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Nonlinear Protocols on Ellipsoids for Multi-agent Systems ⋆

Jiandong Zhu, Jinli Sun

Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, PR China; e-mail: zhujiandong@njnu.edu.cn).

Abstract: This paper investigates the consensus problem on ellipsoids for multi-agent systems. Simple nonlinear protocols are proposed to realize collective behavior on ellipsoids. For the high-dimensional nonlinear multi-agent systems, equilibrium sets are described exactly. By a generalized form of LaSalle invariance principle, some global dynamical properties are obtained. With the linear approximation method, some results on instability of some kinds of equilibria are obtained. Based on the above results, almost global consensus is achieved under some conditions. Simulation results are presented to show the effectiveness of proposed protocols.

Keywords: Multi-agent systems, nonlinear protocols, almost global consensus.

1. INTRODUCTION

Consensus phenomena are widespread in the natural world such as the synchronized motion of pendulum clocks, synchronous flashing of fireflies. For explaining some consensus phenomena, researchers established many kinds of models including the well-known Boid model, Vicsek model, Kuramoto model (see Reynolds (1987), Vicsek et. al. (1995) and Kuramoto (1975)). For continuous-time multi-agent systems, Olfati-Saber and Murray (2004) studied the consensus of single-integrator multi-agent systems based on the strong connectedness assumption on a directed graph. Ren and Beard further generalized the above results and found that the above multi-agent systems can achieve consensus if there exists a spanning tree in the directed graph (Ren (& Beard)). Thereafter, many kinds of distributed protocols were designed for different multi-agent systems (Zhu, Tian, & Kuang (2009), Lin, Francis, & Maggiore (2008), Zhu, Lü, & Yu (2013)). An important nonlinear multi-agent system is Kuramoto model, which describes a collective behavior of a set of agents interconnected by a communication network. In the recent years, Kuramoto model has been paid great attention in the community of control theory (see Jadbabaie, Motee, & Barahona (2004), Olfati-Saber (2006), Chopra, & Spong (2009), Papachristodoulou, & Jadbabaie (2005), Monzón, & Paganini (2005), Canale & Monzón (2008)). In Monzón, & Paganini (2005), Canale & Monzón (2008), properties of equilibria of the Kuramoto model with identical frequencies are investigated and almost global consensus is proved for some special interconnection graphs. The original Kuramoto model actually describes a dynamical behavior on the unit circle. A high-dimensional form of Kuramoto model on unit sphere is proposed in Olfati-Saber (2006). For a complete graph, the synchronization on the unit sphere is achieved. In Zhu (2013), a more general form of the high-dimensional Kuramoto model limited on spheres is established and some results of the original Kuramoto model are generalized. It should be noted that in Olfati-Saber (2006) and Zhu (2013), the initial states of the agents are limited on a sphere and the sphere is an invariant set of the generalized Kuramoto model. An interesting problem is whether some consensus results can be established on ellipsoids without any limitations on the initial states.

In this paper, simple nonlinear protocols for the consensus on ellipsoid are designed for multi-agent systems. The limitation on the initial states are removed and some properties on equilibria and linearized systems are obtained. A global dynamical property are revealed by a generalized form of LaSalle invariance principle. Under some conditions, the almost global synchronization is achieved.

Throughout the paper, we use notations as follows: $\mathbf{R}$ denotes the real number field, $\mathbf{R}^n$ the n-dimensional real linear space, $\otimes$ the Kronecker product, $\mathbf{1}_n$ the $n-$dimensional column vector of all ones, I_n the $n \times n$ identity matrix, $\mathbf{Ker}(A)$ the kernel of a linear operator A, $\mathbf{Span}(v_1, \cdots, v_s)$ the subspace generated by vectors $v_1, \cdots, v_s$ and superscript T the transpose operation.

2. PROBLEM STATEMENT

Consider a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the set of nodes $\mathcal{V} = \{1,\ 2, \cdots,\ m\}$, set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a symmetrical adjacency matrix $\mathcal{A} = (a_{ij}) \in \{0,\ 1\}$. An edge of $\mathcal{G}$ is denoted by (i, j), which means i and j can exchange state information. Adjacency matrix $\mathcal{A}$ is defined such that $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, while $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. We denote the set of neighbors of node i by N_i. The Laplacian matrix of the digraph is defined as $L = (l_{ij})$, where $l_{ii} = \sum_{j=1, j\neq i}^{m} a_{ij}$ and $l_{ij} = -a_{ij}$ $(i \neq j)$. It is obvious that $L\mathbf{1}_n = 0$. Refer to Godsil, & Royle (2001) for more knowledge of algebraic graph theory.

Suppose each node i has the dynamical equation as follows:

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$$\dot{x}_i = f_i(x_i, x_j; j \in N_i). \tag{1}$$

If we regard the right side of (1) as a distributed protocol, then only local information is used in this protocol.

