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The Target Guarding Problem Revisited: Some Interesting Revelations

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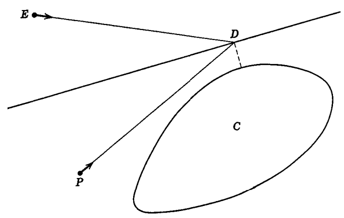
Abstract: In this paper, the target guarding problem, posed by Rufus Isaacs in his seminal textbook is revisited. Unlike the simplified version of the problem solved by Isaacs, optimal strategies are discussed for the pursuer and the evader having simple motions with different speeds. Analysis is also done for the case when the intercept occurs if the pursuer reaches within a certain distance of the evader, instead of a strict coincidence of their positions. The importance of the study is that, it is fundamental to building autonomous systems that would be used for protecting a target, for example, unmanned vehicles used in anti-poaching operations.

1. INTRODUCTION

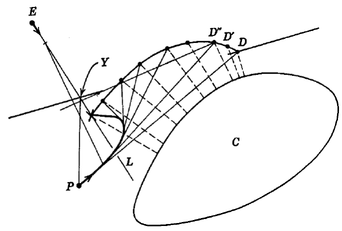
The target guarding problem was first introduced by Isaacs [1965] in his pioneering work on differential games. It consists of a player (the evader E) trying to reach an immobile target T, and another player (the pursuer P) trying to intercept the evader before he reaches the target. The objective of the game for P is to maximize the distance between E and T at intercept, and that for E is to minimize it. Therefore, the best payoff for E would be to reach T without being intercepted.

Isaacs presented the solution to the target guarding problem, assuming equal speeds for P and E. He also assumed that interception corresponds to coincidence of the positions of P and E. When both P and E have equal speeds, capture occurs along the perpendicular bisector of their initial positions. The target to be guarded from E is an area C, which lies in the zone P can reach prior to E. The optimal strategy for either player in this case would be to head to the point closest to C on the perpendicular bisector, as shown in Fig. 1(a). If only P plays optimally, and E chooses to traverse a sub-optimal path, P traverses a curved path resulting in a better payoff for him at intercept, as shown in Fig. 1(b). Similarly, if E plays optimally, but P does not, E could well reach the target C, thus securing a better payoff for himself, as shown in Fig. 1(c).

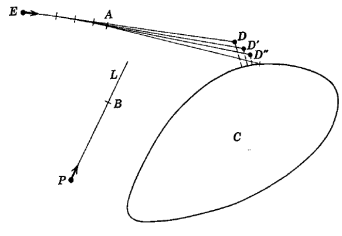
In spite of being one of the earliest problems in differential games, the target guarding problem is one of the least studied in the literature. Unlike the problem solved by Isaacs, in reality, the players are hardly at the same speeds. Usually, the pursuer is faster than the evader. Also, intercept will occur when the pursuer reaches within a certain distance of the evader. The target might also be mobile. These variabilities to the target guarding problem have not been studied in the literature. Guarding a moving target would be the fundamental problem of interest in autonomous agents guarding mobile targets, for example, unmanned vehicles that could be used in anti-poaching operations to protect animals from poachers.



(a) P and E play optimally



(b) P plays optimally, but E does not



(c) E plays optimally, but P does not

Fig. 1. Target guarding problem from Isaacs [1965]

A variant of the problem of guarding a moving target was studied by Rusnak [2005] called “the lady, the bandits and the body guards” game. It consists of the bandits trying to capture the lady, and the body guards trying to intercept the bandits before they catch the lady. In this problem, the lady cooperates with the body guards and actively tries to evade the bandits. However, in the problem of guarding a moving target, the target remains neutral in the game and does not actively cooperate with the rescuers and perform evasive action – this problem applies to, for example, in the case of guarding unsuspecting animals which get ambushed by poachers. Lau and Liu [2013] presented an online path planning algorithm using Rapidly-exploring Random Trees (RRT) for unmanned vehicles in charge of autonomous border patrol, based on Isaacs' target guarding problem. The target considered in the study was static. The concept of RRT was used in the study because solving the differential game to obtain optimal strategies would be computationally intensive and could become infeasible for real-time systems. Therefore, the feasibility of solving the differential game would have to be considered while building real-time systems for anti-poaching operations. If found infeasible, algorithms like RRT should be explored.

