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Feedback linearization of Single-Input and Multi-Input Control System ⋆

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Abstract: A new algorithm is proposed for computing locally the linearizing output of single-input and multi-input nonlinear affine system. The algorithm modifies the extended Goursat normal form to iteratively obtain the successive integrations of one dimensional distributions of control system. The algorithm takes ideas from both vector field approach of feedback linearization and exterior differential system tools, hence the name *Blended Algorithm*. The proposed algorithm leads to a tower like structure depending upon the number of system inputs. Within individual tower, the coordinates are reduced one by one by finding the annihilators of vector fields at each step. The process is repeated till the single vector field is obtained for exact linearizable system. The scheme exhibits reduced computational complexity over the existing methods and can be extended to address feedback linearization of various class of control systems.

Keywords: Feedback linearization, Nonlinear system, Exterior differential system, Normal form

1. INTRODUCTION

The feedback linearization problem involves transforming a nonlinear control system into a linear system with state feedback and change of co-ordinates. Simpler linear control laws can be applied in thus obtained linear system for various control applications. The main problem is in terms of finding the 'output function' which is an necessary and sufficient condition for the existence of a state feedback and a change of coordinates transforming a given system into a linear and controllable one.

The linearization problem is one of the rich research area of control system from last four decades. Although various methods exist to find feedback linearizable form for single input non linear systems (Khalil (2002)), (Pomet and Kupka (1995)), (Respondek and Tall (2006)), (Tall and Respondek (2010)) much work needs to be done in case of multi-input systems.

Existing methods for such problem can be broadly classified into three categories. The first one is the vector field point of view in which the concept of relative degree of system was introduced in (Isidori (1990)) along with linearizing coordinate transformation which is based on linearizing output function. The procedure to find zero dynamics of the system was given in (Isidori (1990)), (Vander Schaft (1990)) and recently explored by Neilson (2009). The control algorithm for specific classes of system were developed by (Krstic (2004)) and (Krstic (2005)) but the method is limited to single input systems. Recent efforts by (Tall (2009)), (Tall (2010a)) are based on extension of explicit solvability of flow box or straightening theorem to Frobenius theorem. This exploits the fact that each system component is dependent on higher variables. The results are generalised for multi input multi output (MIMO) system (Tall (2010b)), (Tall (2010c)). For constant rank distribution, the method provides change of coordinates that simultaneously rectify vector fields of distribution.

The second category of feedback linearization problem have been approached in terms of co-distributions (1-forms) exploiting the tools of exterior differential geometry (Shankar Sastry (1999)). The GS algorithm proposed the exact linearization of nonlinear systems to Brunovsky normal form under non linear feedback (Gardner and Shawick (1991)), (Gardner and Shadwick (1992)). The GS algorithm can be extended to MIMO system but is computationally exhaustive and works only for exact linearization.

The third category is relatively newly developed which combines ideas from both the view points (Mullhaupt (2006)), (Willson *et al.* (2009)), (Willson *et al.* (2011)). In the Quotient Submanifold (QS) method, the one dimensional distributions and projections along these submanifolds are successively integrated. The advantage being the original structure of the distribution is maintained while reduction of base manifold is performed at each step. However, the Quotient Submanifold method is not extended for MIMO systems. The proposed method in the present paper falls in line of third category but provides a different viewpoint from the quotient submanifold method as explained in subsequent sections.

The main contribution of this paper lies in proposing a feedback linearizing algorithm for general control affine systems for which the linearizing output is obtained in a computationally simple way. The *Blended Algorithm* has been shown to find the feedback linearizing coordinates for single input control systems (Rachit *et al.* (2013)). In the present paper, the algorithm is extended for multi-input control affine systems. The system equations are represented in forms using exterior differential geometry tools. The tower structure is developed depending upon number of inputs. Within individual tower, the coordinates

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are reduced one by one finding the annihilators of vector field at each step. The process is repeated to obtain single vector field, if system is exact linearizable. The coordinate transformation in context of exterior differential systems corresponds to coordinate transformation along with state feedback in vector field notation. The algorithm can obtain solutions for all subclasses of feedback linearizable control affine systems, for example strict feedforward forms, strict-feedforward nice and feedforward forms.