Definition 1. Denote a family of ellipsoids in n-dimensional linear space by $\{\Phi_c\}$: $x^{\mathrm{T}}Px = c \quad (c > 0)$, where P is a positive definite matrix and $x \in R^n$. It is said that *consensus on the ellipsoids* $\{\Phi_c\}_{c>0}$ *is achieved*, if there exists a $\bar{c} > 0$, which is possibly dependent on the initial values $x_1(0)$, $x_2(0)$, $\cdots$, $x_m(0)$, such that

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0, \ \lim_{t\to\infty} \mathrm{dist}(x_i(t),\ \Phi_{\bar{c}}) = 0. \tag{2}$$

for all $i, j = 1, 2, \cdots, m$. In particular, if (2) is strengthen as $\lim_{t\to\infty} x_i(t) = C \in \Phi_{\bar{c}}$, it is said that the *static consensus on ellipsoids* $\{\Phi_c\}_{c>0}$ *is achieved*. If (2) holds but $x_i(t)$ does not converge to a constant, we say the *dynamical consensus on ellipsoids* $\{\Phi_c\}_{c>0}$ *is achieved*.

3. MAIN RESULTS

Consider the ellipsoids

$$\Phi_c = \{x \in \mathrm{R}^n \mid x^{\mathrm{T}}Px = c\} \tag{3}$$

with $c > 0$ and P positive definite. For the multi-agent system $\dot{x}_i = u_i$ with state $x_i \in \mathrm{R}^n$ and control $u_i \in \mathrm{R}^n$, we design a nonlinear protocol as

$$u_i = W_iPr_i + \sum_{j\in N_i}(I_n + \frac{r_ir_j^{\mathrm{T}}P}{r_i^{\mathrm{T}}Pr_i})(r_j - r_i), \tag{4}$$

where each W_i is a skew-symmetric matrix. Then the closed-loop can be written as

$$\dot{r}_i = W_iPr_i + \sum_{j\neq i}a_{ij}(I_n + \frac{r_ir_j^{\mathrm{T}}P}{r_i^{\mathrm{T}}Pr_i})(r_j - r_i), \tag{5}$$

where $\mathcal{A} = (a_{ij}) \in R^{m\times m}$ is the adjacency matrix of the interconnection graph and $i = 1, 2, \cdots, m$.

Proposition 1. Consider the nonlinear multi-agent system (5) and ellipsoids $\{\Phi_c\}_{c>0}$ shown in (3) with P positive definite. Assume the interconnection graph $\mathcal{G}$ is connected and W_i is a skew symmetric matrix for each i, i.e. $W_i^{\mathrm{T}} = -W_i$. Let $c_i = r_i(0)^{\mathrm{T}}Pr_i(0)$. Then each $r_i(t)$ converges to $\Phi_{\bar{c}}$ with $\bar{c} = \frac{1}{m}\sum_{j=1}^m c_i$ and

$$S_c = \{(r_1^{\mathrm{T}}, \cdots, r_m^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{R}^{mn} \mid r_i^{\mathrm{T}}Pr_i = c,\ i = 1, 2\cdots, m\} \tag{6}$$

is an invariant set of (5) for any $c > 0$.

Proof. Let $v_i(r_i) = r_i^{\mathrm{T}}Pr_i$. Then

$$\begin{aligned}\dot{v}_i(r_i(t)) &= 2r_i^{\mathrm{T}}(t)P\dot{r}_i(t)\\ &= 2r_i^{\mathrm{T}}(t)PW_iPr_i(t) + 2\sum_{j\neq i}a_{ij}(r_j^{\mathrm{T}} + r_i^{\mathrm{T}})P(r_j - r_i)\\ &= 2\sum_{j\neq i}a_{ij}(r_j^{\mathrm{T}}Pr_j - r_i^{\mathrm{T}}Pr_i)\\ &= 2\sum_{j\neq i}a_{ij}(v_j(r(t)) - v_i(r(t))). \end{aligned} \tag{7}$$

Rewrite (7) as the compact form

$$\dot{v}(r(t)) = -2Lv(r(t)), \tag{8}$$

where L is the Laplacian matrix of the interconnection graph $\mathcal{G}$ and $v(r) = (v_1^{\mathrm{T}}(r_1), v_2^{\mathrm{T}}(r_2), \cdots, v_m^{\mathrm{T}}(r_m))^{\mathrm{T}}$. Since $\mathcal{G}$ is connected, from (8) it follows that $\frac{1}{m}\mathbf{1}_m^{\mathrm{T}}v(r(t))$ is a constant number and $v_i(r_i(t))$ converges to the average value $\frac{1}{m}\mathbf{1}_m^{\mathrm{T}}v(0)$ for every $i = 1, 2, \cdots, m$ as t tends to infinity. Moreover, for any $r(0) \in S_c$, it follows from (8) that $v_i(r(t)) = c$ for any $t \geq 0$ and $i = 1, 2, \cdots, m$. Therefore, $r(t) \in S_c$ for any $t \geq 0$, which means that S_c is an invariant set of (5). □