2. GEOMETRY OF CAPTURE WHEN THE SPEEDS OF P AND E ARE DIFFERENT

As mentioned in the previous section, Isaacs analyzed the target guarding problem when the speeds of P and E are equal and execute simple motions. In this case, the perpendicular bisector of the line connecting their initial positions divides the playing space into regions that either of them can reach prior to the other. In other words, when both P and E travel straight towards the same point on the perpendicular bisector, they arrive at the same instant, and capture occurs. However, if the speeds of P and E are different, capture would not occur on the perpendicular bisector. To find the curve on which capture occurs, let us assume that the ratio of the maximum speeds of P and E be $p : e$. Therefore, let the maximum speed of P be $v_p = pv$, and that of E be $v_e = ev$, where v is some common factor. Let t be the time elapsed until capture. Both P and E will travel straight at their maximum speeds because decreasing their speed or traveling in a curved path would benefit the opponent. Let $E_0\left(x_e, y_e\right)$ be the initial position of E and $P_0\left(x_p, y_p\right)$ be that of P.

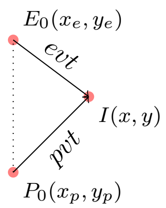


Fig. 2. Game geometry when $v_p \neq v_e$ and capture is the coincidence of P and E

2.1 Capture Occurs when the Positions of P and E Coincide

Let us assume that the condition for capture is the coincidence of positions of P and E. Let $I(x, y)$ be the point where capture occurs, as shown in Fig. 2. Then, the distances traveled by P and E until capture would be pvt and evt respectively, and can be expressed mathematically as:

$$(x - x_e)^2 + (y - y_e)^2 = (v_e t)^2 = (evt)^2 \tag{1}$$

$$(x - x_p)^2 + (y - y_p)^2 = (v_p t)^2 = (pvt)^2 \tag{2}$$

Dividing (1) by (2), we get

$$\frac{(x - x_e)^2 + (y - y_e)^2}{(x - x_p)^2 + (y - y_p)^2} = \frac{e^2}{p^2} \tag{3}$$

$$\begin{aligned}\implies & p^2\left[\left(x^2 - 2xx_e + x_e^2\right) + \left(y^2 - 2yy_e + y_e^2\right)\right] \\ & = e^2\left[\left(x^2 - 2xx_p + x_p^2\right) + \left(y^2 - 2yy_p + y_p^2\right)\right]\end{aligned} \tag{4}$$

$$\begin{aligned}\implies & \left(p^2 - e^2\right)x^2 + \left(p^2 - e^2\right)y^2 \\ & + 2\left(x_p e^2 - x_e p^2\right)x + 2\left(y_p e^2 - y_e p^2\right)y \\ & + p^2\left(x_e^2 + y_e^2\right) - e^2\left(x_p^2 + y_p^2\right) = 0\end{aligned} \tag{5}$$

$$\begin{aligned}\implies & x^2 + 2\frac{\left(x_p e^2 - x_e p^2\right)}{\left(p^2 - e^2\right)}x \\ & + y^2 + 2\frac{\left(y_p e^2 - y_e p^2\right)}{\left(p^2 - e^2\right)}y \\ & + \frac{p^2\left(x_e^2 + y_e^2\right) - e^2\left(x_p^2 + y_p^2\right)}{\left(p^2 - e^2\right)} = 0\end{aligned} \tag{6}$$

Let $x_c = \frac{\left(x_e p^2 - x_p e^2\right)}{\left(p^2 - e^2\right)}$ and $y_c = \frac{\left(y_e p^2 - y_p e^2\right)}{\left(p^2 - e^2\right)}$. Then, (6) becomes

$$\begin{aligned} & x^2 - 2xx_c + y^2 - 2yy_c \\ & + \frac{p^2\left(x_e^2 + y_e^2\right) - e^2\left(x_p^2 + y_p^2\right)}{\left(p^2 - e^2\right)} = 0\end{aligned} \tag{7}$$

$$\begin{aligned}\implies & \left(x - x_c\right)^2 + \left(y - y_c\right)^2 \\ & + \frac{p^2\left(x_e^2 + y_e^2\right) - e^2\left(x_p^2 + y_p^2\right)}{\left(p^2 - e^2\right)} - x_c^2 - y_c^2 = 0\end{aligned} \tag{8}$$

$$\implies \boxed{\left(x - x_c\right)^2 + \left(y - y_c\right)^2 = R^2} \tag{9}$$

