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Switched observers for state and parameter estimation with guaranteed cost

Lie P. Grala Pinto * **Alexandre Trofino** **

* *Department of Automation and System Engineering*
Federal University of Santa Catarina
(e-mail: lie.grala@posgrad.ufsc.br)
** *Department of Automation and System Engineering*
Federal University of Santa Catarina
(e-mail: alexandre.trofino@ufsc.br)

Abstract: This paper addresses the problem of state and parameter estimation for the class of affine systems in the state space representation. The method does not require a specific state representation of the system and consists of designing a switched observer that, under certain conditions given in the paper, allows for the state and parameter estimation errors to converge to zero. Assuming that the parameters to be estimated belong to a given polytope, the idea of the method is to recast the parameter estimation problem as a switching rule design for an auxiliary switched system whose matrices at the equilibrium correspond to the matrices of the system to be estimated. A guaranteed cost is used in the design and the switching rule is based on a max composition of a set of quadratic functions of the observation error. The method is simple and has low computational cost. The main disadvantage regards the amount of information that is needed to have both state and parameters estimated simultaneously. The case when there is no parameter to estimate the method reduces to a standard Luenberger observer with guaranteed cost.

Keywords: State Observers, Guaranteed Cost, Parameter Estimation, Switched Systems, Sliding Mode, Lyapunov Function.

1. INTRODUCTION

In many practical applications, the situation where all state variables are available from measurement is not realistic. A similar situation occur with the parameters of the system, and usually, some of them are not precisely known. Pointed by Postoyan et al. (2012), despite of its importance, the problem of estimating simultaneously the state and parameters of the system is not widely explored as a single problem.

Although some of the current approaches for both state observation and parameter identification can be applied to a wide class of problems (Farza et al., 2009), even in the non-linear case, for example Zhou et al. (2013); Farza et al. (2009). These methods usually lead to complex observer structures found in Zhou et al. (2013); Fridman et al. (2006), or require a large number of auxiliary variables to be estimated Farza et al. (2009). Adaptive observers are often based on a transformation of the original system into some canonical form in which the presence of the unknown parameter is simplified to some extent in Zhang (2005). Few adaptive observers in non canonical form can be found in the literature, for instance Zhang (2005); Tyukin et al. (2009). As described in Fridman et al. (2006), the output injection for both linear feedback or sliding mode is commonly used, but a kind of persistent excitation condition is required and, in general, this condition cannot be easily checked.

In this paper we propose a switched observer approach to cope with the problem of simultaneously estimating the states and parameters of the system. The proposed observer is composed of a set of subsystems, an observer gain and a switching rule. The matrices of the subsystems are obtained from the vertices of the polytope that define the bounds on the parameters to be estimated. The observer gain consists of a switched gain matrix and the switching rule is determined in order to guarantee the convergence of the state and parameter errors to zero with guaranteed cost performance. The sliding mode dynamics of the switched system are represented according to Filippov's results (Filippov, 1988, p. 50). An LMI approach is proposed to solve the problem, i.e. to find the observer gain and the switching rule. The LMI is dependent on the observer state and its feasibility requires the observer state trajectory does not leave a given polytope representing a bound on the observer state. The vertices of this polytope can be adjusted to met this requirement and this condition plays the role of well known persistent excitation requirements found in usual adaptive schemas for parameter estimation problems. The paper is organized as follows. After a notation paragraph, the next section is devoted to some preliminaries and to present the problem formulation. The main result is presented in the Section 3. The results are illustrated in the section 4 through a mechanical system and some concluding remarks end the paper. This paper

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is an extension of Grala Pinto and Trofino (2014) in the sense that a guaranteed cost performance is now included in the design problem.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices; $\|\cdot\|$ stands for the Euclidian norm of vectors and its induced spectral norm of matrices; $\star$ represents block matrix terms that can be deduced from symmetry; 0_n and $0_{m\times n}$ are the $n\times n$ and $m \times n$ matrices of zeros, I_n is the $n \times n$ identity matrix. $\mathbf{1}_n$ and $\mathbf{1}_{m\times n}$ are matrices of dimension $n \times n$ and $m \times n$ where all entries are the unity. For a real matrix S, S^T denotes its transpose and $S > 0$ ($S < 0$) means that S is symmetric and positive-definite (negative-definite). For a set of real numbers $\{v_1, \dots, v_m\}$ we use $\arg\max\{v_1, \dots, v_m\}$ to denote a set of indexes that is the subset of $\{1, \dots, m\}$ associated with the maximum element of $\{v_1, \dots, v_m\}$. $\lambda_{max}(.)$ and $\lambda_{min}(.)$ denotes the maximum and minimum eigenvalue of a symmetric matrix. For a set of integers $\mathcal{M}$ the notation $\mathcal{P}(\mathcal{M})$ denotes its power set.

