Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Stabilization of Inventory System Performance: On/Off Control

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Abstract. Single-product inventory management model with both random and controllable demands and continuous input product flow with fixed uncontrolled rate under finite storage capacity is considered. We use the diffusion approximation of the stock level process. Optimal linear on/off control minimizing the variance of the stock level process in the steady-state case is investigated. The probabilities of the stock-out and overflow are controlled by the base stock-level, involving the storage capacity. The problem of minimizing the variance of the rate of delivering the product to outlets given the probability of the base-stock level exceeding is solved for nonlinear on/off control: continuous and discontinuous control is considered.

Keywords: Inventory Control, Diffusion Approximation, On-Off Control, Stationary Distribution.

1. PROBLEM STATEMENT

A systematic study of inventory models incorporated uncertainly and dynamics began in the early 50s from the works by Arrow, Harris, and Marschak (1951) and Dvoretzky, Kiefer, and Wolfowitz (1953). Nowadays a set of stochastic models are available to solve the inventory control problem under various conditions encountered in practice, for example, Ross (1992), Beyer, Cheng, Sethi, and Taksar (2010), Dolgui, and·Proth (2010), Chopra, and Meindl (2013).

The feature of the system under consideration is exogenous (i.e., outside our control) input product flow.

Our main contribution in this paper is in the field of a risk and its avoidance. Risk management has become an important field of supply chain management; see, for example, Tang and Musa (2010). Under uncontrolled input product flow shortages and overstocks are inevitable due to the uncertainty in the demand. Sometimes the impact of the overflow can have severe consequences, and we need to keep its possibility at some practically negligible level. The trade-off between the probabilities of the overflow and stock-out are also of interest.

The paper is also partly related to consignment stock policy which is increasingly studied as alternative to just-in-time inventory modeling, for more recent papers see Yi and Sarker (2014) and a critical survey by Sarker (2014).

Consider the stochastic inventory model that involves only a few parameters. Let $Q(t)$ be a stock level at time *t*, the product flow be continuous with fixed rate c_0, the demands be a Poisson process with constant intensity λ, the purchases values be independent identically distributed random variables with finite first and second moments equal respectively a_1 and a_2.

If the inventory level $Q(\cdot)$ is above some base-stock level we begin to deliver the product to outlets to prevent the overflow. The output flow is assumed to be continuous with a rate $c^*(Q)$.

The variance of a stochastic process is a widely used measure of risk: the less the variance the more the stability of the system. Our aim is to stabilize of the system's performance in the sense of minimum variance of 1) the stationary process $Q(\cdot)$ under controlled probabilities of the overflow and stock-out; 2) the rate of delivering the product to outlets given the probability of the base-stock level exceeding.

The model can be applied, for example, for water supplies systems.

Denote the rate of inventory movement due to non-random factors

$$c_0 - c^*(Q) = c(Q).$$

We consider Marcovian process $Q(\cdot)$ as diffusion process satisfying the following stochastic differential equation

$$dQ(t) = (c(Q) - a_1\lambda)dt + \sqrt{a_2\lambda}dw(t),$$

where $w(\cdot)$ is the Wiener process.

Diffusion methods have been applied to inventory models in a variety of domains to begin with the papers by Bather (1966) and Puterman (1975).

Because of the boundedness of $Q(\cdot)$ the stationary distribution exists

$$p(x) = C \cdot \exp\left(\frac{2}{a_2 \lambda} \int \left(c(x) - a_1 \lambda \right) dx \right), \qquad (1)$$

where C is the normalization constant.

We assume that the demands that occur during stock-out periods enter a pool of infinite capacity, it corresponds $Q < 0$.

2. LINEAR ON/OFF CONTROL

Let the storage capacity be bounded by $Q_{\max}$, the base-stock level be $Q_{\max} - Q_0$, and the rate of the output flow be proportional to the difference $Q - (Q_{\max} - Q_0)$:

$$c(Q) = \begin{cases} c_0, \text{ if } Q < Q_{\max} - Q_0, \\ c_0 - \beta\left(Q - (Q_{\max} - Q_0)\right), \text{ if } Q > Q_{\max} - Q_0, \end{cases} \qquad (2)$$

and $c_0 > a_1\lambda,\ \beta > 0$.

The condition $c_0 > a_1\lambda$ means that if the inventory level is below the base-stock level, then the stock level is replenished in the mean, that is, the resources are accumulated.