where $R = \sqrt{x_c^2 + y_c^2 - \frac{p^2\left(x_e^2 + y_e^2\right) - e^2\left(x_p^2 + y_p^2\right)}{\left(p^2 - e^2\right)}}$. Equation (9) is that of a circle with center $C(x_c, y_c)$ and radius R. Therefore, capture of E by P occurs on this circle. Also, the slope of the line connecting $C\left(x_c, y_c\right)$ and $E_0\left(x_e, y_e\right)$ is given by,

$$\frac{y_c - y_e}{x_c - x_e} = \frac{\frac{\left(y_e p^2 - y_p e^2\right)}{\left(p^2 - e^2\right)} - y_e}{\frac{\left(x_e p^2 - x_p e^2\right)}{\left(p^2 - e^2\right)} - x_e} = \frac{y_e - y_p}{x_e - x_p} \equiv m_{E_0 P_0} \tag{10}$$

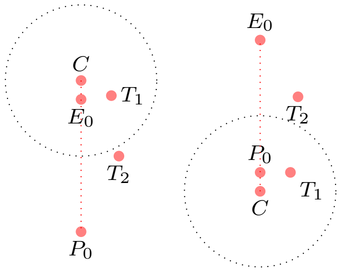
where $m_{E_0 P_0}$ is the slope of the line $E_0 P_0$. Therefore, C lies on $E_0 P_0$. The capture circle looks like as shown in Fig. 3(a) and Fig. 3(b) for $v_e < v_p$ and $v_e > v_p$ respectively. It encloses E_0 when $v_e < v_p$ and P_0 when $v_e > v_p$. C always lies outside $E_0 P_0$. When the speeds of P and E are equal, $\left(p^2 - e^2\right) = 0$, and therefore, (5) reduces to

$$\begin{aligned} & 2\left(x_p - x_e\right)x + 2\left(y_p - y_e\right)y \\ & + \left(x_e^2 + y_e^2\right) - \left(x_p^2 + y_p^2\right) = 0\end{aligned} \tag{11}$$

which is a straight line with slope $m_1 = -\frac{x_p - x_e}{y_p - y_e}$. It can be seen that this line passes through the mid-point of $E_0 P_0$, $\left(\frac{x_e + x_p}{2}, \frac{y_e + y_p}{2}\right)$, and $m_1 m_{E_0 P_0} = -1$. This means that (11) is the perpendicular bisector of $E_0 P_0$, which agrees with the result derived in Isaacs [1965].

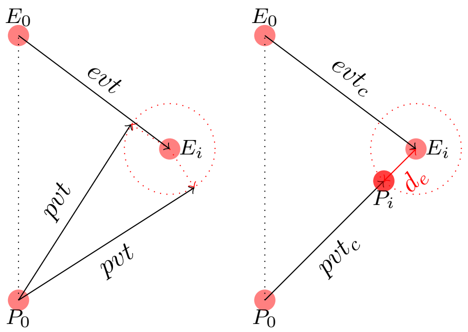
2.2 Capture Occurs when P Reaches within a Distance d_e of E

Let us now assume that the condition for capture is P reaching a distance of d_e from E. At capture, let



(a) $v_e < v_p$ (b) $v_e > v_p$

Fig. 3. Capture curve when $v_p \neq v_e$ and capture is the coincidence of P and E, and possible locations of a point target T



(a) Two possible positions for P (b) Unique position for P

Fig. 4. Game geometry when $v_p \neq v_e$ and capture occurs when P reaches within a distance of d_e from E

the position of P be $P_i\,(x_{pi}, y_{pi})$ and E be $E_i\,(x_{ei}, y_{ei})$. Capture occurs when $|P_i - E_i| = d_e$. Therefore, we have