Proposition 2. Consider the nonlinear multi-agent system (5) with skew symmetric matrices $W_i = W$ $(i = 1, 2, \cdots, m)$ and positive definite matrix P. Let $z_i = \mathrm{e}^{-WPt}r_i(t)$. Then (5) is equivalently transformed to

$$\dot{z}_i = \sum_{j\neq i}a_{ij}(I_n + \frac{z_iz_j^{\mathrm{T}}P}{z_i^{\mathrm{T}}Pz_i})(z_j - z_i), \tag{9}$$

where $z = [z_1^{\mathrm{T}}, z_2^{\mathrm{T}}, \cdots, z_m^{\mathrm{T}}]^{\mathrm{T}}$.

Proof. With simple calculations, we have

$$\begin{aligned}\dot{z}_i &= -WP\mathrm{e}^{-WPt}r_i + \mathrm{e}^{-WPt}\dot{r}_i\\ &= -WPz_i + \mathrm{e}^{-WPt}WP\mathrm{e}^{WPt}z_i\\ &\quad + \mathrm{e}^{-WPt}\sum_{j\neq i}a_{ij}(I_n + \frac{\mathrm{e}^{WPt}z_iz_j^{\mathrm{T}}\mathrm{e}^{-WPt}P}{z_i^{\mathrm{T}}Pz_i})\mathrm{e}^{WPt}(z_j - z_i)\\ &= \sum_{j\neq i}a_{ij}(I_n + \frac{z_iz_j^{\mathrm{T}}P}{z_i^{\mathrm{T}}Pz_i})(z_j - z_i), \end{aligned} \tag{10}$$

where $\mathrm{e}^{-WPt}WP\mathrm{e}^{WPt} = WP$ and $\mathrm{e}^{-WPt}P\mathrm{e}^{WPt} = P$ are used for the last equality of (10). □

We first consider the equilibria of (9). For the convenience of the derivation below, we introduce a well-known result in linear algebra.

Lemma 1. (Theorem 18.1.1. of Harville (2008)) Let R represent an $n\times n$ matrix, S an $n\times m$ matrix, T an $m\times m$ matrix, and U an $m\times n$ matrix. If R and T are nonsingular, then

$$\det(R + STU) = \det R \det T \det(T^{-1} + UR^{-1}S).$$

In particular, for any scalar λ, we have

$$\det(\lambda I_n - SU) = \lambda^{n-m}\det(\lambda I_m - US). \tag{11}$$

Proposition 3. Let $\tilde{z} = (\tilde{z}_1^{\mathrm{T}}, \tilde{z}_2^{\mathrm{T}}, \cdots, \tilde{z}_m^{\mathrm{T}})^{\mathrm{T}} \in \mathbf{R}^{mn}$. Assume the interconnection graph $\mathcal{G}$ is connected. Then $\tilde{z}$ is an equilibrium of (9) if and only if $\tilde{z}_i$ is in parallel with $\sum_{j\in N_i}\tilde{z}_j$ and $\tilde{z}_i^{\mathrm{T}}P\tilde{z}_i = \tilde{z}_j^{\mathrm{T}}P\tilde{z}_j$ holds for any $i \neq j$.

Proof. (Necessity) Assume $\tilde{z}$ is an equilibrium of (9). Then

$$\sum_{j\neq i}a_{ij}(I_n + \frac{\tilde{z}_i\tilde{z}_j^{\mathrm{T}}P}{\tilde{z}_i^{\mathrm{T}}P\tilde{z}_i})(\tilde{z}_j - \tilde{z}_i) = 0. \tag{12}$$

Pre-multiplying (11) by $\tilde{z}_i^{\mathrm{T}}P$ yields

$$\sum_{j\neq i}a_{ij}(\tilde{z}_j^{\mathrm{T}}P\tilde{z}_j - \tilde{z}_i^{\mathrm{T}}P\tilde{z}_i) = 0, \tag{13}$$

which implies $Lv = 0$ with $v = [\tilde{z}_1^{\mathrm{T}}P\tilde{z}_1, \cdots, \tilde{z}_2^{\mathrm{T}}P\tilde{z}_2]^{\mathrm{T}}$. By the connectedness of graph $\mathcal{G}$, we know that $\mathrm{rank}L = m - 1$ and $\mathbf{Ker}(L) = \mathbf{span}\{\mathbf{1}_m\}$. Hence, $Lv = 0$ implies

$\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i = \tilde{z}_j^{\mathrm{T}} P \tilde{z}_j$ for any $i \neq j$. Thus (12) can be rewritten as

$$(I_n - \frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i}) \sum_{j \neq i} a_{ij} \tilde{z}_j = 0. \tag{14}$$

By (11) of Lemma 1, $I_n - \frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i}$ has a simple eigenvalue 0. Since $(I_n - \frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i})\tilde{z}_i = 0$, it follows from (14) that $\tilde{z}_i$ is in parallel with $\sum_{j \in N_i} \tilde{z}_j$.