The paper is organized as follows: Section II presents the brief description of linearization problem for single input and multi-input control systems. The new algorithm is proposed in Section III with subsections covering application to numerical examples. The conclusion is presented in Section IV.

2. PREVIOUS WORK

The present section includes the brief overview of feedback linearization problem in terms of vector field and forms point of view focusing on multi-input systems and some theory on exterior differential systems.

2.1 Vector field viewpoint

Consider a multi-input, multi-output system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y = (y_1, ..., y_m) &= (h_1(x), ..., h_m(x)) \end{aligned} \tag{1}$$

where $x = (x_1, ..., x_n) \in R^n$ are state coordinates, $u = (u_1, ..., u_m) \in R^m$ are control inputs, $f, g_1, ..., g_m$ are smooth vector fields and $h_1, ..., h_m$ are smooth functions. The system has vector relative degree $r = (r_1, ..., r_m)$ at point x_0 if $L_{g_j} L_f^k h_i(x) = 0$ for all $i \le j \le m$ and $0 \le k \le r_i - 2$ with $L_{g_j} L_f^{r_i-1} h_i(x) \ne 0$ The relative degree is exactly the number of times one has to differentiate the output y_i in order to have at least one component of input vector appearing. The system can thus be represented by

$$y^r = L_f^r h(x) + A(x)u$$

where

$$y^r = (y_1^{r_1}, ..., y_m^{r_m})^T$$

$$L_f^r h = (L_f^{r_1} h_1, ...L_f^{r_m} h_m)^T \text{ and } A = (L_{g_i} L_f^{r_j-1} h_j)_{i,j}$$

If the square matrix $A(x)$ is nonsingular at x_0, then the input u can be expressed in terms of new input ν around the point x_0 by

$$u = A^{-1}(x)\nu - A^{-1}(x)L_f^r h(x)$$

For $1 \le i \le m$,

$$\begin{aligned} \phi_1^i(x) &= h_i(x) \\ \phi_2^i(x) &= L_f h_i(x) \\ \vdots &= \vdots \\ \phi_{r_i}^i(x) &= L_f^{r_i-1} h_i(x) \end{aligned}$$

Under this feedback and change of coordinates the original system is transformed into chain of integrators.The system in new coordinates can be written as

$$\begin{aligned} \dot{\xi}_1^i &= \xi_2^i \\ \dot{\xi}_2^i &= \xi_3^i \\ \vdots &= \vdots \\ \dot{\xi}_{r_m}^i &= \nu_m \end{aligned}$$

A necessary and sufficient condition for a multi-input multioutput system to be transformed into a chain of integrators is that it has a vector relative degree $\{r_1, ...r_m\}$ such that $r_1 + ... + r_m = n$. It is not always easy to obtain such coordinate transformation as it involves solving partial differential equation.

2.2 Linearization problem definition in Forms

With a slight change in notation, we define a one-form ω by

$$\omega = \sum_{i=1}^{n} \omega_i dx_i \tag{2}$$

system (1) is feedback linearizable, if and only if there exists a scalar function, namely an integrating factor, $r : M \to \mathbb{R}$ such that

$$dh = r\omega \tag{3}$$

A necessary and sufficient condition for the exactness of the one-form $r\omega$ is $d(r\omega)$=0. Hence feedback linearizability is equivalent to the existence of an integrating factor $r : M \to \mathbb{R}$ such that

$$\frac{\partial r\omega_i}{\partial x_j} = \frac{\partial r\omega_j}{\partial x_i} \qquad 1 \le i < j \le n \tag{4}$$

Given an integrating factor r, one can obtain h from (3) and thus the linearizing feedback and the coordinate transformation map can be easily constructed. Hence, the problem of obtaining the desired coordinate change and feedback reduces to that of obtaining an integrating factor. But solving (4) for r is a difficult problem.