In view of (1) and (2) we get

$$p(x) = C \exp\left(\frac{2}{a_2\lambda}(c_0 - a_1\lambda)\left(x - (Q_{\max} - Q_0)\right) \right),$$

$$\text{if } x < Q_{\max} - Q_0;$$

and

$$p(x) = C \exp\left(\frac{2}{a_2\lambda}\left((c_0 - a_1\lambda)\left(x - (Q_{\max} - Q_0)\right) - \frac{\beta\left(x - (Q_{\max} - Q_0)\right)^2}{2} \right)\right), \text{if } x > Q_{\max} - Q_0,$$

where

$$C^{-1} = \int_{-\infty}^{Q_{\max} - Q_0} \exp\left(\frac{2}{a_2\lambda}(c_0 - a_1\lambda)\left(x - (Q_{\max} - Q_0)\right) \right) dx + \int_{Q_{\max} - Q_0}^{\infty} \exp\left(\frac{2}{a_2\lambda}\left((c_0 - a_1\lambda)\left(x - (Q_{\max} - Q_0)\right) - \frac{\beta\left(x - (Q_{\max} - Q_0)\right)^2}{2} \right)\right) dx = \frac{1 - 2b\Phi(b)\exp\left(b^2\right)}{2d},$$

$$d = \frac{c_0 - a_1\lambda}{a_2\lambda} > 0,\ b = -d\sqrt{\frac{a_2\lambda}{\beta}} < 0,\ \Phi(b) = \int_b^{\infty} \exp\left(-t^2\right) dt.$$

The expectation is $E(Q) =$

$$= C \int_{-\infty}^{Q_{\max} - Q_0} x \exp\left(\frac{2}{a_2\lambda}(c_0 - a_1\lambda)\left(x - (Q_{\max} - Q_0)\right) \right) dx + C \int_{Q_{\max} - Q_0}^{\infty} x \exp\left(\frac{2}{a_2\lambda}\left((c_0 - a_1\lambda)\left(x - (Q_{\max} - Q_0)\right) - \frac{\beta\left(x - (Q_{\max} - Q_0)\right)^2}{2} \right)\right) dx = \frac{1}{1 - 2b\Phi(b)\exp\left(b^2\right)} \Big(Q_{\max} - Q_0 - \frac{1}{2d} - 2b\left(\left(Q_{\max} - Q_0 - \frac{b^2}{d} \right)\Phi(b)\exp\left(b^2\right) - \frac{b}{2d} \right) \Big).$$

The variance is

$$Var(Q) = \frac{1}{2d^2}\left(b^2 + \frac{1 + b^2}{1 - 2b\Phi(b)\exp\left(b^2\right)} \right) = \frac{g(b)}{2d^2}. \qquad (3)$$

To obtain the value b giving minimal variance (3) we need to solve the following equation

$$\left(2\Phi(b)\exp(b^2) + b\right)\left(1 - b^2 + 2b^3\Phi(b)\exp(b^2)\right) = 0.$$

It is easy to see that under condition $b < 0$ the unique root b_0 exists.

Numerical computations give the following results

$$b_0 \approx -0.563, \quad g(b_0) \approx 0.734. \qquad (4)$$

Optimal value of parameter β is

$$\beta_0 = \frac{d(c_0 - a_1\lambda)}{b_0^2} = -\frac{d^2 a_2\lambda}{b_0}.$$

Consider the probability of the overflow

$$\alpha = P\left(Q > Q_{\max}\right) =$$

$$= C \int_{Q_{\max}}^{\infty} \exp\left(\frac{2}{a_2\lambda}\left((c_0 - a_1\lambda)\left(x - \left(Q_{\max} - Q_0\right)\right) - \right.\right.$$

$$\left.\left. - \frac{\beta\left(x - \left(Q_{\max} - Q_0\right)\right)^2}{2} \right)\right) dx = \frac{2b\Phi\left(b - \frac{d}{b}Q_0\right)\exp\left(b^2\right)}{2b\Phi(b)\exp\left(b^2\right) - 1}. \quad (5)$$

We can control probability $\alpha = \alpha\left(Q_0\right)$ given b_0 by choosing the value Q_0. We can find optimal value Q_0 given some α_0 by using (5), and since $1 - {b_0}^2 + 2{b_0}^3\Phi(b_0)\exp\left({b_0}^2\right) = 0$ we get

$$2{b_0}^3\Phi\left(b_0 - \frac{d}{b_0}Q_0\right)\exp\left({b_0}^2\right) = -\alpha_0.$$

The probability (5) decreases monotonically and takes maximal value

$$\alpha_{\max} = 1 + \frac{1}{2b_0\Phi(b_0)\exp\left(b_0^2\right) - 1}$$

for $Q_0 = 0$.

