathbf{w}},\\&=\mathbf{h}(t,\mathbf{e})+\delta_{\mathbf{u}}-\mathbf{A}\delta_{\mathbf{x}}-\delta_{\mathbf{w}}-\delta_{\mathbf{h}}=\mathbf{h}(t,\mathbf{e})+\xi,\end{aligned}\tag{15}$$

where $\xi:=\delta_{\mathbf{u}}-\mathbf{A}\delta_{\mathbf{x}}-\delta_{\mathbf{w}}-\delta_{\mathbf{h}}$, which defines the total design error that lumps the control-estimate-induced bias and the errors caused by inexact estimation of $\mathbf{x}$ and $\mathbf{w}$. Sufficient conditions for the tracking error to be bounded are given in the next theorem.

Theorem 4. (Weak closed-loop stability) Let the fictitious model be specified such that $\delta_{\mathbf{u}} \equiv 0$. The closed-loop system described by (15) is stable in the sense that the state tracking error $\mathbf{e}$ is bounded, if the three conditions are satisfied: 1) the desired error dynamics, $\dot{\mathbf{e}} = \mathbf{h}(t, \mathbf{e})$, is globally exponentially stable at the origin; 2) the function $\mathbf{h}(t, \mathbf{e})$ is continuously differentiable, and there exists a positive scalar $l_{\mathbf{h}}$ such that $\|\mathbf{h}(t, \mathbf{e}_1) - \mathbf{h}(t, \mathbf{e}_2)\| \leq l_{\mathbf{h}} \|\mathbf{e}_1 - \mathbf{e}_2\|$ for any $t \geq t_0$ and $\mathbf{e}_1, \mathbf{e}_2$ in the admissible domain; 3) the disturbance, measurement noise and observer gain satisfy the conditions in Lemma 3. More precisely, under these conditions there exists a finite time T_2 ($\geq T_1 \geq t_0$) and a positive constant c such that the tracking error is bounded as

$$\|\mathbf{e}\| \leq c\,\|\xi\| \leq c\gamma_{\tilde{\mathbf{A}}}(c_{\dot{\mathbf{w}}} + c_{\mathbf{v}} \left\|\bar{\mathbf{L}}\right\|)(l_{\mathbf{h}} + \|[\mathbf{A}\ \mathbf{I}_n]\|), \quad (16)$$

for all $t \geq T_2$, where $\gamma_{\tilde{\mathbf{A}}}$, $c_{\dot{\mathbf{w}}}$ and $c_{\mathbf{v}}$ are given in Lemma 3.

Proof. Conditions 1) and 2) imply that the tracking error dynamics described in (15) is input-to-state stable with the input being the total design error ξ (Lemma 4.6 in (Khalil, 2002)). To prove the theorem, it is sufficient to show that ξ is bounded. Condition 2) implies that $\|\delta_{\mathbf{h}}\| = \|\mathbf{h}(t, \mathbf{e}) - \mathbf{h}(t, \hat{\mathbf{e}})\| \leq l_{\mathbf{h}} \|\delta_{\mathbf{x}}\| \leq l_{\mathbf{h}} \|\delta_{\bar{\mathbf{x}}}\|$. With $\xi = -[\mathbf{A}\ \mathbf{I}_n]\delta_{\bar{\mathbf{x}}} - \delta_{\mathbf{h}}$ and condition 3), it follows from Lemma 3 that there exists a finite time T_1 ($\geq t_0$) such that $\|\xi\| \leq 2\gamma_{\tilde{\mathbf{A}}}(c_{\dot{\mathbf{w}}} + c_{\mathbf{v}} \left\|\bar{\mathbf{L}}\right\|)(l_{\mathbf{h}} + \|[\mathbf{A}\ \mathbf{I}_n]\|)$ for all $t \geq T_1$. This establishes the boundedness of ξ and thus proves the boundedness of the state tracking error.

The specific bound of the tracking error can be derived by referring to the proof of Lemma 4.6 in (Khalil, 2002). Condition 1) implies that, for the unperturbed system $\dot{\mathbf{e}} = \mathbf{h}(t, \mathbf{e})$, there exists a Lyapunov function $V(t, \mathbf{e})$ satisfying $c_1 \|\mathbf{e}\|^2 \leq V(t, \mathbf{e}) \leq c_2 \|\mathbf{e}\|^2$, $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{e}}\mathbf{h}(t, \mathbf{e}) \leq -c_3 \|\mathbf{e}\|^2$ and $\left\|\frac{\partial V}{\partial \mathbf{e}}\right\| \leq c_4 \|\mathbf{e}\|$ for some positive constants c_1, c_2, c_3 and c_4 (Theorem 4.14 in (Khalil, 2002)). Then the derivative of $V(t, \mathbf{e})$ with respect to the perturbed system $\dot{\mathbf{e}} = \mathbf{h}(t, \mathbf{e}) + \xi$ satisfies