2.3 Exterior Differential Systems

The theory and results of exterior differential system can be applied to the problem of feedback linearization. The Pfaffian systems represents a set of first order ordinary differential equations.

Single input control system representation:

The control system of the form

$$\dot{x} = f(x) + g(x)u, \qquad x \in M \tag{5}$$

can also be thought of as a Pfaffian system of codimension 2 in R^{n+2}. The corresponding ideal is generated by the codistribution

$$I = \{dx_i - f_i(x,u)dt : i = 1, 2, ...n\}$$

The $n+2$ variables for the Pfaffian system corresponds to the n states,one input and time t. For the special case of the affine system (5) the co-distribution becomes

$$I = \{dx_i - (f_i(x) + g_i(x)u)dt : i = 1, 2, ...n\}$$

The differential ideal is the subset of algebraic ideal generated by Pfaffian system which satisfy the Frobenius condition. This differential subideal can be found by taking the derived flag of the Pfaffian system. Let $I^{(0)} = \{\omega_1, ..., \omega_s\}$ be the algebraic ideal generated by independent 1-forms $\omega_1, ..., \omega_s$. We define $I^{(1)}$ as

$$I^{(1)} = \{\lambda \in I^{(0)} : d\lambda \equiv 0 \ mod \ I^{(0)}\} \subset I^{(0)}$$

the ideal $I^{(1)}$ is called the *first derived system*. If $I^{(0)} = (\Delta_0)^{\perp}$, then $I^{(1)} = (\Delta_0 + [\Delta_0, \Delta_0])^{\perp}$. one may inductively continue this procedure of obtaining derived systems and define

$$I^{(2)} = \{\lambda \in I^{(1)} : d\lambda \equiv 0 \ mod \ I^{(1)}\} \subset I^{(1)}$$

or in general, $I^{(k+1)} = \{\lambda \in I^{(k)} : d\lambda \equiv 0 \ mod \ I^{(k)}\} \subset I^{(k)}$ this procedure results in a nested sequence of codistributions

$$...I^{(k)} \subset I^{(k-1)} \subset ...... \subset I^{(1)} \subset I^{(0)} \tag{6}$$

Decomposition of system I into a tower gives the derived flag which has the tower like structure. The structure of jth tower is as follows

$I^{(0)}$:	ω_1	ω_2	....	...	ω_s
$I^{(1)}$:	ω_1	ω_2	...	ω_{s-1}	
	..	..	..		
	..	..			
	ω_1	ω_2			
$I^{(s)}$:	ω_1				

If we define $\Delta_0 = (I^{(0)})^{\perp}$, $\Delta_1 = (I^{(1)})^{\perp}$, and in general $\Delta_k = (I^{(k)})^{\perp}$, then $I^{(k)} = \Delta_k^{\perp}$ then $I^{k+1} = (\Delta_k + [\Delta_k, \Delta_k])^{\perp}$. The sequence of decreasing codistributions (6), called the *derived flag* of $I^{(0)}$, is associated with the sequence of increasing distributions, called the *filtration* of Δ_0,

$$...\Delta_k \supset \Delta_{k-1} \supset \cdots \supset \Delta_1 \supset \Delta_0 \tag{7}$$

If the dimension of each codistribution is constant then there will be an integer N such that $I^{(N)} = I^{(N+1)}$. This integer N is called the derived length of I. A basis of 1-forms ω_i for I is said to be *adapted to the derived flag* if a basis for each derived system $I^{(j)}$ can be chosen to be some subset of the ω_i's. The codistribution $I^{(N)}$ is always integrable by definition since

$$dI^{(N)} \equiv 0 \ mod \ I^N$$

The codistribution $I^{(N)}$ is the *largest integrable subsystem* in I. Therefore, if $I^{(N)} \neq \{0\}$ then there exist functions $h_1, \cdots, h_r$ such that $\{dh_1, \cdots, dh_r\} \subset I$.