$$(x_{ei} - x_e)^2 + (y_{ei} - y_e)^2 = (evt)^2 \tag{12}$$

$$(x_{pi} - x_p)^2 + (y_{pi} - y_p)^2 = (pvt)^2 \tag{13}$$

$$(x_{ei} - x_{pi})^2 + (y_{ei} - y_{pi})^2 = d_e^2 \tag{14}$$

The above equations present us with a geometry as shown in Fig. 4(a). At capture, we have a disc of radius d_e centered at E_i, and P_i can be anywhere on the periphery of this disc. However, as P started from P_0 and traversed a distance of pvt, with P_0 as center and pvt as radius, we draw an arc that intersects the disc at two points, which are the possible locations of P_i. For a finite duration after the game begins, pvt would not be big enough to touch the disc centered at E. At some critical time t_c, pvt touches the disc and later, it intersects the disc at two points, which is shown in Fig. 4(a). As P desires to minimize the distance E travels, he should capture E at the critical time t_c, when the arc of radius pvt_c centered at P_0 just touches the disc of radius d_e centered at E. Any longer than t_c, E would have traveled further, and P would then face the dilemma of capturing E at two possible positions. Therefore, let us assume that capture occurs at $t = t_c$ henceforth. The geometry of the game would then look like as shown in Fig. 4(b). From the figure, it is clear that at capture, P_i lies on the line P_0E_i. So we have the additional condition:

$$(x_{ei} - x_p)^2 + (y_{ei} - y_p)^2 = (pvt + d_e)^2 \tag{15}$$

Dividing (12) by (13) and cross-multiplying the terms, we have

$$\begin{aligned} &p^2\left[(x_{ei} - x_e)^2 + (y_{ei} - y_e)^2\right] \\ &= e^2\left[(x_{pi} - x_p)^2 + (y_{pi} - y_p)^2\right] \end{aligned} \tag{16}$$

Equations (14) - (16) do not simplify any further.

To gain more insight into the locus of points where capture would occur, without loss of generality, let us fix the origin of the coordinate system at P_0, and let the y-axis coincide with P_0E_0. Therefore, the coordinates of the initial positions of P and E would now be $P_0(0,0)$ and $E_0(0, y_e)$. Like before, the coordinates of the positions of P and E at capture would remain as $P_i(x_{pi}, y_{pi})$ and $E_i(x_{ei}, y_{ei})$. If E were to travel straight vertically down from E_0, P would travel vertically up from P_0, and capture would occur when P and E have traveled distances of $d_1 = p\left(\frac{y_e - d_e}{p+e}\right)$ and $d_2 = e\left(\frac{y_e - d_e}{p+e}\right)$ respectively. This is because the triangle of Fig. 4(b) would flatten out to the line segment P_0E_0 in such a case. Therefore, if E takes any other path than vertically down, he will get to travel at least the distance d_2 before running the risk of being captured, and P will have to travel at least the distance d_1 before getting a chance to capture. Let $vt = l$. Therefore, in the new coordinate system, (12) - (15) become:

$$x_{ei}^2 + (y_{ei} - y_e)^2 = e^2 l^2 \tag{17}$$

$$x_{pi}^2 + y_{pi}^2 = p^2 l^2 \tag{18}$$

$$x_{ei}^2 + y_{ei}^2 = (pl + d_e)^2 \tag{19}$$

$$(x_{ei} - x_{pi})^2 + (y_{ei} - y_{pi})^2 = d_e^2 \tag{20}$$

Calculating l from (17) and substituting in (19), we have

$$x_{ei}^2 + y_{ei}^2 = \left(\frac{p}{e}\sqrt{x_{ei}^2 + (y_{ei} - y_e)^2} + d_e\right)^2 \tag{21}$$

As E_i lies on the line joining P_i and P_0, the slope of the line is given by

$$m = \frac{y_{ei}}{x_{ei}} = \frac{y_{pi}}{x_{pi}} = \frac{y_{ei} - y_{pi}}{x_{ei} - x_{pi}} \tag{22}$$

If the locus of the point P_i is desired, subtracting (19) from (17) yields

$$(pl + d_e)^2 - 2y_e y_{ei} + y_e^2 = e^2 l^2 \tag{23}$$

$$\implies y_{ei} = \frac{(pl + d_e)^2 + y_e^2 - e^2 l^2}{2y_e} \tag{24}$$

$$\begin{aligned} \implies x_{ei} &= \frac{y_{ei}}{m} = \frac{y_{ei}}{y_{pi}} x_{pi} \\ &= \frac{x_{pi}}{y_{pi}}\left[\frac{(pl + d_e)^2 + y_e^2 - e^2 l^2}{2y_e}\right] \end{aligned} \tag{25}$$

where we have substituted the expression for y_{ei} from (24). Substituting (24) and (25) in (20), we have

$$\left(\frac{y_{ei}}{m} - x_{pi}\right)^2 + (y_{ei} - y_{pi})^2 = d_e^2 \tag{26}$$

$$\begin{aligned} &\implies \left(\frac{x_{pi}}{y_{pi}}\left[\frac{(pl + d_e)^2 + y_e^2 - e^2 l^2}{2y_e}\right] - x_{pi}\right)^2 \\ &+ \left(\left[\frac{(pl + d_e)^2 + y_e^2 - e^2 l^2}{2y_e}\right] - y_{pi}\right)^2 = d_e^2 \end{aligned} \tag{27}$$