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{e}}\mathbf{h}(t, \mathbf{e}) + \frac{\partial V}{\partial \mathbf{e}}\xi \leq -c_3 \|\mathbf{e}\|^2 + c_4 \|\mathbf{e}\| \|\xi\| .$$

Then $\dot{V} < 0$ for all $\|\mathbf{e}\| > \frac{c_4}{c_3}\|\xi\|$. This means that there exists a finite time T_2 ($\geq T_1 \geq t_0$) such that $\|\mathbf{e}\| \leq \frac{c_4}{c_3}\|\xi\|$ for all $t \geq T_2$. By using the preceding bound of $\|\xi\|$, this establishes the bound of the tracking error as given in (16) and thus completes the proof.

An important implication follows on the value of having a good system model: It is useful to have a more accurate model if it contributes to a smaller total design error ξ. This is likely to be true because a more accurate model and hence a smaller or smoother $\mathbf{w}$ would contribute to a smaller estimation error $\delta_{\mathbf{w}}$ and consequently a smaller total design error ξ. This reveals that the value of having a good model remains in the current approach, although its marginal benefit is probably not obvious because of the online feedforward compensation embedded.

As remarked before, the assumption of $\delta_{\mathbf{u}} \equiv 0$ imposes certain assumptions on the true system dynamics and the fictitious model. Also the assumption of bounded $\dot{\mathbf{w}}(t, \mathbf{x}, \hat{\mathbf{u}}, \mathbf{w}_0)$ and $\mathbf{v}(t, \mathbf{x}, \hat{\mathbf{u}}, \mathbf{v}_0)$ has limitation because these vectors are normally dependent on the control applied and their boundedness are hard to know beforehand. For these reasons, it is desirable to establish a more general stability result without these restrictive assumptions.

For tractability, we consider a measurement model with $\mathbf{D} = \mathbf{0}$ and the reference tracking error dynamics having a linear form of $\dot{\mathbf{e}} = \mathbf{h}(t, \mathbf{e}) := \mathbf{K}\mathbf{e}$, where $\mathbf{K} \in \mathbb{R}^{n\times n}$ is a given matrix. Then the estimation and tracking error dynamics which determines the closed-loop stability is derived as follows (after tedious deductions):

$$\begin{bmatrix} \dot{\delta}_{\bar{\mathbf{x}}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{bmatrix} \delta_{\bar{\mathbf{x}}} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \quad (17)$$

where the related matrices and vectors are given in (18)-(20), with $\tilde{\mathbf{B}} := \mathbf{I}_n - \mathbf{B}\mathbf{B}^\dagger$ and $\mathbf{M} := \begin{bmatrix} -\mathbf{I}_n & \mathbf{0} \\ \mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) & -\tilde{\mathbf{B}} \end{bmatrix}$ (as frequently used later on). Define $\bar{\mathbf{e}} = \begin{bmatrix} \delta_{\bar{\mathbf{x}}} \\ \mathbf{e} \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}$ and $\bar{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$. Then the error dynamics becomes concise as $\dot{\bar{\mathbf{e}}} = \mathbf{H}\bar{\mathbf{e}} + \bar{\delta}$, from which the closed-loop stability is deduced. For convenience, let $\gamma_{\mathbf{H}} := \left\|\int_{t_0}^{\infty} e^{\mathbf{H}^T\tau} e^{\mathbf{H}\tau} d\tau\right\|$.

Theorem 5. (Closed-loop stability) The closed-loop system described by (17) is stable in the sense that the estimation and the tracking errors are bounded if the following conditions are satisfied:
1) the true state function $\mathbf{f}_0(t, \mathbf{x}, \mathbf{u}, \mathbf{w})$ is continuously differentiable, and $\mathbf{f}_0(t, \mathbf{x}, \mathbf{u}, \mathbf{w})$ and $\dot{\mathbf{f}}_0(t, \mathbf{x}, \mathbf{u}, \mathbf{w})$ are uniformly globally Lipchitz in $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{w}$ and so is the true measurement function $\mathbf{g}_0(t, \mathbf{x}, \mathbf{u}, \mathbf{v})$ in $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{v}$, i.e., there exist non-negative constants $l_{\mathbf{f}0}^{\mathbf{x}/\mathbf{u}/\mathbf{w}}$, $l_{\dot{\mathbf{f}}0}^{\mathbf{x}/\mathbf{u}/\mathbf{w}}$ and $l_{\mathbf{g}0}^{\mathbf{x}/\mathbf{u}/\mathbf{v}}$ such that