Multi-input control system representation:
The multi-input control system of the form (1) can be represented as a Pfaffian system of codimension $m+1$ in R^{n+m+1}. The $n+m+1$ variables for the Pfaffian system corresponds to the n states, m input and time t. For the special case of the affine system (1) the co-distribution becomes,

$I = \{dx_i - (f_i(x) + \sum_{j=1}^{m} g_{ij}(x)u_j)dt$: $i = 1, 2, ...n\}$

The decomposition of system into a tower gives the derived flags of the system. For system (1) the system will be decomposed in m towers. Collecting all the m towers for system, the top rows taken together generate I and succeeding rows taken together generate the derived flag of I.

$I^{(0)}-$	ω_1^1	ω_2^1	..	$\omega_{k_j}^1$	..	..	ω_1^m	ω_2^m	..	$\omega_{k_m}^m$
:	:							:		
:							ω_1^m			
$I^{(s-1)}-$	ω_1^1	ω_2^1								
$I^{(s)}-$	ω_1^1									

Extended Goursat Normal Form. Let I be a Pfaffian system on R_{n+m+1} of codimension $n+m+1$. If there exists a set of generators, α_i^j; $i = 1, \cdots, s_j; j = 1, \cdots, m$ for I and an integrable one form $\pi (= dt)$ such that

$$\begin{array}{ll} d\alpha_i^j \equiv -\alpha_{i+1}^j \wedge \pi \text{ mod } I^{(s_j - i)}, & 1 \leq i \leq s_j - 1, \\ d\alpha_{s_j}^j \not\equiv 0 \text{ mod } I, & \end{array} \tag{8}$$

Then there exists a coordinate system $z_1, z_2, \cdots, z_s$ such that I is in extended Goursat normal form:

$$I = dz_i^j - z_{i+1}^j dz_0; \ \ i = 1, \cdots, s_j; j = 1, \cdots, m$$
$$\text{and } dz_0 = \pi$$

With the above conditions the Pfaffian system can be transformed to extended Goursat normal form and can be viewed as linearization theorem. The Goursat normal form can be thought of as a single chain of integrators, while the extended Goursat normal form consists of many chains of such integrators (Shankar Sastry (1999)), (Gardner and Shawick (1991)) and (Gardner and Shadwick (1992)).

3. EXTENDED BLENDED ALGORITHM

The **Blended Algorithm** for single input non linear control system has been earlier used to sucessfully obtain linear coordinates via coordinate transformation (Rachit *et al.* (2013)).

In this section the algorithm is extended for multi-input control system. The procedure is iterative in nature and starts with the representation of the control system as Pfaffian system $I^{(0)}$

3.1 Proposed Algorithm

(1) **System Representation in forms:**
Consider a multi-input control system (1). The control system defines an associated Pfaffian system of codimension $m+1$ in $\mathbb{R}^{n+m+1}$. The corresponding ideal $I^{(0)}$ is generated by the codistribution as follows:
$I^{(0)} = (\Delta_0)^{\perp}$
$= \{dx_i - (f_i(x) + \sum_{j=1}^{m} g_{ij}(x)u_j)dt : i = 1, 2, ...n\}$
$= \{\omega_1 \quad \omega_2 \quad ... \quad ... \quad \omega_n\}$
For simplicity, let us consider the case $m = 2$. The pfaffian system $I^{(0)}$ and its annihilating distribution Δ_0 are given by
$I^{(0)} = \{dx_i - (f_i(x) + g_{i1}(x)u_1 + g_{i2}(x)u_2)dt : i = 1, 2, ...n\}$
and

$$\Delta_0 = (I^{(0)})^{\perp} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ (f + g_1 u_1 + g_2 u_2) \end{pmatrix} \right\} \tag{9}$$

(2) **Division of Towers:**
In this step the derived flag of $I^{(0)}$ is constructed and divided into different towers. The system decomposition into various towers depends on system input. The numbers of towers formed will be equal to the number of inputs of system. This encodes the derived flags of the system.
For a two input system, the two towers will be formed as shown below.
$I^{(0)} = \quad \{\omega_1^1 \ \omega_2^1 \ \ ... \ \ \omega_k^1 \qquad \omega_1^2 \ \ \omega_2^2 \ \ ... \ \ \omega_l^2\}$
Note: In some cases the ω_i's are annihilators of both the inputs. Such ω_i's shall be placed in the tower based upon

generation of derived flag.