2. PROBLEM STATEMENT

Consider the system

$$\dot{x}(t) = Ax(t) + b + Dr(t) \quad , \quad y(t) = Cx(t) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^{n_y}$ is the measurement vector and the input $r(t) \in \mathbb{R}^{n_r}$ is a known excitation signal. It is assumed that the system is exponentially stable. The matrices $C \in \mathbb{R}^{n_y\times n}$ and $D \in \mathbb{R}^{n\times n_r}$ are assumed to be known. The matrix $A \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^n$ are unknown and it is assumed that (A, b) is an element (to be found) of the convex hull

$$(A, b) \in \underset{i\in\mathcal{M}}{\mathbf{Co}}\{(A_i, b_i)\} = \sum_{i=1}^{m} \theta_i (A_i, b_i)$$

where $\underset{i\in\mathcal{M}}{\mathbf{Co}}\{.\}$ denotes the convex hull, $\mathcal{M} := \{1, \dots, m\}$ is a set of integers, (A_i, b_i) are given matrices, and the vector $\theta = [\theta_1 \ \dots \ \theta_m]^T$ is an element of the m-dimensional unity simplex

$$\Theta = \left\{ \theta \in \mathbb{R}^m : \theta_i \geq 0, \ \forall i \text{ and } \sum_{i=1}^{m} \theta_i = 1 \right\},$$

From now on we use the notation

$$A = A_{\bar\theta} = \sum_{i=1}^{m} \bar\theta_i A_i, \qquad b = b_{\bar\theta} = \sum_{i=1}^{m} \bar\theta_i b_i, \qquad (2)$$

where $\bar\theta \in \Theta$ is a parameter to be found, characterizing the system matrices A, b of (1) as an element of the above convex hull. Observe $A_{\bar\theta}$ is Hurwitz from assumption.

Problem 1. Given the matrices A_i, b_i, C, D, the measurement signal $y(t)$ and the input signal $r(t)$, for $t \geq 0$, the problem of concern is to find a switching rule and observer gains L_i such that the following switched observer

$$\dot{z}(t) = A_\theta z(t) + b_\theta + L_\theta(y(t) - Cz(t)) + Dr(t) \qquad (3)$$

$$(A_\theta, b_\theta, L_\theta) = \sum_{i=1}^{m} \theta_i(x, z)(A_i, b_i, L_i) \qquad (4)$$

satisfies the following convergence properties:

$$\lim_{t\to\infty} z(t) = x(t) \quad , \quad \lim_{t\to\infty} \theta(x(t), z(t)) = \bar\theta \qquad (5)$$

and minimizes an upper bound of the following guaranteed cost

$$J = \min_{L_\theta, \sigma(\varepsilon)} \ \max_{\substack{e_0\in\mathcal{E}_0, \\ \bar\theta, \theta\in\Theta}} \int_0^\infty \xi^T(t)\, \xi(t)\, dt \qquad (6)$$

where $\theta(x, z) \in \Theta$ is a piecewise continuous multivalued function defined according to Filippov's results (Filippov, 1988, p. 50) for discontinuous right hand side equations. $\mathcal{E}_0$ denotes a given set of initial conditions $e_0 = x(0) - z(0)$ and $\xi(t)$ is the performance output

$$\xi(t) = C_p e(t) + D_p \delta(t) \qquad (7)$$

where C_p, D_p are given weighting matrices, and

$\varepsilon(t) = y(t) - Cz(t)$, $e(t) = x(t) - z(t)$, $\delta(t) = \bar\theta - \theta(z(t), x(t))$

are, respectively, the output estimation error, state estimation error and parameter estimation error, and the switching rule is represented by a piecewise constant set valued function $\sigma(\varepsilon(t)) \subseteq \mathcal{P}(\mathcal{M})$. □

Note that $\varepsilon = Ce$. Recall that when $\sigma(\varepsilon(t))$ is a singleton, namely when $\sigma(\varepsilon(t)) = \{i\}$, the parameter $\theta(x(t), z(t))$ is such that $\theta_i(x(t), z(t)) = 1$ and thus $\theta_j(x(t), z(t)) = 0, \ \forall j \neq i$. When $\sigma(\varepsilon(t))$ is not a singleton and a sliding mode is occurring, the role of $\theta(x(t), z(t))$ is to keep the system vector field on the tangent hyperplane of the switching surface where the sliding motion is taking place. See (Filippov, 1988, p. 50) for details.