$$\|\mathbf{f}_0(t, \mathbf{x}_1, \mathbf{u}_1, \mathbf{w}_1) - \mathbf{f}_0(t, \mathbf{x}_2, \mathbf{u}_2, \mathbf{w}_2)\|$$
$$\leq l_{\mathbf{f}0}^{\mathbf{x}} \|\mathbf{x}_1 - \mathbf{x}_2\| + l_{\mathbf{f}0}^{\mathbf{u}} \|\mathbf{u}_1 - \mathbf{u}_2\| + l_{\mathbf{f}0}^{\mathbf{w}} \|\mathbf{w}_1 - \mathbf{w}_2\|$$
$$\left\|\dot{\mathbf{f}}_0(t, \mathbf{x}_1, \mathbf{u}_1, \mathbf{w}_1) - \dot{\mathbf{f}}_0(t, \mathbf{x}_2, \mathbf{u}_2, \mathbf{w}_2)\right\|$$
$$\leq l_{\dot{\mathbf{f}}0}^{\mathbf{x}} \|\mathbf{x}_1 - \mathbf{x}_2\| + l_{\dot{\mathbf{f}}0}^{\mathbf{u}} \|\mathbf{u}_1 - \mathbf{u}_2\| + l_{\dot{\mathbf{f}}0}^{\mathbf{w}} \|\mathbf{w}_1 - \mathbf{w}_2\|$$
$$\|\mathbf{g}_0(t, \mathbf{x}_1, \mathbf{u}_1, \mathbf{v}_1) - \mathbf{g}_0(t, \mathbf{x}_2, \mathbf{u}_2, \mathbf{v}_2)\|$$
$$\leq l_{\mathbf{g}0}^{\mathbf{x}} \|\mathbf{x}_1 - \mathbf{x}_2\| + l_{\mathbf{g}0}^{\mathbf{u}} \|\mathbf{u}_1 - \mathbf{u}_2\| + l_{\mathbf{g}0}^{\mathbf{v}} \|\mathbf{v}_1 - \mathbf{v}_2\|$$

for all $t \geq t_0$ and the variables in the admissible domain;
2) the reference state function $\mathbf{f}_r(t, \mathbf{x}, \mathbf{u})$ is continuously differentiable, and $\mathbf{f}_r(t, \mathbf{x}, \mathbf{u})$ and $\dot{\mathbf{f}}_r(t, \mathbf{x}, \mathbf{u})$ are uniformly globally Lipchitz in $\mathbf{x}$, and $\mathbf{u}$, i.e., there exist non-negative constants $l_{\mathbf{f}r}^{\mathbf{x}/\mathbf{u}}$ and $l_{\dot{\mathbf{f}}r}^{\mathbf{x}/\mathbf{u}}$ such that

$$\|\mathbf{f}_r(t, \mathbf{x}_1, \mathbf{u}_1) - \mathbf{f}_r(t, \mathbf{x}_2, \mathbf{u}_2)\| \leq l_{\mathbf{f}r}^{\mathbf{x}} \|\mathbf{x}_1 - \mathbf{x}_2\| + l_{\mathbf{f}r}^{\mathbf{u}} \|\mathbf{u}_1 - \mathbf{u}_2\|$$
$$\left\|\dot{\mathbf{f}}_r(t, \mathbf{x}_1, \mathbf{u}_1) - \dot{\mathbf{f}}_r(t, \mathbf{x}_2, \mathbf{u}_2)\right\| \leq l_{\dot{\mathbf{f}}r}^{\mathbf{x}} \|\mathbf{x}_1 - \mathbf{x}_2\| + l_{\dot{\mathbf{f}}r}^{\mathbf{u}} \|\mathbf{u}_1 - \mathbf{u}_2\|$$