(3) **Reduction:**
Once the tower structure is formed, the towers are reduced individually and independently from one another.
The appropriate generators (basis) of $I^{(0)}$ are found that are adapted to the derived flag. If the basis of $I^{(0)}$ satisfies the congruence condition (8), then one gets the derived ideal, $I^{(1)}$ ($I^{(1)} \subset I^{(0)}$). Assuming that $I^{(1)}$ exists and if $\omega_i \in I^{(0)}$ does not satisfy the congruence condition (8), then it is suitably modified to $\overline{\omega_i}$, where $\overline{\omega_i}$ is written as a linear combination of generators ω_j of $I^{(0)}$ satisfying the congruence condition (8).
In the vector field context it means the ω_i fails to annihilate $(\Delta_0 + [\Delta_0, \Delta_0])^{\perp}$, hence it is replaced with $\overline{\omega_i}$. Once the derived ideal ($I^{(1)} \subset I^{(0)}$) has been fixed along with its generators $\{\overline{\omega_1}, \overline{\omega_2}, \cdots, \overline{\omega_{s-1}}\}$, the above process is repeated to obtain the successive derived ideals $I^{(i)}$ and complete the filtration process. The decomposition of the ideal $I^{(0)}$ into a tower structure is thus obtained.
A step by step illustration of above process for a 5-dimensional, two input system is explained below.
- Tower structure for 5 dimension system with two inputs will be as follows:

$$
\begin{array}{llllll}
I^{(0)} = & \omega_1^1 & \omega_2^1 & \omega_3^1 & \omega_1^2 & \omega_2^2 \\
I^{(1)} = & \overline{\omega_1^1} & \overline{\omega_2^1} & & \overline{\omega_1^2} & \\
I^{(2)} = & \overline{\overline{\omega_1^1}} & & & &
\end{array}
$$

- For first tower, the derived flag $I^{(1)}$ consist of annihilators of elements of input g_1. Similarly, for second tower the derived flag $I^{(1)}$ consist of annihilators of elements of input g_2. This forms the input to next iteration.
- For first tower, the derived flag $I^{(2)}$ consist of annihilators of elements of g_1 and $[f, g_1]$.
- The process is continued until single element is remaining.

(4) **Linearizing output:** For Tower-1, the final modified (adapted) generators of $I^{(0)}$ = { $\overline{\overline{w_1^1}}$, $\overline{w_2^1}$, w_3^1}. Here $\overline{\overline{w_1}}$ is the exact one form dh which annihilates the distribution g_1 and $[f, g_1]$. Solving for dh implies integrating an exact 1-form to get the linearizing output $h(x)$ e.g. refer (Forsyth (1959)).

$$
\phi_1(x) = \begin{pmatrix} h_1(x) \\ L_f h_1(x) \\ L_f^2 h_1(x) \end{pmatrix} \tag{10}
$$

The complete linearizing coordinates for the previous example considered shall be
$\phi_1(x) = (h_1(x), L_f h_1(x), L_f^2 h_1(x), h_2(x), L_f h_2(x))$

Linearizing coordinates for general system can be obtained by taking the transformation

$$
\phi(x) = \mathrm{col}(\underbrace{\phi_1^1, \cdots, \phi_{r_1}^1}_{Input-1}, \cdots, \underbrace{\phi_1^m, \cdots, \phi_{r_m}^m}_{Input-m})
$$

where

$$
\phi^i(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \\ \cdots \\ L_f^{r_1} h(x) \end{pmatrix} \tag{11}
$$