(Sufficiency) Assume $\sum_{j \in N_i} \tilde{z}_j = d_i \tilde{z}_i$. Since $\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i = \tilde{z}_j^{\mathrm{T}} P \tilde{z}_j$, we have

$$\begin{aligned}
&\sum_{j \neq i} a_{ij} (I_n + \frac{\tilde{z}_i \tilde{z}_j^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i})(\tilde{z}_j - \tilde{z}_i) \\
&= (I_n - \frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i}) \sum_{j \neq i} a_{ij} \tilde{z}_j \\
&= (I_n - \frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i}) d_i \tilde{z}_i = 0.
\end{aligned} \tag{15}$$

□

By Proposition 4, we can obtain some special kinds of equilibrium points just like the discussion in Monzón, & Paganini (2005) or Canale & Monzón (2008).

Definition 2. If $r_1 = r_2 = \cdots = r_m$, we call r a *consensus point*. If r is not a consensus point and $r_i = \pm r_1$ for all $i = 1, 2, \cdots, m$, we call r a *partial consensus point*.

By Proposition 3, we can see all the consensus points and partial consensus points are equilibria.

Definition 3. All the equilibria except the consensus points and the partial consensus points are called *non-consensus equilibria*.

Proposition 4. The following statements hold:

1) If the graph is a tree, then any equilibrium is a consensus point or a partial consensus point.

2) Assume the graph is a complete graph and r is neither a consensus point nor a partial consensus point. Then r is a equilibrium if and only if $\sum_{i=1}^{m} r_i = 0$.

Proof. The results are directly obtained from Proposition 3. □

In the following, we investigate the global dynamical properties of (9). Before the main result, we introduce a generalized form of LaSalles invariance principle. As a matter of fact, the following lemma is a special case of Theorem 6 of Arsie, & Ebenbauer (2010).

Lemma 2 (Theorem 6 of Arsie, & Ebenbauer (2010)). Consider the locally Lipschitz continuous system $\dot{x} = f(x)$ with $x \in \mathrm{R}^N$. Assume solution $x(t, x(0))$ is bounded and the $\omega-$limit set $\Omega(x(0))$ is contained in a closed submanifold S$\subset \mathrm{R}^N$. Suppose there exists a smooth function $W : \mathrm{R}^n \to \mathrm{R}$ such that the derivative of $W(x)$ along the flow (Lie derivative) $\dot{W}(x) \leq 0$ on S. Let $E := \{x \in \mathrm{S} : \dot{W}(x) = 0\}$. Then $\Omega(x(0)) \subset E$, i.e. $x(t, x(0))$ converges to S for any initial vector $x(0) \in \mathrm{R}^N$.

Remark 1. Compared with the original LaSalle invariance principle, Lemma 2 only requires $\dot{W}(x) \leq 0$ on S. So, generally speaking, $\dot{W}(x)$ is not a nonnegative definite function on R^n.

Theorem 1. Consider the nonlinear multi-agent system (9) with positive definite matrix P. Assume the graph $\mathcal{G}$ is connected. Then, for any the initial condition $z_i(0) \neq 0$ $(i = 1, 2, \cdots, m)$, the solution $z(t, z(0))$ converges to the equilibrium set.

Proof. Rewrite (9) as

$$\dot{z}_i = (I_n - \frac{z_i z_i^{\mathrm{T}} P}{z_i^{\mathrm{T}} P z_i}) \sum_{j=1}^{m} a_{ij} z_j + \sum_{j=1}^{m} a_{ij} (\frac{z_j^{\mathrm{T}} P z_j}{z_i^{\mathrm{T}} P z_i} - 1) z_i \tag{16}$$

for every $i = 1, 2, \cdots, m$. Let

$$\phi_i(z) = \sum_{j=1}^{m} a_{ij} (\frac{z_j^{\mathrm{T}} P z_j}{z_i^{\mathrm{T}} P z_i} - 1), \tag{17}$$

$$\phi(z) = [\phi_1^{\mathrm{T}}(z),\ \phi_2^{\mathrm{T}}(z), \cdots, \phi_m^{\mathrm{T}}(z)]^{\mathrm{T}}, \tag{18}$$

$$D(z) = \mathrm{diag}\{I_n - \frac{z_i z_i^{\mathrm{T}} P}{z_i^{\mathrm{T}} P z_i} \mid i = 1, 2, \cdots, m\}. \tag{19}$$