for all $t \geq t_0$ and the variables in the admissible domain;
3) the control actions are bounded, i.e., $\|\hat{\mathbf{u}}\| \leq c_{\mathbf{u}}$ and $\|\mathbf{u}_r\| \leq c_{\mathbf{u}r}$ for some positive constants $c_{\mathbf{u}}$ and $c_{\mathbf{u}r}$, and so are the reference state $\mathbf{x}_r$ and the actual disturbance $\mathbf{w}_0$ and noise $\mathbf{v}_0$, i.e., $\|\mathbf{x}_r\| \leq c_{\mathbf{x}r}$, $\|\mathbf{w}_0\| \leq c_{\mathbf{w}0}$, and $\|\mathbf{v}_0\| \leq c_{\mathbf{v}0}$ for some non-negative constant $c_{\mathbf{x}r}$, $c_{\mathbf{w}0}$ and $c_{\mathbf{v}0}$;
4) there exist a reference tracking error gain matrix $\mathbf{K}$ and an observer gain matrix $\bar{\mathbf{L}}$ such that $\mathbf{H}$ is Hurwitz and that

$$2\gamma_{\mathbf{H}} \left(\begin{array}{c} l_{\mathbf{g}0}^{\mathbf{x}} \left\|\mathbf{M}\bar{\mathbf{L}}\right\| \\ +l_{\mathbf{f}0}^{\mathbf{x}} \left(\left\| \left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} \right\| + \left\|\tilde{\mathbf{B}}\right\| \right) + l_{\dot{\mathbf{f}}0}^{\mathbf{x}} \end{array} \right) < 1. \quad (21)$$

More precisely, under these conditions there exists a finite time $T \geq t_0$ such that the estimation and tracking errors are bounded as $\|\bar{\mathbf{e}}\| \leq c_{\bar{\mathbf{e}}}$, with $c_{\bar{\mathbf{e}}}$ computed from (22).

$$\mathbf{H}_{11} = \tilde{\mathbf{A}} + \begin{bmatrix} \mathbf{0} \\ \left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)[\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K})\ \mathbf{B}\mathbf{B}^\dagger] - [\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K})\ \mathbf{0}]\tilde{\mathbf{A}} + [\mathbf{0}\ \mathbf{B}\mathbf{B}^\dagger][\bar{\mathbf{L}}\mathbf{C}\ \mathbf{0}] \end{bmatrix}, \tag{18}$$

$$\mathbf{H}_{12} = \begin{bmatrix} \mathbf{0} \\ [\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K})\ \mathbf{B}\mathbf{B}^\dagger]\bar{\mathbf{L}}\mathbf{C} + \left(\mathbf{A} - \mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K})\right)\mathbf{B}\mathbf{B}^\dagger\mathbf{K} \end{bmatrix} - \bar{\mathbf{L}}\mathbf{C},\ \mathbf{H}_{21} = -[\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K})\ \mathbf{B}\mathbf{B}^\dagger],\quad \mathbf{H}_{22} = \mathbf{B}\mathbf{B}^\dagger\mathbf{K}, \tag{19}$$

$$\delta_1 = \mathbf{M}\bar{\mathbf{L}}\left(\mathbf{g}_0 - \mathbf{C}\mathbf{x}_r\right) + \begin{bmatrix} \mathbf{0} \\ \left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} \end{bmatrix}\mathbf{f}_0 - \begin{bmatrix} \mathbf{0} \\ \left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\mathbf{A} \end{bmatrix}\mathbf{f}_r + \begin{bmatrix} \mathbf{0} \\ \dot{\mathbf{f}}_0 - \mathbf{B}\mathbf{B}^\dagger\dot{\mathbf{f}}_r \end{bmatrix},\quad \delta_2 = \tilde{\mathbf{B}}\left(\mathbf{f}_r - \mathbf{f}_0\right). \tag{20}$$

$$c_{\bar{\mathbf{e}}} := \frac{2\gamma_{\mathbf{H}}\left(c_{\mathbf{x}r}\beta_1 + c_{\mathbf{u}r}\beta_2 + c_{\mathbf{u}}\beta_3 + c_{\mathbf{w}0}\beta_4 + c_{\mathbf{v}0}\beta_5\right)}{1 - 2\gamma_{\mathbf{H}}\left(l^{\mathbf{x}}_{\mathbf{g}0}\left\|\mathbf{M}\bar{\mathbf{L}}\right\| + l^{\mathbf{x}}_{\mathbf{f}0}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) + l^{\mathbf{x}}_{\dot{\mathbf{f}}0}\right)}, \tag{22}$$

where

$$\beta_1 = \begin{pmatrix} l^{\mathbf{x}}_{\mathbf{g}0}\left\|\mathbf{M}\bar{\mathbf{L}}\right\| + \left\|\mathbf{M}\bar{\mathbf{L}}\mathbf{C}\right\| + l^{\mathbf{x}}_{\mathbf{f}0}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) \\ +l^{\mathbf{x}}_{\mathbf{f}r}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\mathbf{A}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) + l^{\mathbf{x}}_{\dot{\mathbf{f}}0} + l^{\mathbf{x}}_{\dot{\mathbf{f}}r}\left\|\mathbf{B}\mathbf{B}^\dagger\right\| \end{pmatrix},\ \beta_2 = l^{\mathbf{u}}_{\mathbf{f}r}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\mathbf{A}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) + l^{\mathbf{u}}_{\dot{\mathbf{f}}r}\left\|\mathbf{B}\mathbf{B}^\dagger\right\|,$$