Substituting the expression for l from (18) in (27), we have

$$\Longrightarrow \left(\frac{x_{pi}}{y_{pi}}\left[\frac{\left(\sqrt{x_{pi}^2+y_{pi}^2}+d_e\right)^2+y_e^2-\frac{e^2}{p^2}\left(x_{pi}^2+y_{pi}^2\right)}{2y_e}\right]-x_{pi}\right)^2 + \left(\left[\frac{\left(\sqrt{x_{pi}^2+y_{pi}^2}+d_e\right)^2+y_e^2-\frac{e^2}{p^2}\left(x_{pi}^2+y_{pi}^2\right)}{2y_e}\right]-y_{pi}\right)^2 = d_e^2 \tag{28}$$

Equation (28) is a 12$^{\text{th}}$ degree polynomial. When $p = e$, it reduces to a 8$^{\text{th}}$ degree polynomial.

And when $d_e = 0$, (28) further reduces to

$$\left(\frac{x_{pi}}{y_{pi}}\underbrace{\left[\frac{\left(x_{pi}^2+y_{pi}^2\right)+y_e^2-\frac{e^2}{p^2}\left(x_{pi}^2+y_{pi}^2\right)}{2y_e}\right]}_{A}-x_{pi}\right)^2 + \left(\underbrace{\left[\frac{\left(x_{pi}^2+y_{pi}^2\right)+y_e^2-\frac{e^2}{p^2}\left(x_{pi}^2+y_{pi}^2\right)}{2y_e}\right]}_{A}-y_{pi}\right)^2 = d_e^2 \tag{29}$$

$$\Longrightarrow \left(\frac{x_{pi}^2}{y_{pi}^2}+1\right)\left(A-y_{pi}\right)^2 = 0 \tag{30}$$

$$\Longrightarrow \boxed{A - y_{pi} = 0} \tag{31}$$

$$\therefore \left(1-\frac{e^2}{p^2}\right)\left(x_{pi}^2+y_{pi}^2\right)+y_e^2-2y_e y_{pi} = 0 \tag{32}$$

$$\Longrightarrow \left(x_{pi}^2+y_{pi}^2\right)+\frac{y_e^2-2y_e y_{pi}}{\left(1-\frac{e^2}{p^2}\right)} = 0 \tag{33}$$

$$\Longrightarrow x_{pi}^2+\left(y_{pi}-\frac{y_e}{1-\frac{e^2}{p^2}}\right)^2 = \left(\frac{e}{p}\frac{y_e}{1-\frac{e^2}{p^2}}\right)^2 \tag{34}$$

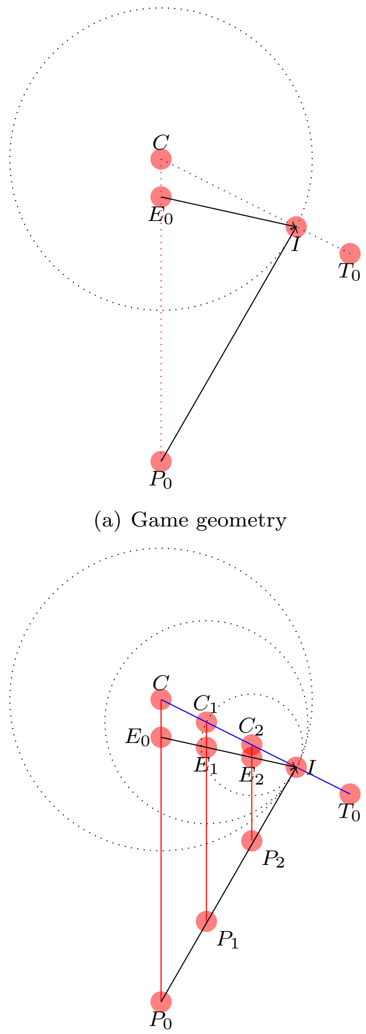
Equation (34) is that of a circle with center at C$\left(0, \frac{y_e}{1-\frac{e^2}{p^2}}\right)$ and radius $R = \frac{e}{p}\frac{y_e}{1-\frac{e^2}{p^2}}$, which is the circle of (9) in the new coordinate system, where $x_p = y_p = x_e = 0$. When $e < p$, the circle encloses $E_0\,(0, y_e)$ and when $e > p$, the circle encloses the origin $P_0(0,0)$.