Then (16) can be rewritten as the compact form as

$$\dot{z} = -D(z)(L \otimes I_n) z + \phi(z), \tag{20}$$

where L is the Laplacian matrix of the interconnected graph $\mathcal{G}$. Letting $W(z) = z^{\mathrm{T}}(L \otimes I_n) z/2$ yields

$$\dot{W}(z) = -z^{\mathrm{T}}(L \otimes I_n) D(z)(L \otimes I_n) z + z^{\mathrm{T}}(L \otimes I_n)\phi(z). \tag{21}$$

Let $\mathrm{S} = \{z = (z_1^{\mathrm{T}}, z_2^{\mathrm{T}}, \cdots, z_m^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{R}^{mn} \mid z_1^{\mathrm{T}} P z_1 = z_2^{\mathrm{T}} P z_2 = \cdots = z_m^{\mathrm{T}} P z_m\}$. Then, by Proposition 1, S is a closed invariant submanifold of R^{mn}. By (17) and (18), we have $\phi(z) = 0$ for any $z \in S$. Hence

$$\dot{W}(z) = -z^{\mathrm{T}}(L \otimes I_n) D(z)(L \otimes I_n) z \leq 0 \quad \forall\ z \in \mathrm{S}. \tag{22}$$

Moreover,

$$E := \{z \in \mathrm{S} \mid \dot{W}(z) = 0\} = \{z \in \mathrm{S} \mid D(z)(L \otimes I_n) z = 0\}. \tag{23}$$

Thus, by Lemma 2, $z(t)$ converges to E. From (23) and Proposition 3, we can see that the equilibrium set is exactly E.

Remark 2. From (21), we cannot guarantee that $\dot{W}(z)$ is negative semi-definite. So the traditional LaSalle invariance principle is invalid in this case. But using Lemma 2, i.e. the generalized form of LaSalle invariance principle, a global dynamical property is obtained. Theorem 1 shows that there is no any periodic solution and any chaotic phenomenon for system (21).

Theorem 1 has shown that the trajectories must converge to the equilibrium set. It is still unsolved that whether the consensus can be achieved. However, if all the equilibria except consensus points are unstable, then the almost global consensus can be realized. For analyzing the instability of an equilibrium, we consider the linear approximation of the nonlinear multi-agent system (9) around an equilibrium.

Theorem 2. Let $\tilde{z} = (\tilde{z}_1^{\mathrm{T}}, \tilde{z}_2^{\mathrm{T}}, \cdots, \tilde{z}_m^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{R}^{mn}$ be an equilibrium of (9) satisfying $\sum_{j \neq N_i} a_{ij} \tilde{z}_j = d_i \tilde{z}_i$ and $\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i = \tilde{z}_j^{\mathrm{T}} P \tilde{z}_j$ for all $i \neq j$, where P is a positive definite matrix. Assume the graph $\mathcal{G}$ is connected. Then the following statements hold:

1) If there is at least one $d_i < 0$, then $\tilde{z}$ is unstable;

2) If $\tilde{r}$ is a partial synchronization point, then $\tilde{r}$ is unstable.

Proof. Denote the righthand side function of (9) by $f_i(z)$. Then with simple calculations, we have

$$\frac{\partial f_i}{\partial z_k} = a_{ik}(I_n + \frac{2z_i z_k^{\mathrm{T}} P}{z_i^{\mathrm{T}} P z_i} - \frac{z_i z_i^{\mathrm{T}} P}{z_i^{\mathrm{T}} P z_i}), \quad (i \neq k) \tag{24}$$

$$\frac{\partial f_i}{\partial z_i} = \sum_{j\neq i} a_{ij}(-I_n + \frac{z_j^{\mathrm{T}} P z_j}{z_i^{\mathrm{T}} P z_i} I_n + (z_j^{\mathrm{T}} P z_j) z_i \frac{-2z_i^{\mathrm{T}} P}{(z_i^{\mathrm{T}} P z_i)^2} + \frac{z_i^{\mathrm{T}} P z_j}{z_i^{\mathrm{T}} P z_i} I_n - \frac{z_i z_j^{\mathrm{T}} P}{z_i^{\mathrm{T}} P z_i} + z_i (z_i^{\mathrm{T}} P z_j) \frac{2z_i^{\mathrm{T}} P}{(z_i^{\mathrm{T}} P z_i)^2}). \tag{25}$$