$$\beta_3 = \left(l^{\mathbf{u}}_{\mathbf{g}0}\left\|\mathbf{M}\bar{\mathbf{L}}\right\| + l^{\mathbf{u}}_{\mathbf{f}0}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) + l^{\mathbf{u}}_{\dot{\mathbf{f}}0}\right),\ \beta_4 = l^{\mathbf{w}}_{\mathbf{f}0}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) + l^{\mathbf{w}}_{\dot{\mathbf{f}}0},\quad \beta_5 = l^{\mathbf{v}}_{\mathbf{g}0}\left\|\mathbf{M}\bar{\mathbf{L}}\right\|. \tag{23}$$

Proof. Consider the closed-loop system $\dot{\bar{\mathbf{e}}} = \mathbf{H}\bar{\mathbf{e}} + \bar{\delta}$. Let the Lyapunov function be $V(t, \mathbf{e}) = \bar{\mathbf{e}}^T\mathbf{P}\bar{\mathbf{e}}$, where $\mathbf{P} = \int_{t_0}^{\infty} e^{\mathbf{H}^T\tau}e^{\mathbf{H}\tau}d\tau$. Since $\mathbf{H}$ is Hurwitz, the positive definite matrix $\mathbf{P}$ is a unique solution to the Lyapunov equation $\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H} = -\mathbf{I}_{3n}$. The proof proceeds as in (24). Then by condition 4), $\dot{V} < 0$ if $\|\bar{\mathbf{e}}\| > c_{\bar{\mathbf{e}}}$, where $c_{\bar{\mathbf{e}}}$ is defined in (22). Therefore there exists a finite time T such that $\|\bar{\mathbf{e}}\| \le c_{\bar{\mathbf{e}}}$ for all $t \ge T$, which completes the proof.

The Liptchiz conditions in 1) and 2) on the system and the reference dynamics are normally required for control of nonlinear systems (Khalil, 2002). Similar conditions on their derivatives are reasonable as the variables should not change too fast if a system is practically controllable. The various bounds imposed by condition 3) are common in normal operations. Condition 4) is thus the key constraint to enable closed-loop stability. The condition roughly means that the poles of the observer (as determined by the observer gain matrix $\bar{\mathbf{L}}$) and the poles of the target tracking error dynamics (as determined by matrix $\mathbf{K}$) should both be far from the imaginary axis such that $\gamma_{\mathbf{H}}$ is small enough. However, this is limited by the simultaneous inflation of the other factor, to which $\gamma_{\mathbf{H}}$ times. In other words, while a high gain observer and fast target dynamics are desirable, they are useful only if they do not inflate the (general) measurement noise too much at the same time.

In particular, if the true system dynamics can be represented in Han's canonical form (Han, 1981; Youcef-Toumi and Ito, 1988; Fliess, 1990) , then it is feasible to specify the system model also in a canonical form. In this case, all terms related to $\mathbf{B}\mathbf{B}^\dagger$ in the closed-loop analysis will disappear (cf. Lemma 1). Consequently $\mathbf{B}\mathbf{B}^\dagger$ and $\tilde{\mathbf{B}}$ are treated as identity and zero matrices, respectively. Then the closed-loop matrices in (18)-(20) simplify into

$$\mathbf{H}_{11} = \tilde{\mathbf{A}} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{K}[\mathbf{A}-\mathbf{K}\ \mathbf{I}_n] - [\mathbf{A}-\mathbf{K}\ \mathbf{0}]\tilde{\mathbf{A}} + [\mathbf{0}\ \mathbf{I}_n][\bar{\mathbf{L}}\mathbf{C}\ \mathbf{0}] \end{bmatrix},$$

$$\mathbf{H}_{12} = \begin{bmatrix} \mathbf{0} \\ [\mathbf{A}-\mathbf{K}\ \mathbf{I}_n]\bar{\mathbf{L}}\mathbf{C} + \mathbf{K}^2 \end{bmatrix} - \bar{\mathbf{L}}\mathbf{C},\ \mathbf{H}_{21} = -[\mathbf{A}-\mathbf{K}\ \mathbf{I}_n],\ \mathbf{H}_{22} = \mathbf{K},$$