We use (4) to compute $\alpha_{\max} \approx 0,68$.

Minimal value of α under optimal control is

$$\alpha_{\min} = -2{b_0}^3\Phi\left(b_0 - \frac{d}{b_0}Q_{\max}\right)\exp\left({b_0}^2\right) \approx$$

$$\approx 0.48\Phi\left(-0.563 + \sqrt{\frac{\beta_0}{a_2\lambda}}Q_{\max}\right).$$

It follows that the storage capacity given desirable $\alpha_{\min}^0$ is

$$Q_{\max}^{0\alpha} \approx \left(\Psi\left(2.08\alpha_{\min}^0\right) + 0.563\right)\sqrt{\frac{a_2\lambda}{\beta_0}}, \quad (6)$$

where $\Psi(\cdot)$ is the inverse function of $\Phi(\cdot)$.

Obtain the probability of the stock-out $\gamma = P(Q < 0) =$

$$= C\int_{-\infty}^{0} \exp\left(\frac{2}{a_2\lambda}(c_0 - a_1\lambda)\left(x - \left(Q_{\max} - Q_0\right)\right)\right)dx =$$

$$= \frac{\exp\left(-2d\left(Q_{\max} - Q_0\right)\right)}{1 - 2b\Phi(b)\exp\left(b^2\right)}.$$

We can find optimal base-stock level given γ_0

$$Q_{\max} - Q_0 = \frac{2\ln b_0 - \ln\gamma_0}{2d}.$$

Maximal value of γ is

$$\gamma_{\max} = \frac{1}{1 - 2b_0\Phi(b_0)\exp\left(b_0^2\right)} \approx 0.316$$

for $Q_0 = Q_{\max}$.

Minimal value of γ is

$$\gamma_{\min} = \frac{\exp\left(-2dQ_{\max}\right)}{1 - 2b_0\Phi(b_0)\exp\left(b_0^2\right)} \approx 0.316\exp\left(-2dQ_{\max}\right).$$

It follows that the storage capacity given desirable $\gamma_{\min}^0$ is

$$Q_{\max}^{0\gamma} \approx -\frac{\ln\left(3.16\gamma_{\min}^0\right)}{2d}. \quad (7)$$

Combining (6) and (7) we obtain the storage capacity $Q_{\max}^0$ giving it possible to choose $\alpha \in \left[\alpha_{\min}; 0.68\right]$ and $\gamma \in \left[\gamma_{\min}; 0.316\right]$

$$Q_{\max}^0 = \max\left(\left(\Psi\left(2.08\alpha_{\min}\right) + 0.563\right)\sqrt{\frac{a_2\lambda}{\beta_0}}, -\frac{\ln\left(3.16\gamma_{\min}\right)}{2d}\right).$$

Then, we need to compute the corresponding level Q_0. Our choice of the probabilities depends on some loss function defining our preferences.

Note that the probability of overflow $\alpha_{\max} = 0.68$ is more than twice the probability of stock-out $\gamma_{\max} = 0.316$ under optimal control. So consider more complicated, nonlinear models, of controlled output flow.

3. NONLINEAR ON/OFF CONTROL

Let us assume that the density function of the stock level Q satisfies (1).

Consider the problem of minimizing of the variance of the rate of delivering the product to outlets given the probability of the base-stock level exceeding

$$Var(c(Q)) \to \min_{c(Q)} \quad \text{given} \quad \mathrm{P}(\mathrm{Q} > Q_0) = \pi_1 . \qquad (8)$$

Here we do not need to assume beforehand the boundedness of the storage capacity.

The probability density function of Q is

$$p(s) = \begin{cases} C \exp \dfrac{2}{a_2\lambda}(c_0 - a_1\lambda)(s - Q_0), & s < Q_0, \\ Cg(s), & s > Q_0, \end{cases}$$

where $C = \left(\dfrac{1}{2\tilde{c}} + \int_{Q_0}^{\infty} g(s)ds \right)^{-1}$, $\tilde{c} = \dfrac{c_0 - a_1\lambda}{a_2\lambda}$,

$$g(s) = \exp\left(\frac{2}{a_2\lambda} \int_{Q_0}^{s} \left(c(x) - a_1\lambda\right)dx \right).$$