Remarks: The drawback with the GS algorithm which is based on the Goursat normal form is of obtaining the linearizing coordinates $(z_1, ...z_n)$ to represent the adapted generators of $I^{(0)}$. In the Blended Algorithm, the linearizing coordinates are obtained once the linearizing output function $h(x)$ is known (3),(4). As compared to the vector field approach to linearization of control system the evaluation of the output function $h(x)$ in the Blended Algorithm is made simpler because of the tower formation. The output function $h(x)$ appears in the generator of $I^{(s-1)}$

$$
\begin{array}{lllllll}
I^{(s-1)}: & dh(x) & & & & & \\
I^{(s-2)}: & dh(x) & dL_f h(x) & & & & \\
\vdots & .. & .. & .. & & & \\
\vdots & .. & .. & .. & .. & & \\
\vdots & .. & .. & .. & .. & .. & \\
I^{(0)}: & dh(x) & dL_f h(x) .... & & ... & ... & dL_f^{s-1} h(x)
\end{array}
$$

3.2 Nature of coordinate transformation

The coordinate transformation achieved in the context of exterior differential systems corresponds to coordinate transformation along with state feedback in vector field notation. The state space R^{n+m+1} does not differentiates between states, input and time. However if congruence condition is satisfied a time invariant state feedback and coordinate change can be found.

3.3 Numerical examples

A few examples solved by the proposed algorithm are illustrated as follows. To motivate the examples for multi-input system and for better understanding of the algorithm, we first present an example of single input system.

Example 1: Single input control system example

Let us consider the system in Feedforward form (Tall (2009)),

$$
\begin{aligned}
\dot{x_1} &= x_2 + (\frac{1}{2}x_2 - \frac{1}{12}x_3x_4)u \\
\dot{x_2} &= x_3 + \frac{1}{2}x_3 u \\
\dot{x_3} &= x_4 + x_4 u \\
\dot{x_4} &= u
\end{aligned} \tag{12}
$$

where

$$
f(x) = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} \qquad g(x) = \begin{pmatrix} \frac{1}{2}x_2 - \frac{1}{12}x_3x_4 \\ \frac{1}{2}x_3 \\ x_4 \\ 1 \end{pmatrix} \tag{13}
$$

The form representation of above system is given as

$$
\begin{aligned}
\omega_1 &= dx_1 - [x_2 + (\frac{1}{2}x_2 - \frac{1}{12}x_3x_4)u]dt \\
\omega_2 &= dx_2 - [x_3 + \frac{1}{2}x_3 u]dt \\
\omega_3 &= dx_3 - [x_4 + x_4 u]dt \\
\omega_4 &= dx_4 - u dt
\end{aligned} \tag{14}
$$

The annihilator of distribution are given by

$$
(\Delta_0)^{\perp} = [\omega_1, \omega_2, \omega_3, \omega_4] \tag{15}
$$

The annihilators of g is given by

$$\begin{aligned}\overline{\omega_1} &= \omega_1 - \frac{1}{2}x_2\omega_4 + \frac{1}{6}x_4\omega_2 \\ \overline{\omega_2} &= \omega_2 - \frac{1}{2}x_3\omega_4 \\ \overline{\omega_3} &= \omega_3 - x_4\omega_4\end{aligned} \tag{16}$$

The annihilators of $[f,g]$ is given by

$$\begin{aligned}\overline{\overline{\omega_1}} &= \overline{\omega_1} - \frac{1}{2}x_4\overline{\omega_2} \\ \overline{\overline{\omega_2}} &= \overline{\omega_2} - \frac{1}{2}x_4\overline{\omega_3}\end{aligned} \tag{17}$$

The annihilators of $[f,[f,g]]$ is given by, $\overline{\overline{\overline{\omega_1}}} = \overline{\overline{\omega_1}} - \frac{1}{3}x_4\overline{\overline{\omega_2}}$

$$\overline{\overline{\overline{\omega_1}}} = \omega_1 - \frac{1}{2}x_4\omega_2 + \frac{1}{6}x_4^2\omega_3 + [\frac{x_4x_3}{3} - \frac{x_4^3}{6} - \frac{1}{2}x_2]\omega_4 \tag{18}$$

$$\overline{\overline{\overline{\omega_1}}} = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \frac{\partial h}{\partial x_4} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} \tag{19}$$

$$\overline{\overline{\overline{\omega_1}}} = \begin{pmatrix} 1 & \frac{-x_4}{2} & \frac{x_4^2}{6} & \frac{x_4x_3}{3} - \frac{x_4^3}{6} - \frac{x_2}{2} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} \tag{20}$$