Let $\tilde{z} = (\tilde{z}_1^{\mathrm{T}}, \tilde{z}_2^{\mathrm{T}}, \cdots, \tilde{z}_m^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{R}^{mn}$ be an equilibrium of (9). Then, by Proposition 3, we know that $z_i^{\mathrm{T}} P z_i = z_j^{\mathrm{T}} P z_j$ and there exists d_i such that $d_i \tilde{z}_i = \sum_{j\neq i} a_{ij}\tilde{z}_j$. Thus it follows from (25) that

$$\left.\frac{\partial f_i}{\partial z_i}\right|_{z=\tilde{z}} = (d_i - 2l_i)\frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i} - d_i I_n, \tag{26}$$

where $l_i = \sum_{j\neq i} a_{ij}$. Let $A_{ij} = \left.\frac{\partial f_i}{\partial z_i}\right|_{z=\tilde{z}}$ for any $i, j = 1, 2, \cdots, m$. Then the block matrix $A = (A_{ij})$ is just the coefficient matrix of the linear approximation of (9) around the equilibrium $\tilde{z}$. Since P is positive definite, we can write $P = C^{\mathrm{T}} C$ with C nonsingular. Let $C\tilde{z}_i/\|C\tilde{z}_i\| = \bar{z}_i$. Then we have

$$\left.\frac{\partial f_i}{\partial z_i}\right|_{z=\tilde{z}} = (d_i - 2l_i)\frac{\tilde{z}_i \tilde{z}_i^{\mathrm{T}} P}{\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i} - d_i I_n, \tag{27}$$

Taking a similar transformation to A, we have

$$(I_m \otimes C)A(I_m \otimes C^{-1}) = (B_{ij}) = B, \tag{28}$$

where

$$B_{ii} = (d_i - 2l_i)\bar{z}_i \bar{z}_i^{\mathrm{T}} - d_i I_n, \tag{29}$$

$$B_{ik} = a_{ik}(I_n + 2\bar{z}_i \bar{z}_k^{\mathrm{T}} - \bar{z}_i \bar{z}_i^{\mathrm{T}}), \quad i \neq k. \tag{30}$$

Since $\bar{z}_i$ is a unit vector, we can construct an orthogonal matrix $[U_i,\ \bar{z}_i]$ satisfying $U_i^{\mathrm{T}} U_i = I_{n-1}$ and $U_i^{\mathrm{T}} \bar{z}_i = 0$. It is easy to check that, for all $i \neq k$,

$$U_i^{\mathrm{T}} B_{ik} U_k = a_{ik} U_i^{\mathrm{T}} U_k,\ \bar{z}_i^{\mathrm{T}} B_{ik} U_k = 0,\ \bar{z}_i^{\mathrm{T}} B_{ik} \bar{z}_k = 2a_{ik}, \tag{31}$$

$$U_i^{\mathrm{T}} B_{ii} U_i = -d_i I_{n-1},\ \bar{z}_i^{\mathrm{T}} B_{ii} \bar{z}_i = -2l_i. \tag{32}$$

Let

$$T = [U, Z] = \left[\begin{array}{ccc|ccc} U_1 & & & \bar{z}_1 & & \\ & \ddots & & & \ddots & \\ & & U_m & & & \bar{z}_m \end{array}\right]. \tag{33}$$

Then it follows from (31)-(33) that T is an orthogonal matrix satisfying

$$T^{\mathrm{T}} B T = \begin{bmatrix} U^{\mathrm{T}} B U & U^{\mathrm{T}} B Z \\ Z^{\mathrm{T}} B U & Z^{\mathrm{T}} B Z \end{bmatrix}$$

$$= \left[\begin{array}{cccc|c} -d_1 I_{n-1} & a_{12} U_1^{\mathrm{T}} U_2 & \cdots & a_{1m} U_1^{\mathrm{T}} U_m & \\ a_{21} U_2^{\mathrm{T}} U_1 & -d_2 I_{n-1} & \cdots & a_{2m} U_2^{\mathrm{T}} U_m & \\ \vdots & \vdots & \ddots & \vdots & * \\ a_{m1} U_m^{\mathrm{T}} U_1 & a_{m2} U_m^{\mathrm{T}} U_2 & \cdots & -d_m I_{n-1} & \\ \hline & & 0 & & -2L \end{array}\right] \tag{34}$$

It follows from (34) that $U^{\mathrm{T}} B U$ is a symmetrical matrix, although B may be not. Since there exists a $d_i < 0$, it follows from (34) that $U^{\mathrm{T}} B U$ is not negative semi-definite, which implies that it has at least one positive eigenvalue. Therefore, the equilibrium $\tilde{z}$ is unstable.

2) Since $\tilde{z}_i = \pm\tilde{z}_j$, we have $\bar{z}_i = \pm\bar{z}_j$ for any $i \neq j$. Hence one can let $U_i = U_1$ such that $T_i = [U_1\ \ \bar{z}_i]$ is an orthogonal matrix for every $i = 1, 2, \cdots, m$. Thus it follows from (34) that $U^{\mathrm{T}} B U = \bar{L} \otimes I_{n-1}$, where

$$\bar{L} = \begin{bmatrix} -d_1 & a_{12} & \cdots & a_{1m} \\ a_{21} & -d_2 & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & -d_m \end{bmatrix}. \tag{35}$$