Note that when $\sigma(\varepsilon(t)) = \{i\}$ is a singleton the observer (3) takes the form

$$\dot{z}(t) = A_i z(t) + b_i + L_i(y(t) - Cz(t)) + Dr(t) \qquad (8)$$

From the states of the system, the observer and the decompositions (2), (4) we get the dynamics of the estimation error as follows.

$$\dot{e}(t) = (A_{\bar\theta} - L_\theta C)\, e(t) + (A_{\bar\theta} - A_\theta)\, z(t) + (b_{\bar\theta} - b_\theta) \quad (9)$$

Recall that $\bar\theta$ is an unknown element of the unity simplex.

In order to estimate $\bar\theta$ and the system state with the convergence properties (5), let us consider the following set of m auxiliary functions $v_i(e, \bar\theta)$.

$$\begin{aligned} v_i(e, \bar\theta) &= e^T C^T P_i C e + 2e^T C^T S_i - 2e^T C^T S_{\bar\theta} + e^T Q_{\bar\theta} e \\ &= \varepsilon^T P_i \varepsilon + 2\varepsilon^T S_i - 2\varepsilon^T S_{\bar\theta} + e^T Q_{\bar\theta} e \end{aligned} \qquad (10)$$

where $S_{\bar\theta} = \sum_{i=1}^{m} \bar\theta_i S_i$, $Q_{\bar\theta} = \sum_{i=1}^{m} \bar\theta_i Q_i$.

Based on the above auxiliary functions $v_i(e, \bar\theta)$ consider a switching rule characterized by the set-valued function

$$\begin{aligned} \sigma(\varepsilon) &= \arg\max_{i\in\mathcal{M}} \{v_i(e, \bar\theta)\} \\ &= \arg\max_{i\in\mathcal{M}} \{\varepsilon^T P_i \varepsilon + 2\varepsilon^T S_i\} \end{aligned} \qquad (11)$$

Observe that despite the dependence of v_i with respect to $(e, \bar\theta)$, the switching signal σ is a function of ε only.

The following result, known as Finsler's Lemma, found in Boyd et al. (1997), is of interest to this paper.

Lemma 1. (Finsler Lemma). Let $\mathcal{W} \subseteq \mathbb{R}^s$ be a given polytopic set, $M(.) : \mathcal{W} \longmapsto \mathbb{R}^{q\times q}$, $G(.) : \mathcal{W} \longmapsto \mathbb{R}^{r\times q}$ be given matrix functions, with $M(.)$ symmetric. Let $Q(w)$ be a basis for the null space of $G(w)$. Then the following are equivalent:

(i) $\forall w \in \mathcal{W}$ the condition $z^T M(w) z > 0$ is satisfied $\forall z \in \mathbb{R}^q : G(w) z = 0$.
(ii) $\forall w \in \mathcal{W}$ there exists a matrix function $L(.) : \mathcal{W} \longmapsto \mathbb{R}^{q\times r}$ such that $M(w) + L(w) G(w) + G(w)^T L(w)^T > 0$.
(iii) $\forall w \in \mathcal{W}$ the condition $Q(w)^T M(w) Q(w) > 0$ is satisfied.
(iv) $\forall w \in \mathcal{W}$, $\exists \alpha \in \mathbb{R}$ such that the condition $M(w) + \alpha G(w)^T G(w) > 0$ is satisfied.

3. MAIN RESULTS

In order to show the desired convergence (5), in this paper we propose the use of a Lyapunov function based on the max composition of the set of auxiliary quadratic functions in (10) as indicated below.

$$\begin{aligned} V(e,\bar\theta) &= \max_i\{v_i(e,\bar\theta)\} \qquad (12)\\ &= \max_i\{e^T C^T P_i C e + 2e^T C^T S_i\} - 2e^T C^T S_{\bar\theta} + e^T Q_{\bar\theta} e \end{aligned}$$

Note that $V(e,\bar\theta)$ is locally Lipchitz but not differentiable everywhere and thus a special attention will be devoted to show the decreasing of $V(e,\bar\theta)$ based on the directional derivative. Moreover, from (11) and (12) it follows that $v_i(e,\bar\theta) = v_j(e,\bar\theta)$ $\forall i,j \in \sigma(\varepsilon)$. As $\theta_i(x,z) = 0$ if $i \notin \sigma(\varepsilon)$ we can represent $V(e,\bar\theta)$ in the form

$$\begin{aligned} V(e,\bar\theta) &= \sum_{i=1}^m \theta_i(x,z) v_i(e,\bar\theta) \qquad (13)\\ &= e^T C^T P_\theta C e + 2e^T C^T (S_\theta - S_{\bar\theta}) + e^T Q_{\bar\theta} e \end{aligned}$$

where $P_\theta = \sum_{i=1}^m \theta_i(x,z) P_i$, $S_\theta = \sum_{i=1}^m \theta_i(x,z) S_i$ and $S_{\bar\theta}$, $Q_{\bar\theta}$ were previously defined.