The expectation is

$$E(c(Q)) = Ca_1\lambda, \text{ and } P(Q > Q_0) = C\int_{Q_0}^{\infty} g(s)ds .$$

Since $c(s) = \dfrac{a_2\lambda}{2}\dfrac{g'(s)}{g(s)} + a_1\lambda$ we get that

$$Var(c(Q)) + C^2 a_1^2\lambda^2 =$$

$$= C\left(\frac{c_0^2}{2\tilde{c}} + \int_{Q_0}^{\infty} \left(\frac{a_2^2\lambda^2}{4}\frac{g'^2(s)}{g(s)} + a_1 a_2\lambda^2 g'(s) + a_1^2\lambda^2 g(s) \right) ds \right).$$

So we have the following minimization problem

$$\int_{Q_0}^{\infty} \left(\frac{a_2^2\lambda^2}{4}\frac{g'^2(s)}{g(s)} + a_1 a_2\lambda^2 g'(s) + a_1^2\lambda^2 g(s) \right) ds \to \min_{g(s)}$$

given $\int_{Q_0}^{\infty} g(s)\,ds$.

The Euler equation is

$$\phi' + \phi^2 = \frac{a_1^2\lambda^2 + \lambda^*}{a_2^2\lambda^2}, \qquad (9)$$

where $\phi = \dfrac{g'(s)}{2g(s)}$, λ^* is a Lagrange multiplier.

Clearly, if $\dfrac{a_1^2\lambda^2 + \lambda^*}{a_2^2\lambda^2} > 0$ then function $g(\cdot)$ satisfying both (9) and the constraint does not exist.

Let $\dfrac{a_1^2\lambda^2 + \lambda^*}{a_2^2\lambda^2} = -\beta^2$ then the solution of (9) is $\varphi = \beta\,\mathrm{tg}(\overline{c} - \beta s), \quad \beta > 0$, and

$$c(s) = a_2\lambda\beta\,\mathrm{tg}(\overline{c} - \beta s) + a_1\lambda , \qquad (10)$$

where $\overline{c}$ is a constant.

3.1 Continuous nonlinear control

Consider the case of continuous control, that is, let us find the constant $\overline{c}$ from condition $c(Q_0) = c_0$. We get $\overline{c} = \mathrm{arctg}\dfrac{\tilde{c}}{\beta} + \beta Q_0$, and maximal value of Q is

$$Q_{\max} = \frac{\overline{c} + \pi/2}{\beta} = \frac{1}{\beta}\left(\mathrm{arctg}\frac{\tilde{c}}{\beta} + \frac{\pi}{2} \right).$$

Since $g(Q_0) = 1$ function $g(\cdot)$ is

$$g(s) = \frac{\cos^2(\overline{c} - \beta s)}{\cos^2(\overline{c} - \beta Q_0)} = \left(1 + \gamma^2\right)\cos^2\left[\mathrm{arctg}\,\gamma - \beta\left(s - Q_0\right)\right],$$

where $\gamma = \tilde{c}/\beta$.

Then, we find

$$\int_{Q_0}^{Q_{\max}} g(s)ds = \int_{Q_0}^{Q_{\max}} (1+\cos 2(\overline{c}-\beta s))ds =$$

$$= \frac{1+\gamma^2}{2\beta}\left(\operatorname{arctg}\gamma + \frac{\pi}{2} + \frac{\gamma}{1+\gamma^2}\right),$$

and the constraint is

$$\gamma\left(1+\gamma^2\right)(\operatorname{arctg}\gamma + \pi/2) + \gamma^2 = \frac{\pi_1}{1-\pi_1}. \quad (11)$$

The equation has only one positive root $0 < \gamma_0 < \sqrt{\frac{\pi_1}{1-\pi_1}}$.

So optimal rate of continuously controlled output flow is

$$c^*(s) = \frac{a_2\lambda\tilde{c}\left(1+\gamma_0^2\right)}{\gamma_0} \frac{\operatorname{tg}\left(\frac{\tilde{c}}{\gamma_0}(s-Q_0)\right)}{1+\gamma_0 \operatorname{tg}\left(\frac{\tilde{c}}{\gamma_0}(s-Q_0)\right)}, \quad Q_{\max} > s > Q_0.$$

Note that the needed storage capacity is

$$Q_{\max} = Q_0 + \frac{\gamma_0}{\tilde{c}}\left(\operatorname{arctg}\gamma_0 + \frac{\pi}{2}\right).$$

3.2 Discontinuous nonlinear control

Let us find a constant $\overline{c}$ in (10) from condition $c(Q_0) = c_1$.