Since $\sum_{j\neq i} a_{ij}\tilde{z}_j = d_i \tilde{z}_i$ and $\tilde{z}_i^{\mathrm{T}} P \tilde{z}_i = \tilde{z}_i^{\mathrm{T}} P \tilde{z}_i$, we have $\sum_{j\neq i} a_{ij}\bar{z}_j = d_i \bar{z}_i$, which implies $d_i = \sum_{j\neq i} a_{ij}\bar{z}_i^{\mathrm{T}}\bar{z}_j$. Thus, by (35), we have

$$\mathbf{1}_m^{\mathrm{T}} \bar{L} \mathbf{1}_m = \sum_{i=1}^{m}(\sum_{j\neq i} a_{ij} - d_i) = \sum_{i=1}^{m}\sum_{j\neq i}(a_{ij}(1 - \bar{z}_i^{\mathrm{T}}\bar{z}_j)) \geq 0. \tag{36}$$

We claim that $\mathbf{1}_m^{\mathrm{T}} \bar{L} \mathbf{1}_m > 0$. Otherwise, by (36), we can see that, if $a_{ij} \neq 0$, then $1 - \bar{z}_i^{\mathrm{T}}\bar{z}_j = 0$, i.e. $\bar{z}_i^{\mathrm{T}}\bar{r}_j = 1$. Thus, by the second part of Lemma 2, $\bar{z}_i$ and $\bar{z}_j$ are linear dependent, that is, there exists l such that $\bar{z}_i = l\bar{z}_j$. Substituting $\bar{z}_i = l\bar{z}_j$ into $\bar{z}_i^{\mathrm{T}}\bar{z}_j = 1$ yields $l = 1$. So $\bar{z}_i = \bar{z}_j$ as (i, j) is an edge. Thus, by the connectedness of the graph, we have $\bar{z}_1 = \bar{z}_2 = \cdots = \bar{z}_m$. This contradicts that $\bar{z}$ is a partial consensus equilibrium. Therefore, it is proved that $\mathbf{1}_m^{\mathrm{T}} \bar{L} \mathbf{1}_m > 0$, which implies $\bar{L}$ has a positive eigenvalue. From (34), it follows that $T^{\mathrm{T}} B T$ has a positive eigenvalue, which implies that A has a positive eigenvalue due to (28). Therefore, the equilibrium $\tilde{z}$ is unstable. □

Theorem 3. Consider the nonlinear multi-agent system (9) with positive definite matrix P. If the interconnection graph is a tree or a complete graph, then the almost globally static consensus on the ellipsoid family $\{\Phi_c\}_{c>0}$ is achieved.

Proof. By Theorem 1, the trajectories of (9) converge to the equilibrium set. If all the equilibria except the consensus points are unstable, then the theorem is proved. As the interconnection graph is a tree, by Proposition 4, all the equilibria except the consensus points are partial consensus points, which are unstable due to the second part of Theorem 2. As the graph is a complete graph, we have $a_{ij} = 1$ for any $i \neq j$. By Proposition 2, for any given i there exists a k_i such that $\sum_{j\neq i}\tilde{z}_j = k_i \tilde{z}_i$, which implies

$$\sum_{j=1}^{m}\tilde{z}_j = (k_i + 1)\tilde{z}_i. \tag{37}$$

If $\tilde{r}$ is a partial synchronization point, then it is unstable for the second part of Theorem 2. If $\tilde{r}$ is a non-synchronization point, then there exist $\tilde{r}_\mu$ and $\tilde{r}_\nu$ such that

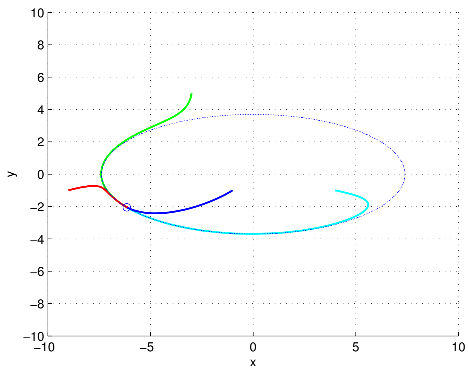

Fig. 1. Static consensus on an ellipse.

they are linear independent. From (37), we obtain that $(k_\mu + 1)\tilde{r}_\mu = (k_\nu + 1)\tilde{r}_\nu$. Thus $k_\mu = k_n u = -1 < 0$. Consequently, by the first part of Theorem 2, the equilibrium $\tilde{z}$ is unstable.

Therefore, either for a tree graph or a complete graph, the nonlinear multi-agent system (9) achieves almost global static consensus. $\square$

By Proposition 2 and Theorem 3, we obtain the following Corollary.

Corollary 1. Consider the nonlinear multi-agent system (5) with positive definite matrix P. If the interconnection graph is a tree or a complete graph, then the almost globally dynamical consensus on the ellipsoid family $\{\Phi_c\}_{c>0}$ is achieved.