Before presenting the main result of this paper, which establishes conditions for the convergence requirements (5), let us introduce some auxiliary notation.

Let $\aleph_\theta : \mathbb{R}^m \mapsto \mathbb{R}^{r\times m}$ be a linear annihilator of θ as in the Definition 1 of Trofino et al. (2011), i.e. $\aleph_\theta$ is a linear function and $\aleph_\theta \theta = 0, \forall \theta \in \Theta$, let $\alpha \in \mathbb{R}^+$ be a given positive scalar, $\mathcal{Z}$ and $\mathcal{E}_0$ given polytopes.

Consider the following LMI conditions:

$$\Psi + G_c \aleph_{e_a} + \aleph_{e_a}^T G_c^T \geq 0,\ e(0) \in \mathcal{E}_0,\ \bar\theta,\theta \in \Theta \qquad (14)$$

$$C^T P_i C + Q_i > 0,\ i \in \mathcal{M} \qquad (15)$$

$$N^T \left(\Gamma_d + G_d C_d + C_d^T G_d^T\right) N < 0,\ z \in \mathcal{Z},\ \bar\theta,\theta \in \Theta \qquad (16)$$

where

$$\Psi = \begin{bmatrix} -C^T P_\theta C - Q_{\bar\theta} & \star \\ S_{\bar\theta}^T C - S_\theta^T C & \gamma \end{bmatrix},$$

$$\Gamma_d = \begin{bmatrix} \Gamma + G_a C_a + C_a^T G_a^T & \star \\ 0_{m(3n+m)\times 3n+m} & I_m \otimes (\Gamma + G_a C_a + C_a^T G_a^T) \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \alpha C^T (P_\theta - P_{\bar\theta}) C + C_p^T C_p & \star & \star & \star \\ C^T P_\theta C + Q_{\bar\theta} & \mathbf{0}_n & \star & \star \\ C^T P_\theta C + Q_{\bar\theta} & \mathbf{0}_n & \mathbf{0}_n & \star \\ -\alpha S^T C + D_p^T C_p & -S^T C & \mathbf{0}_{m\times n} & \Phi \end{bmatrix} \qquad (17)$$

$$\Phi(z) = -S^T C (A_z + B) - (A_z + B)^T C^T S + D_p^T D_p$$

$$C_a(\bar\theta,\theta) = [\, G A_{\bar\theta} - H_\theta C \ \ -G \ \ \mathbf{0}_n \ \ \mathbf{0}_{n\times m} \,],$$

$$C_b(\bar\theta,\theta,z) = \begin{bmatrix} \mathbf{0}_n & \mathbf{0}_n & -I_n & A_z + B \\ \mathbf{0}_{r\times n} & \mathbf{0}_{r\times n} & \mathbf{0}_{r\times n} & \aleph_{\bar\theta-\theta} \end{bmatrix}$$

$$C_d(\bar\theta,\theta,z) = \begin{bmatrix} \bar\theta \otimes I_{3n+m} & -I_{m(3n+m)} \\ \mathbf{0}_{r(3n+m)\times 3n+m} & \aleph_{\bar\theta} \otimes I_{3n+m} \\ C_b & \mathbf{0}_{n+r\times m(3n+m)} \\ 0_{m(n+r)\times 3n+m} & I_m \otimes C_b \end{bmatrix}$$

$$\tilde N = \begin{bmatrix} -I_{3n+m} & I_{3n+m} \ \dots \ I_{3n+m} \\ [\, \mathbf{0}_{1\times 3n} \ \mathbf{1}_{1\times m} \,] & \mathbf{0}_{1\times m(3n+m)} \\ \mathbf{0}_{m\times 3n+m} & I_m \otimes [\, \mathbf{0}_{1\times 3n} \ \mathbf{1}_{1\times m} \,] \end{bmatrix}$$

$$N = \mathrm{null}\left(\tilde N\right),$$

$$G_a = [\, I_n \ I_n \ \mathbf{0}_n \ \mathbf{0}_{n\times m} \,]^T, \quad A_z = [\, A_1 z \ A_2 z \ \dots \ A_m z \,], \quad B = [\, b_1 \ b_2 \ \dots \ b_m \,], \quad e_a = \begin{bmatrix} e(0) \\ 1 \end{bmatrix}$$

Define $\aleph_{e_a}$ as the linear annihilator of e_a.