Denote $\tilde{c}_1 = \frac{c_1 - a_1\lambda}{a_2\lambda}, \gamma_1 = \frac{\tilde{c}_1}{\beta}$, we get

$$\overline{c} = \operatorname{arctg}\gamma_1 + \beta Q_0,\ Q_0 = \frac{\operatorname{arctg}\gamma_1 + \pi/2}{\beta},$$

$$g(s) = \left(1+\gamma_1^2\right)\cos^2(\overline{c}-\beta s).$$

The constraint is

$$\gamma\left(1+\gamma_1^2\right)\left(\operatorname{arctg}\gamma_1 + \pi/2\right) + \gamma\gamma_1 = \frac{\pi_1}{1-\pi_1}. \quad (12)$$

The variance of $c(s)$ is

$$Var(c(Q)) = a_1^2\lambda^2\pi_1 +$$

$$+(1-\pi_1)\left(c_0^2 - 2\tilde{c}a_1a_2\lambda^2 + a_2^2\lambda^2\tilde{c}^2 \frac{\left(1+\gamma_1^2\right)\left(\operatorname{arctg}\gamma_1 + \pi/2\right) - \gamma_1}{\gamma}\right). \quad (13)$$

Obtain the minimal value of $Var(c(Q))$ as the function of γ_1 (we express γ from (12) and substitute it into (13)). The task reduces to computation the minimum of the function

$$f(\gamma_1) = \left(1+\gamma_1^2\right)^2(\operatorname{arctg}\gamma_1 + \pi/2)^2 - \gamma_1^2.$$

Clearly, the function increases monotonically, and $\lim_{\gamma_1\to-\infty} f(\gamma_1) \approx 1{,}33$. In order to satisfy the condition $\gamma_1 \to -\infty$, we take $c_1 < a_1\lambda$ and require β tends to zero.

Compute

$$\lim_{\gamma_1\to-\infty} c(s) = a_1\lambda + a_2\lambda\tilde{c}_1 \lim_{\gamma_1\to-\infty} \frac{\operatorname{tg}\left(\operatorname{arctg}\gamma_1 - \tilde{c}_1(s-Q_0)/\gamma_1\right)}{\gamma_1} =$$

$$= a_1\lambda + a_2\lambda\tilde{c}_1\left(1+\tilde{c}_1(s-Q_0)\right)^{-1}.$$

The equation (12) can be rewritten as

$$\gamma_1\left(1+\gamma_1^2\right)\left(\operatorname{arctg}\gamma_1 + \pi/2\right) + \gamma_1^2 = \frac{\pi_1}{1-\pi_1}\frac{c_1 - a_1\lambda}{c_0 - a_1\lambda}.$$

Compute the limit

$$\lim_{\gamma_1\to-\infty}\left(\gamma_1\left(1+\gamma_1^2\right)\left(\operatorname{arctg}\gamma_1 + \pi/2\right) + \gamma_1^2\right) = -2/3,$$

it follows that $\tilde{c}_1 = \frac{2}{3}\tilde{c}\left(1 - \frac{1}{\pi_1}\right)$.

So the optimal rate of controlled output flow is

$$c^*(s) = a_2\lambda\tilde{c}\left(1 - \left(1.5\frac{\pi_1}{\pi_1 - 1} + \tilde{c}(s-Q_0)\right)^{-1}\right),\ Q_{\max} > s \geq Q_0.$$

The needed storage capacity is

$$Q_{\max} = Q_0 + \frac{1.5\pi_1}{\tilde{c}(1-\pi_1)}.$$

So abandoning the continuity we can reduce the variance of $c(Q)$ under the probability of the base-stock level exceeding being fixed. At the point $s = Q_0$ we have the jump

$$\frac{a_2\lambda\tilde{c}}{3}\left(1+\frac{2}{\pi_1}\right) = \frac{c_0 - a_1\lambda}{3}\left(1+\frac{2}{\pi_1}\right).$$

Given the probability of stock-out $\overline{\gamma}$

$$P(Q<0) = \frac{\exp(-2\tilde{c}Q_0)}{1-\pi_1} = \overline{\gamma}$$

we get the base stock level

$$Q_0 = -\frac{\ln(1-\pi_1) + \ln\overline{\gamma}}{2\tilde{c}}.$$

ACKNOWLEGEMENTS

The author would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

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