In addition, when $p = e$, $A = \frac{y_e}{2}$ and (31) becomes

$$\frac{y_e}{2} - y_{pi} = 0 \tag{35}$$

which is the perpendicular bisector of P_0E_0 in the new coordinate system.

From the above analysis, it is clear that if the criterion for capture is a proximity of d_e between P and E, the locus of P_i is a nontrivial 12$^{\text{th}}$ degree polynomial. Even when $v_p = v_e$, locus of P_i is not a simple curve. Only when the criterion for capture is a simple coincidence of positions, the locus of P_i (and E_i) simplify to a circle. And when $v_p = v_e$, the locus further simplifies to the perpendicular bisector of P_0E_0.



(a) Game geometry

(b) Game dynamics

Fig. 5. Optimal strategies when T is immobile, for $v_e < v_p$

The above analysis has some interesting revelations about the target guarding problem. To derive optimal strategies for P and E, it is essential to understand the loci of P_i and E_i. These loci are simple only when d_e can be neglected in (27), i.e., when $d_e \ll pl$, so that $pl + d_e \approx pl$. In other words, when the proximity for capture d_e is negligible compared to the distance P travels till capture (pl), it is relatively easier to calculate optimal strategies. However, if d_e is comparable to pl for the problem in hand, even when $p = e$, the locus of P_i is not the perpendicular bisector of P_0E_0 as derived in Isaacs [1965].

3. OPTIMAL STRATEGIES FOR P AND E WHEN T IS IMMOBILE

With a picture of the loci of P_i and E_i, it is possible to derive optimal strategies for P and E when playing with a point target T. As shown in the previous section, if d_e cannot be neglected compared to pl, the loci are complicated and therefore, the derivation of optimal strategies is not straight forward. In many practical scenarios, the assumption that $d_e \ll pl$ could well be satisfactory, in which case, P_i and E_i would coalesce into the capture point I. Therefore, the locus of I would be the capture circle (referred to as C_I henceforth) given by (9), which will be used in the following sections.

3.1 Case $v_e < v_p$

In this case, C_I encloses E_0. T could be in two possible locations - within C_I at T_1 or outside C_I at T_2 as shown in Fig. 3(a). If T is at T_1, E would reach T_1 before P and the game ends with T being sacrificed. However, if T is at T_2, he could be guarded. Let T be located at a position T_0 outside C_I as shown in Fig. 5(a). P strives to keep E as away from T as possible and E strives to do the opposite. E will choose to head to the point closest to T_0 on C_I to make sure that even if captured, he is at the minimum possible distance from T at capture. This point I is the intersection of the radial line CT_0 with C_I, as shown in the figure. P too will head to I expecting E to play optimally.

When the game proceeds optimally on both sides, the dynamics of the game look like as shown in Fig. 5(b). The locations of P and E at two instants during the course of the game are shown as P_1, P_2 and E_1, E_2 respectively. The instantaneous capture circles C_{I_1} and C_{I_2} are drawn at these two instants and their centers are marked as C_1 and C_2 respectively. As both P and E have been playing optimally, i.e., traveling straight along P_0I and E_0I respectively right from the beginning, the center C of the instantaneous capture circle C_I also moves along the line CI. In fact, $P_0E_0 \parallel P_1E_1 \parallel P_2E_2$ because $\triangle E_0P_0I \sim \triangle E_1P_1I \sim \triangle E_2P_2I$. And by (11), C_1 and C_2 should lie along P_1E_1 and P_2E_2 respectively. The instantaneous capture circle C_{I_t} (the subscript I's subscript t being the time instant under consideration) reduces in size as the game proceeds, finally coalescing to the capture point I. Also, all the instantaneous capture circles C_{I_t}'s touch each other at I. The optimal strategies for P and E remain the same at all instants during the game. Both of them head towards the point of intersection of the instantaneous radial line C_tT_0 with the instantaneous capture circle C_{I_t}, which remains as I if P and E play optimally through out the game.