The decision variables of the LMI problem (15) and (16) are

$$P_i \in \mathbb{R}^{n_y\times n_y},\quad Q_i \in \mathbb{R}^{n\times n},\quad G \in \mathbb{R}^{n\times n},$$
$$H_i \in \mathbb{R}^{n_y\times n},\quad G_c \in \mathbb{R}^{n+1\times r_1},\quad \gamma \in \mathbb{R},$$
$$G_d \in \mathbb{R}^{(m+1)(3n+m)\times(m+r)(3n+m)+(m+1)(n+r)},$$
$$S = [\, S_1 \ S_2 \ \dots \ S_m \,] \in \mathbb{R}^{n\times m}.$$

Considering the switching rule (11) and the above notation we can establish the following theorem.

Theorem 1. Suppose the system (1) is stable and satisfies the decomposition (2). Let $\mathcal{Z}$ and $\mathcal{E}_0$ be given positively invariant polytopes for the system (3). Suppose there exist matrices P_i, Q_i, S_i, H_i, $i \in \mathcal{M}$, G_c, G_d and G satisfying the LMI conditions (14), (15) and (16) where γ is minimized and define $L_i = G^{-1} H_i$. Then, γ is an upper bound for the guaranteed cost functional (6), the switched observer (3) under the switching rule (11) leads the error convergence (5) to be satisfied and (12) is a Lyapunov function for the error system (9).

Proof 1. The proof consists of showing that if the LMIs (14)-(16) are satisfied, then the locally Lipschitz function (12) satisfies the conditions

$$\phi_1(e) \leq V\left(e,\bar\theta\right) \leq \phi_2(e), \qquad (18)$$

$$D_{f_\theta} V(e,\bar\theta) \leq -\phi_3(e), \qquad (19)$$

$$\int_0^\infty \xi^T(t)\,\xi(t)\,dt \leq \gamma,\quad \forall e_0 \in \mathcal{E}_0,\ \bar\theta,\theta \in \Theta \qquad (20)$$

where $\phi_1(e)$, $\phi_2(e)$, and $\phi_3(e)$, are continuous positive definite functions, $D_h V\left(e(t),\bar\theta\right)$ is the Dini's directional derivative of V in the direction h and f_θ denotes the vector field of the error system (9), i.e.

$$f_\theta = (A_{\bar\theta} - L_\theta C) e - A_\theta z - b_\theta + A_{\bar\theta} z + b_{\bar\theta} \qquad (21)$$

In particular, for V in (12) it follows from (Lasdon, 1970, p.420) that

$$D_h V\left(e,\bar\theta\right) := \max_{i\in\sigma(e)} \nabla v_i\left(e,\bar\theta\right) h \qquad (22)$$

where $\nabla v_i\left(e,\bar\theta\right)$ is a row vector denoting the gradient of $v_i\left(e,\bar\theta\right)$. The local asymptotic stability follows from (18), (19) using the same arguments in (Filippov, 1988, p.155).

The proof that (15) implies (18) and (16) implies (19) is identical to the proof of the Theorem 1 of Grala Pinto and Trofino (2014) and will be omitted here due to space limitation. In fact, with the results in this reference it is easy to show that (16) implies that

$$D_{f_\theta} V\left(e,\bar{\theta}\right)+\alpha\left(V(e,\bar{\theta})-\bar{V}(e,\bar{\theta})\right)+\xi^T\xi<0 \qquad (23)$$

In the sequel, the above expression, jointly with the LMI (14), are used to show the performance requirement as indicated in (20).

Since $V\left(e,\bar{\theta}\right)$ is locally Lipschitz, it follows that for almost all $t\in[0,\infty)$ the time derivative of $V\left(e,\bar{\theta}\right)$ exists and coincides with the directional derivative, i.e.

$$D_{f_\theta} V\left(e,\bar{\theta}\right)=\frac{d}{dt}V\left(e,\bar{\theta}\right) \qquad (24)$$

Therefore we have

$$\lim_{t\to\infty} V\left(e(t),\bar{\theta}\right)-V\left(e(0),\bar{\theta}\right)=\lim_{t\to\infty}\int_0^t \frac{d}{ds}V\left(e(s),\bar{\theta}\right)ds \qquad (25)$$

As $\lim_{t\to\infty} V\left(e(t),\bar{\theta}\right)=0$ in view of (18) and (19) and $\alpha\left(V\left(e,\bar{\theta}\right)-\bar{V}\left(e,\bar{\theta}\right)\right)$ is non-negative, see Grala Pinto and Trofino (2014) for details, we conclude from (23),(24),(25) that

$$\int_0^\infty \xi^T(t)\,\xi(t)\,dt\le V\left(e(0),\bar{\theta}\right) \qquad (26)$$