$$\delta_1 = \begin{bmatrix} -\mathbf{I}_n & \mathbf{0} \\ \mathbf{A}-\mathbf{K} & \mathbf{0} \end{bmatrix}\bar{\mathbf{L}}\left(\mathbf{g}_0 - \mathbf{C}\mathbf{x}_r\right) + \begin{bmatrix} \mathbf{0} \\ \dot{\mathbf{f}}_0 - \dot{\mathbf{f}}_r \end{bmatrix},\ \delta_2 = \mathbf{0},$$

and the key stability condition in (21) reduces to

$$2\gamma_{\mathbf{H}}\left(l^{\mathbf{x}}_{\mathbf{g}0}\left\|\begin{bmatrix} -\mathbf{I}_n & \mathbf{0} \\ \mathbf{A}-\mathbf{K} & \mathbf{0} \end{bmatrix}\bar{\mathbf{L}}\right\| + l^{\mathbf{x}}_{\dot{\mathbf{f}}0}\right) < 1.$$

4. NUMERICAL EXAMPLE

Consider a normalized model of the pendulum when the control input is the acceleration of the pivot (Åström et al., 2008):

$$\dot{x}_1 = x_2,\ \dot{x}_2 = \sin x_1 - u\cos x_1, \tag{25}$$

where x_1 is the angular position of the pendulum with the origin at the upright position and x_2 is the angular velocity of the pendulum. The goal is to design a controller based on the measurable states x_1 and x_2 that is able to swing-up the pendulum from all or constrained initial conditions and maintain it at the upright position.

We specify the reference closed-loop model as: $\dot{x}_{r,1} = x_{r,2}$ and $\dot{x}_{r,2} = -k_1x_{r,1} - k_2x_{r,2}$, where k_1 and k_2 are positive scalars, and the desired tracking error dynamics as: $\dot{e}_1 = e_2$ and $\dot{e}_2 = -k_1e_1 - k_2e_2$, where $e_1 := x_{r,1} - x_1$ and $e_2 := x_{r,2} - x_2$. Depending on the system model used, different controllers can be designed by the proposed approach.

Case A: The design bases on the ideal model (25), leading to an exact control as

$$u = \frac{k_1x_1 + k_2x_2 + \sin x_1}{\cos x_1}, \tag{26}$$

which recovers the control law obtained by an input-output linearization technique (Srinivasan et al., 2009). The singularity of the control at $x_1 = \frac{2k+1}{2}\pi$ for $k = 0, 1, 2, ...$, can be resolved by bounding the input and meanwhile switching the reference value of x_1 properly (Srinivasan et al., 2009). This controller is treated as a reference controller without implementing online feedforward compensation for unknown disturbances.

Case B: While the dynamic model of x_1 is exact as in (25), the dynamic model of x_2 is replaced by a fictitious model per $\dot{x}_2 = -u + w$, where w contains any model mismatch (i.e., $w = \sin x_1 + u(1 - \cos x_1)$). If a Type-I estimator is applied, it leads to an estimated control:

$$\hat{u} = \mathcal{L}^{-1}\left(\frac{k_1X_1 + (k_2 + sF_x)X_2}{1 - F_u}\right), \tag{27}$$

where F_x and F_u are filters for estimating $\dot{x}_2$ and u, respectively. Note that the disturbance w has been estimated and compensated in an implicit manner. If a Type-II estimator is used instead, then the estimate $\hat{w}$ is obtained from an ESO defined in (14) and the observer gain matrix