The capture point $I(x_i, y_i)$ can be found in the following way. Let the coordinates of T be $T_0(x_t, y_t)$. I lies on the line CT_0. Therefore, it's slope

$$m = \frac{y_i - y_t}{x_i - x_t} = \frac{y_c - y_t}{x_c - x_t} \tag{36}$$

$$\implies y_i = m\,(x_i - x_t) + y_t \tag{37}$$

As I also lies on C_I given by (9),

$$(x_i - x_c)^2 + (y_i - y_c)^2 = R^2 \tag{38}$$

$$\tag{39}$$

Substituting for y_i from (37) in (38), we have

$$(x_i - x_c)^2 + [m\,(x_i - x_t) + y_t - y_c]^2 = R^2 \tag{40}$$

$$\implies \left(1 + m^2\right) x_i^2 + 2\,(mk - x_c)\,x_i + x_c^2 + k^2 - R^2 = 0 \tag{41}$$

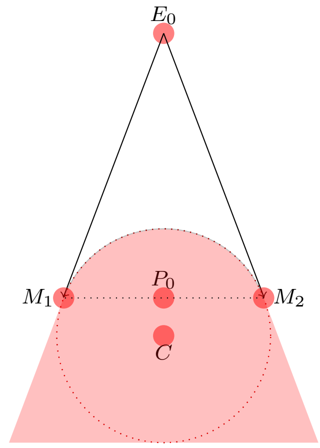
where $k = y_t - y_c - mx_t$. Solving (41), we get

$$x_i = \frac{-\,(mk - x_c) \pm \sqrt{(1+m^2)\,R^2 - (mx_c + k)^2}}{1 + m^2} \tag{42}$$

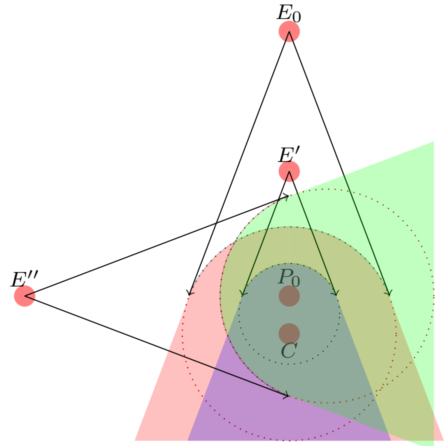
Each x_i has a corresponding y_i. Out of these two points, the point closest to the target $T(x_t, y_t)$ is I.

3.2 Case $v_e > v_p$

The capture circle C_I encloses P_0 in this case, and like before, T could be either at T_1 inside C_I or at T_2 outside as shown in Fig. 3(b). It is quite obvious that if T is at T_2, E can reach there before P and T is lost. However, if T is at T_1, one might be tempted to say that P can save T. But it turns out that E will certainly reach T wherever he is located in the plane, as shown later. Therefore, the best P can do is to reach T's location and wait for E to capture him there.



(a) Range of locations E can reach



(b) Range of locations E can reach depending on position and orientation w.r.t P

Fig. 6. Game dynamics when both P and E play optimally

To understand the dynamics of the game when $v_e > v_p$, it is important to understand the range of positions that E can reach for a given position of P. Let E be at $E_0(0, y_e)$ and P at $P_0(0, 0)$ as shown in Fig. 6(a), in accordance with the simplified coordinate system used in section 2.2. C_I is then given by (34), which is shown in the figure. The tangents drawn from E_0 to C_I meet at M_1 and M_2. Now, the range of locations that E can reach safely without any chance of being captured by P is the unshaded region in Fig. 6. This is because if E tries to reach any location within the shaded region, he faces the possibility of being intercepted by P on the arc M_1M_2. The tangents can be found in the following way: Let the slope of the tangent be m_t. As the tangent passes through $E_0(0, y_e)$, equation of the tangent is

$$y - y_e = m_t x \tag{43}$$

The point of tangency $M_1(x_{m_1}, y_{m_1})$ lies on this tangent and also on C_I, therefore

$$x_{m_1}^2 + (y_{m_1} - y_c)^2 = R^2 \tag{44}$$

$$\implies x_{m_1}^2 + (y_e + m_t x_{m_1} - y_c)^2 = R^2 \tag{45}$$

$$\implies (1 + m_t^2)x_{m_1}^2 + 2m_t(y_e - y_c)x_{m_1} + (y_e - y_c)^2 - R^2 = 0 \tag{46}$$