Keeping in mind that $\aleph_{e_a}e_a=0$, by multiplying the left hand side of (14) by e_a to the right and by its transpose to the left we get

$$\begin{aligned}0\le e_a^T\Psi e_a =&e^T(0)\left(-C^TP_\theta C-Q_{\bar{\theta}}\right)e(0)+\\ &+2e^T(0)\left(S_{\bar{\theta}}^TC-S_\theta^TC\right)+\gamma \qquad \forall e(0)\in\mathcal{E}_0\end{aligned}$$

The above expression can be rewritten as

$$V(e(0),\bar{\theta})\le\gamma, \quad \forall e(0)\in\mathcal{E}_0 \qquad (27)$$

and from (26),(27) we get (20), completing the proof. □

Remark 1. Note that the equilibrium $e(t)=\dot{e}(t)=0$, $\theta=\bar{\theta}$ is always enforced by a sliding mode when the system parameters are in the interior of $\underset{i\in\mathcal{M}}{\mathbf{Co}}\{[A_i,b_i]\}$. This implies that, if sliding modes are to be avoided, no exact parameter characterization can be found in general.

Remark 2. As mentioned in the previous remark, no exact parameter characterization is expected if the switching frequency is bounded. As in practice this situation is always the case, we present in the sequel a procedure to get an approximation of the Filippov's convex parameter $\theta(x,z)$ used in the switched observer (3). The idea is usual in PWM based models Sira-Ramírez (1993) and consists of replacing the ideal sliding mode dynamics associated with unbounded switching frequency, with a bounded but sufficiently high switching frequency. For this purpose it is required that the switching frequency must be higher than the spectrum of the subsystems, in the sense that the switching period is associated with a time scale where the subsystems vector fields can be considered almost constant in this small time interval. In this case, the Filippov's convex parameter $\theta(x,z)$ can be approximated by the average value of a logical variable. To illustrate the ideas, suppose that $f_i(e(t))$ are Lipschitz continuous functions representing the vector fields of the subsystems and $f(e(t))=\sum_{i=1}^m\theta_i(e(t))f_i(e(t))$ is the vector field of the switched system where $\theta_i(e(t))$ is the convex combination parameter defined according to Filippov's results (Filippov, 1988, p.50). Consider the following approximation:

$$f(e(t))=\sum_{i=1}^m\theta_i(e(t))f_i(e(t))\cong\frac{1}{T}\int_{t-T}^t\sum_{i=1}^m\mu_i(t)f_i(e(t))dt \qquad (28)$$

where $T>0$ is a sufficiently small time interval, $\mu_i(t)$ are logical variables defined as

$$\begin{cases}\mu_i(t)=1 & \text{for some } i\in\sigma(e(t))\\ \mu_j(t)=0 & \text{for } j\neq i\end{cases} \qquad (29)$$

and $\sigma(e(t))$ is a set valued function defining the switching rule in the ideal scenario (arbitrarily fast switchings). As the functions $f_i(e(t))$ are Lipschitz, the more T is reduced, the more $f_i(e(t))$ approaches a fixed value in the interval, in the sense that $f(e(t))$ is practically constant in the interval $[t-T,\ t]$. Thus, for sufficiently small $T>0$, the right hand side of (28) can be approximated using the expression

$$\int_{t-T}^t\sum_{i=1}^m\mu_i(t)f_i(e(t))\,dt\cong\sum_{i=1}^m\left(\int_{t-T}^t\mu_i(t)dt\right)f_i(e(t)) \qquad (30)$$

that in turn yields the approximation

$$\theta_i(e(t))\cong\frac{1}{T}\int_{t-T}^t\mu_i(t)dt \qquad (31)$$

that is valid for a sufficiently small $T>0$. Observe that (31) express an approximation based on the average value of the logical variables $\mu_i(t)$ in the interval T. This approximation can be used to get a duty cycle for the switching devices.

Remark 3. Observe that one of the requirements for the convergence (5) to be true is that the polytope $\mathcal{Z}$ must be positively invariant for the observer dynamics (3). LMI conditions for a given polytope to be positively invariant can be found in Trofino and Dezuo (2013) and can be used here to check the positively invariance requirement. This condition can be viewed as a type of persistent excitation condition that appears in parameter estimation problems. The nice feature of this condition is that it is easy to check online if this condition is satisfied or not during the parameter identification experiment. Recall that the persistent excitation condition appearing in several parameter identification methods is difficult to be checked Farza et al. (2009). An interesting point that we are investigating is to find the best polytope and input $r(t)$ that maximizes the chances of the LMI to be feasible. In this direction the function $\Phi(z)$ in (17) plays an important role.