$$
\begin{aligned}
\dot{V} &= \bar{\mathbf{e}}^T(\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H})\bar{\mathbf{e}} + 2\bar{\mathbf{e}}^T\mathbf{P}\bar{\delta} = -\bar{\mathbf{e}}^T\bar{\mathbf{e}} + 2\bar{\mathbf{e}}^T\mathbf{P}\bar{\delta} \leq -\|\bar{\mathbf{e}}\|^2 + 2\gamma_{\mathbf{H}}\|\bar{\mathbf{e}}\|\left(\|\delta_1\| + \|\delta_2\|\right) \\
&\leq -\|\bar{\mathbf{e}}\|^2 + 2\gamma_{\mathbf{H}}\|\bar{\mathbf{e}}\| \times \left(\begin{array}{c} \left\|\mathbf{M}\bar{\mathbf{L}}\right\|\|\mathbf{g}_0\| + \left\|\mathbf{M}\bar{\mathbf{L}}\mathbf{C}\right\|\|\mathbf{x}_r\| \\ + \left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\|\|\mathbf{f}_0\| + \left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\mathbf{A}\right\|\|\mathbf{f}_r\| \\ + \left\|\dot{\mathbf{f}}_0\right\| + \left\|\mathbf{B}\mathbf{B}^\dagger\right\|\left\|\dot{\mathbf{f}}_r\right\| + \left\|\tilde{\mathbf{B}}\right\|\|\mathbf{f}_r\| + \left\|\tilde{\mathbf{B}}\right\|\|\mathbf{f}_0\| \end{array}\right) \\
&\leq -\|\bar{\mathbf{e}}\|^2 + 2\gamma_{\mathbf{H}}\|\bar{\mathbf{e}}\| \times \left(\begin{array}{c} \left\|\mathbf{M}\bar{\mathbf{L}}\right\|\left(l^{\mathbf{x}}_{\mathbf{g}0}\|\mathbf{x}\| + l^{\mathbf{u}}_{\mathbf{g}0}\|\hat{\mathbf{u}}\| + l^{\mathbf{v}}_{\mathbf{g}0}\|\mathbf{v}_0\|\right) + \left\|\mathbf{M}\bar{\mathbf{L}}\mathbf{C}\right\|\|\mathbf{x}_r\| \\ + \left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right)\left(l^{\mathbf{x}}_{\mathbf{f}0}\|\mathbf{x}\| + l^{\mathbf{u}}_{\mathbf{f}0}\|\hat{\mathbf{u}}\| + l^{\mathbf{w}}_{\mathbf{f}0}\|\mathbf{w}_0\|\right) \\ + \left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\mathbf{A}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right)\left(l^{\mathbf{x}}_{\mathbf{f}r}\|\mathbf{x}_r\| + l^{\mathbf{u}}_{\mathbf{f}r}\|\mathbf{u}_r\|\right) \\ + l^{\mathbf{x}}_{\dot{\mathbf{f}}0}\|\mathbf{x}\| + l^{\mathbf{u}}_{\dot{\mathbf{f}}0}\|\hat{\mathbf{u}}\| + l^{\mathbf{w}}_{\dot{\mathbf{f}}0}\|\mathbf{w}_0\| + \left\|\mathbf{B}\mathbf{B}^\dagger\right\|\left(l^{\mathbf{x}}_{\dot{\mathbf{f}}r}\|\mathbf{x}_r\| + l^{\mathbf{u}}_{\dot{\mathbf{f}}r}\|\mathbf{u}_r\|\right) \end{array}\right) \\
&\leq -\left(1 - 2\gamma_{\mathbf{H}}\left(l^{\mathbf{x}}_{\mathbf{g}0}\left\|\mathbf{M}\bar{\mathbf{L}}\right\| + l^{\mathbf{x}}_{\mathbf{f}0}\left(\left\|\left(\mathbf{B}\mathbf{B}^\dagger(\mathbf{A}-\mathbf{K}) - \mathbf{A}\right)\tilde{\mathbf{B}}\right\| + \left\|\tilde{\mathbf{B}}\right\|\right) + l^{\mathbf{x}}_{\dot{\mathbf{f}}0}\right)\right)\|\mathbf{e}\|^2 \\
&\quad + 2\gamma_{\mathbf{H}}\left(c_{\mathbf{x}r}\beta_1 + c_{\mathbf{u}r}\beta_2 + c_{\mathbf{u}}\beta_3 + c_{\mathbf{w}0}\beta_4 + c_{\mathbf{v}0}\beta_5\right)\|\bar{\mathbf{e}}\|,
\end{aligned} \tag{24}
$$

where β_i, $i = 1, 2, ..., 5$ are defined in (23).

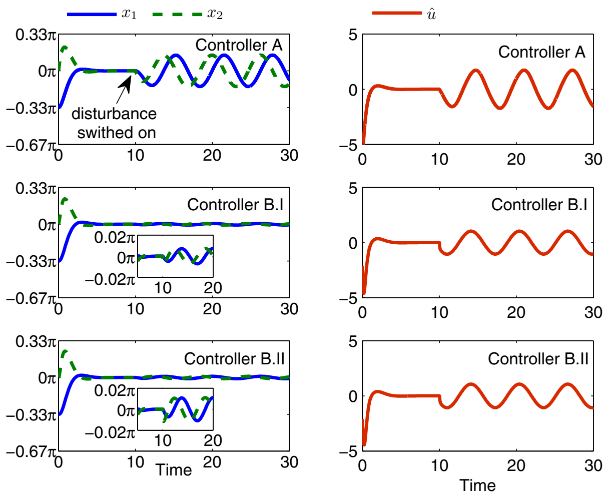

Fig. 1. Performances of three controllers.