Now, for (43) to be a tangent at $M_1(x_{m_1}, y_{m_1})$, (46) should have a double root. Therefore,

$$4m_t^2(y_e - y_c)^2 - 4(1 + m_t^2)\left[(y_e - y_c)^2 - R^2\right] = 0 \tag{47}$$

$$\implies R^2m_t^2 + R^2 - (y_e - y_c)^2 = 0 \tag{48}$$

$$\implies m_t = \pm\sqrt{\left(\frac{y_e - y_c}{R}\right)^2 - 1} \equiv m_{t_\pm} \tag{49}$$

From (34), we have $y_c = \frac{y_e}{1-\frac{e^2}{p^2}}$ and $R = -\frac{e}{p}\frac{y_e}{1-\frac{e^2}{p^2}}$. Therefore, (49) becomes

$$m_{t_\pm} = \pm\sqrt{\frac{e^2}{p^2} - 1} \tag{50}$$

which is independent of the instantaneous locations of E and P! The result means that the radius of C_I varies with the location E in such a way that the tangents to it from E always make a constant angle with the line EP. The points of tangency M_1, M_2 can be found as the intersection of the line (43) and the radial line of C_I perpendicular to it, which can be written as

$$y - y_c = -\frac{1}{m_{t_\pm}}x \tag{51}$$

Solving (43) and (51), we get the points of tangency M_1, M_2 as $\left(\frac{m_{t_+}}{1-\frac{e^2}{p^2}}y_e, 0\right)$ and $\left(\frac{m_{t_-}}{1-\frac{e^2}{p^2}}y_e, 0\right)$ respectively, which lie on the line perpendicular to E_0P_0 and passing through P_0. Therefore, $M_1P_0 \perp E_0P_0$ and $M_1P_0 \parallel M_1M_2$, as shown in Fig. 6(a). As m_t is independent of the instantaneous locations of P and E, the line connecting the instantaneous points of tangency always is perpendicular to EP and passes through P.

As only the relative positions of E and P matter, by fixing P to P_0 and moving E w.r.t him, it is easier to analyze the range of locations that E can reach mathematically. Fig. 6(b) shows the inaccessible regions for E, depending on his location w.r.t P_0. If E is at E_0, his inaccessible region is the region shaded in red. However, if E is located a little closer to P, like at E', the region inaccessible to him (indicated by the region in purple) is lesser than that at E_0. Therefore, as E moves closer to P, the range of locations that he can reach without being intercepted by P increases. If E is located at E'', the inaccessible region to him (indicated by the region shaded in green) is oriented differently, opening up locations which were inaccessible to him while at E_0. Therefore, E can reach any location in the plane by suitably positioning himself w.r.t P, merely by virtue of his higher speed than P. To save T, P's only option then is to reach T's position and wait to capture E there. This would be P's optimal strategy. Any effort by P to intercept E at a different location could result in the loss of T. Similarly, the optimal strategy for E would be to head straight to T. If P plays sub-optimally and happens to be on his way, he should first get arbitrarily close to P and then keep traveling along one of the tangents to the instantaneous C_I, till the tangent points in the direction of T. At this point, he should leave along this tangent, thus avoiding capture and acquiring T.

4. CONCLUSIONS

In this paper, the target guarding problem was solved for the pursuer and evader having simple motions with different speeds. The target guarding problem is fundamental to systems that protect a target from enemies, for example, unmanned vehicles trying to protect endangered animals from poachers. Results discussed in this paper could be applied to a single drone trying to protect a static animal, under threat from a single poacher. The optimal strategy for the drone would depend on the ratio of its speed to that of the poacher. If the drone's speed is higher than that of the poacher (which is usually the case), it would be possible for the drone to keep the poacher at bay, depending on their initial locations. However, if the poacher's speed is more than that of the drone, an interesting result from the analysis done in this paper shows that it would not be possible for the drone to keep the poacher away from the animal if the poacher plays optimally. The best that the drone could do in such a case is to go to the animal's location and wait there for the poacher to arrive.

To build field-fit drones that would help in anti-poaching operations, the analysis done in this paper has to be extended for mobile targets because, in reality, the animals would be moving. Also, if interception is achieved when the drone gets within a certain proximity of the poacher, calculations for optimal strategies could become intense, and it may not be possible to do such time consuming mathematical computations in real-time. Therefore, algorithms like Rapidly exploring Random Trees (RRT) will have to be explored.

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