4. NUMERICAL EXAMPLE

Consider the mechanical system shown in the Fig. 1 where the blocks have masses m_a,m_b, the springs have constants k_a,k_b and the movement is subjected to viscous friction with coefficients b_a,b_b respectively. This system can be represented as in (1) with

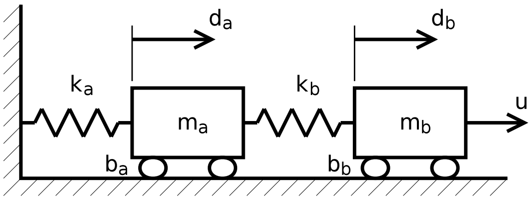

Fig. 1. Two mass-spring-dumper example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_a + k_b}{m_a} & -\dfrac{b_b}{m_a} & \dfrac{k_a}{m_a} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_a}{m_b} & 0 & -\dfrac{k_a}{m_b} & -\dfrac{b_a}{m_b} \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{F}{m_b} \end{bmatrix} \tag{32}$$

$$C = [\, 0_2 \ I_2 \,] \ , \ D = 0 \tag{33}$$

The system state is $x = [\, d_a \ v_a \ d_b \ v_b \,]$, which corresponds to the blocks displacement (d_a, d_b) with respect to their equilibrium condition and velocities (v_a, v_b) of the blocks. The external force $u = F_0 + F$ is applied to the block b, where F is the force deviation and F_0 is the force at the equilibrium. The external force is supposeed to be constant. The measurement vector corresponds to the position and velocity of the block b. The weighing matrices for the performance output have been chosen as $C_p = [I_4 \ 0_{4\times 2}]^T$ and $D_p = [0_{2\times 4} \ 0_2]^T$, i.e. $\xi = e$.

The first problem to be considered is to estimate the states and the spring constant k_a, given the others parameters. The only information that will be used about k_a is that it is a bounded parameter $k_a \in [1.5;\ 2.5]$. The remaining parameters are $k_b = 3$, $b_a = 3$, $b_b = 3$, $m_a = 3$ and $m_b = 4$, $F = 2$. The units are in the MKS system.

The LMIs (14), (15) and (16) were solved for $\alpha = 1$ and the polytope $\mathcal{Z}$ such that $z_1 \in [0.3, 0.6]$, $z_2, z_4 \in [-0.1, 0.1]$ and $z_3 \in [1.0, 1.5]$. To illustrate the impact of the set of initial conditions in the guaranteed cost we have considered four different polytopes $\mathcal{E}_0$. The simulation results are shown in the Table 1 where $e_i(0)$ denotes the i-th entry of the initial error vector $e(0) = x(0) - z(0)$. To illustrate the degree of conservatism of the guaranteed cost upper bound proposed by the theorem we have used a grid technique on $\mathcal{E}_0$ to get, by simulation, the energy of the performance output during the state and parameter estimation experiment for each initial condition e_0 in the grid. In the column $\|\xi(t)\|_2^2$ of the table, is indicated the largest energy obtained from the points (initial condition) in the grid.

To emphasize how much the energy of the performance output can be reduced by taking the guaranteed cost into account when designing the switching rule we have also indicated, in the last column of the table, the energy of the performance output obtained with the same grid technique applied to the results in Grala Pinto and Trofino (2014) that does not take into account any performance criterion.

The measurement signal was obtained by simulation considering the true model with the nominal value of the spring constant $k_a = 2$. In this case the decomposition (2) results $\bar{\theta} = [\, 0.5 \ 0.5 \,]^T$. Observe from the Table 1 that the bound on the guaranteed cost proposed by the theorem and the one obtained by simulation are very close in the

Table 1. Guaranteed cost index

#	$\mathcal{E}_0$	γ	$\|\xi\|_2^2$	$\|\xi\|_2^2$
1	$\{e_i(0) = -1\}$	2.282	1.999	8.012
2	$\{e_i(0) \in [-1.0, 0.0]\}$	6.443	2.933	8.012
3	$\{e_i(0) \in [-1.0, 1.0]\}$	8.582	4.796	8.366
4	$\{e_i(0) \in [-1.5, 1.5]\}$	18.36	10.82	18.82

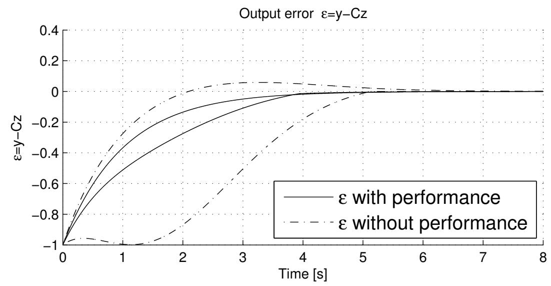

Fig. 2. Output error convergence with and without guaranteed cost in the design.