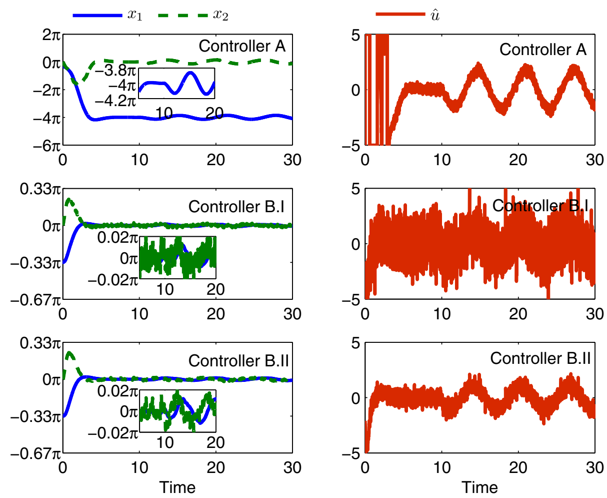

Fig. 3. Control performance under noisy measurements.

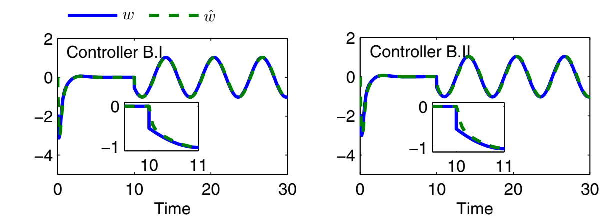

Fig. 2. Disturbance estimates vs. the true disturbances.

$\bar{\mathbf{L}} \in \mathbb{R}^{3\times 2}$ is designed such that $\bar{\mathbf{A}} - \bar{\mathbf{L}}\bar{\mathbf{C}}$ is Hurwitz. The control then takes the form of

$$\hat{u} = k_1 x_1 + k_2 x_2 + \hat{w}. \tag{28}$$

The three controllers in (26), (27) and (28) are named as controller A, B.I and B.II, respectively. The design parameters of the controllers are specified as: $k_1 = k_2 = 2$, $|u_{\max}| = 5$ (bounded control), $F_x = F_u = \frac{1}{0.05s+1}$, and $\bar{\mathbf{L}} = \begin{bmatrix} 20 & 1 \\ 0 & 60 \\ 0 & 800 \end{bmatrix}$. The filters F_x and F_u are such that the derivative of state x_2 can well be estimated, and the observer gain matrix $\bar{\mathbf{L}}$ is such that the three poles of the ESO are placed at -20, -20 and -40, which are ten or more times faster than the actual state dynamics.

With $(x_1(0), x_2(0)) = (-\frac{\pi}{3}, 0)$, the simulation results, when an *additive sinusoid disturbance*, $\sin t$, entering into the dynamics of x_2 is present or absent, are shown in Fig. 1. In the absence of the disturbance, all three controllers are able to stabilize the pendulum pretty well with comparable performances. The implicit and explicit estimates of the total disturbances used by controllers B.I and B.II are shown in Fig. 2, which are accurate and hence imply good online compensations of the disturbances. When the sinusoid disturbance is switched on, controller A becomes unacceptable, leading to large oscillating tracking errors. In contrast, controllers B.I and B.II maintain small tracking errors.

When the states x_1 and x_2 are both measured with additive zero-mean Gaussian noises having a variance of $(\frac{\pi}{60})^2$ (and so 97% of the errors are within ±9 degrees and ±9 degrees/second, respectively), the control performances for a single noise realization are shown in Fig. 3. Regarding each controller, the resultant state dynamics is similar to the previous scenario, except that it becomes noisy. The change is more obvious in the control signals: They become very noisy, which is most severe with controller B.I. Nevertheless, this can be alleviated if the measurements are filtered before use. For example, if both the measurements of x_1 and x_2 are filtered by the filter $F_y(s) = \frac{1}{0.05s+1}$, then the performance of controller B.I becomes much smoother and is very close to that of controller B.II. The results are not shown due to space limit.

5. CONCLUSIONS

This work presented a feedforward and feedback approach for controlling a dynamic system to track a reference state trajectory. Sufficient conditions for assuring stability of the closed-loop system were provided.

Future research will be conducted to refine the analyses and provide comprehensive numerical and experimental validations, and to further investigate the interplay between a model used in control design and the total disturbance to compensate.

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