Finally, we give some simulations on the nonlinear multi-agent system (9). Let $n = 2$, $m = 4$ and

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}. \tag{38}$$

The initial states of the agents are $r_1(0) = [-1, -1]^{\mathrm{T}}$, $r_2(0) = [-9, -1]^{\mathrm{T}}$, $r_3(0) = [-3, 5]^{\mathrm{T}}$ and $r_4(0) = [4, -1]^{\mathrm{T}}$. The static consensus is shown in Fig.1 and Fig.2 with $W = 0$. The dynamical consensus is shown in Fig.3 and Fig.4 with $W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. For the case $n = 3$ and $m = 4$, the dynamical consensus on an ellipsoid for (9) is displayed in Fig.5 and Fig.6 with

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \tag{39}$$

and the initial values $r_1(0) = [-0.5, -0.1, -0.1]^{\mathrm{T}}$, $r_2(0) = [-1, -1, 1]^{\mathrm{T}}$, $r_3(0) = [-1, -0.1, 2]^{\mathrm{T}}$ and $r_4(0) = [-1, 3, -1]^{\mathrm{T}}$. The simulation results has validated the effectiveness of our protocol.

4. CONCLUSIONS

In the present paper, we have proposed a simple nonlinear protocol for multi-agent systems. For the closed-loop systems, the equilibrium set has been exactly described and some sufficient conditions for the instability of some kinds of equilibrium points has been obtained. Under some conditions, the almost global consensus on ellipsoids has been

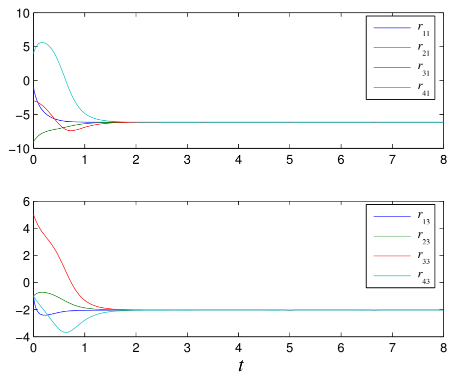

Fig. 2. Time-respond curves for static consensus on the ellipse.

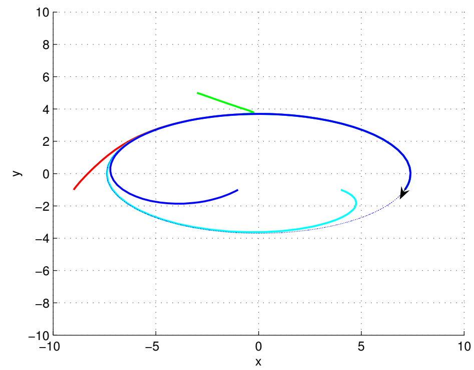

Fig. 3. Dynamical consensus on an ellipse.

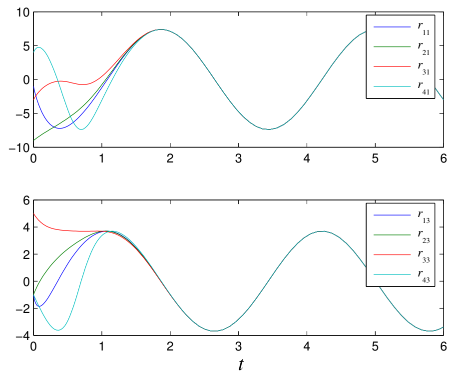

Fig. 4. Time-respond curves for dynamical consensus on the ellipse.

achieved. Simulations are given to validate the theoretical results.

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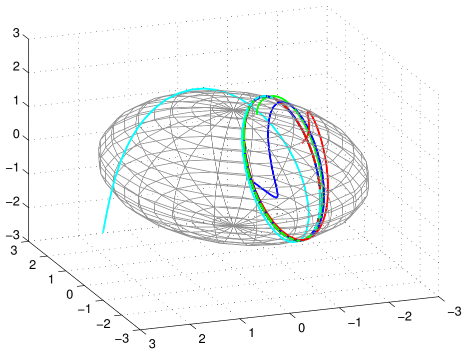

Fig. 5. Dynamical consensus on an ellipsoid.

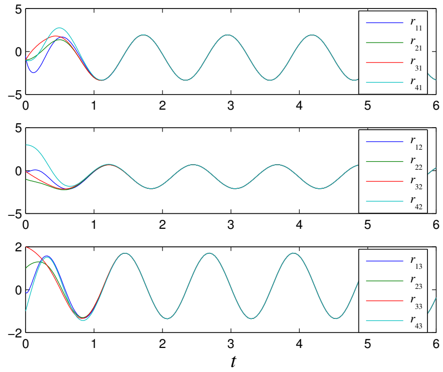

Fig. 6. Time-respond curves for dynamical consensus on the ellipsoid.

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