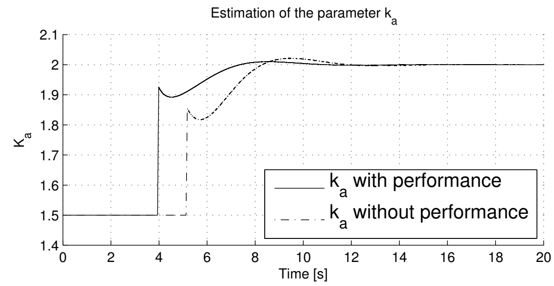

Fig. 3. Parameter convergence with and without guaranteed cost in the design. $k_a = \theta_1 1.5 + \theta_2 2.5$.

case 1. The gap in the cases 2,3,4 can be reduced by taking a grid with more points. In this example the grid was obtained by considering a precision of 0.1 for each component e_i inside the polytope. The output estimation errors, corresponding to the case 2 of Table 1, is indicated in the figure 2. Recall that in addition to the state estimation we are also estimating the parameter k_a. The parameter error convergence in this case is shown in the Figure 3. Observe that the output estimation error is practically zero after 5 seconds, but the parameter estimation error takes more time to converge to zero. The fast convergence of ε is a consequence of the choice of the weighting matrices C_p, D_p leading to $\xi = e$.

The second problem to be considered in this example is to estimate the states and the unknown input force F, given the others parameters. The only information that will be used on F is that the input force is bounded $F \in [1.5;\ 2.5]$. The remaining parameters are $k_a = 2$, $k_b = 2$, $b_a = 3$, $b_b = 3$, $m_a = 3$ and $m_b = 4$. The units are in the MKS system.

The LMIs (14), (15) and (16) were solved for the same previous α and polytope $\mathcal{Z}$. The guaranteed cost was obtained considering the same polytope $\mathcal{E}_0$ of case 2 in the Table 1. The simulation results are shown in the Figure 4 where the unknown input is piecewise constant in the interval $F \in [1.5;\ 2.5]$. To illustrate the impact of the weighting matrices charactering the performance output

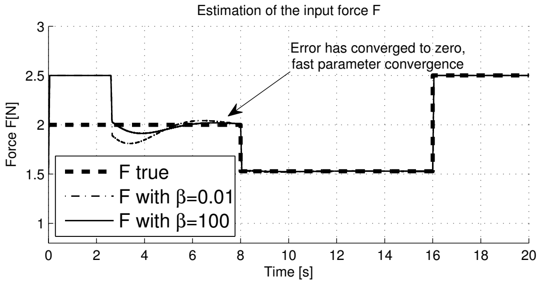

Fig. 4. Unknown input force estimation with and without guaranteed cost in the design. $F = \theta_1 1.5 + \theta_2 2.5$.

we have considered two different weighting matrix D_p and the same C_p of the previous case. In the first case we have a small weighting on the parameter estimation error with the choice $D_p = [0_{2\times4}\ \beta I_2]^T$ with $\beta = 0.01$ and a large weighting in the second case with $\beta = 100$. As expected we have a faster parameter convergence in the second case.

Finally, as mentioned in the Remark 3, one interesting feature of the proposed design condition is that it is easy to check on line if the positively invariance condition on the polytope $\mathcal{Z}$ is satisfied during the identification experiment. In fact, during all the simulation results this invariance condition was satisfied and thus the parameter estimation error is guaranteed to converge to zero.

5. CONCLUSION

A switched observer design with guaranteed cost to cope with the problem of estimating simultaneously the states and parameters of an affine system is proposed in this paper. An LMI approach is used to minimize an upper bound of the cost, for a set of initial conditions, and to get the observer gains and switching rule leading the state and parameter estimation errors to converge to zero. The LMI depends on the observer state and its feasibility requires the observer state trajectory to belong to a positively invariant polytope $\mathcal{Z}$ whose vertices can be adjusted. This condition can be viewed as a type of persistent excitation condition that appears in parameter estimation problems. The nice feature of this condition is that it is easy to check online if this condition is satisfied or not during the parameter identification experiment. Recall that the persistent excitation condition appearing in several parameter identification methods are difficult to be checked (Farza et al., 2009). An interesting point that we are investigating is to find the best polytope that maximizes the chances of the LMI to be feasible and input $r(t)$ leading the positively invariance condition to be satisfied during the identification experiment. In this direction the function $\Phi(z)$ in (17) plays an important role.

A numerical example based on a mechanical system is used to illustrate the joint state and parameter estimations.

We are currently investigating the extension of the method to consider the design problem with H_∞ performance and measurement noises. We are also investigating additional relations that could be used to reduce the conservativeness of the LMIs.

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