Solving equation (27) for h, the linearizing output is given as $h(x) = x_1 - \frac{x_2x_4}{2} + \frac{x_3x_4^2}{6} - \frac{x_4^4}{24}$

Example 2: Multi-input control system example
Let us consider the system (Tall (2010c)),

$$\begin{aligned}\dot{x_1} &= x_2(1+x_3) \\ \dot{x_2} &= x_3(1+x_1) - x_2u_1 \\ \dot{x_3} &= x_1 + x_5 + x_1^2 + (1+x_3)u_1 \\ \dot{x_4} &= x_5 + x_1^2 \\ \dot{x_5} &= u_2\end{aligned} \tag{21}$$

where

$$f(x) = \begin{pmatrix} x_2(1+x_3) \\ x_3(1+x_1) \\ x_1+x_5+x_1^2 \\ x_5+x_1^2 \\ 0 \end{pmatrix}$$
$$g_1(x) = \begin{pmatrix} 0 \\ -x_2 \\ 1+x_3 \\ 0 \\ 0 \end{pmatrix} \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{22}$$

The form representation of above system is given as

$$\begin{aligned}\omega_1 &= dx_1 - x_2(1+x_3)dt \\ \omega_2 &= dx_2 - [x_3(1+x_1) - x_2u_1]dt \\ \omega_3 &= dx_3 - [x_1 + x_5 + x_1^2 + (1+x_3)u_1]dt \\ \omega_4 &= dx_4 - [x_5 + x_1^2]dt \\ \omega_5 &= dx_5 - u_2dt\end{aligned} \tag{23}$$

The annihilator of distribution is given by

$$(\Delta_0)^{\perp} = [\omega_1, \omega_2, \omega_3, \omega_4, \omega_5] \tag{24}$$

Division of towers based on the number of inputs (u_1, u_2) :

- $(\omega_1, \omega_2, \omega_3)$ belong to Tower-1 based on input u_1
- (ω_4, ω_5) belong to Tower-2 based on input u_2

$$\begin{aligned}I^{(0)} &= \omega_1^1 \quad \omega_2^1 \quad \omega_3^1 \quad \omega_4^2 \quad \omega_5^2 \\ I^{(1)} &= \overline{\omega_1^1} \quad \overline{\omega_2^1} \quad \overline{\omega_4^2} \\ I^{(2)} &= \overline{\overline{\omega_1^1}}\end{aligned}$$

The annihilators of g_1 is given by

$$\begin{aligned}\overline{\omega_1^1} &= \omega_1^1 \\ \overline{\omega_2^1} &= \omega_2^1 + \frac{x_2}{1+x_3}\omega_3^1\end{aligned} \tag{25}$$

The annihilators of g_2 is given by, $\overline{\omega_4^2} = \omega_4^2$

The annihilators of $[f,g_1]$ is given by, $\overline{\overline{\omega_1^1}} = \overline{\omega_1^1}$

$$\overline{\overline{\omega_1^1}} = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \frac{\partial h}{\partial x_4} & \frac{\partial h}{\partial x_5} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix} \tag{26}$$

$$\overline{\overline{\omega_1^1}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix} \tag{27}$$

Solving equation (27) for h, the linearizing output is given as $h(x) = x_1$. The linearizing coordinates for complete system can be obtained as

$$\phi(x) = \begin{pmatrix} \overline{\overline{\omega_1^1}} \\ \overline{\omega_2^1} \\ \overline{\omega_3^1} \\ \overline{\omega_4^2} \\ \omega_5^2 \end{pmatrix} \tag{28}$$

Example 3: Multi-input control system example
Let us consider the system (Tall (2010c)),

$$\begin{aligned}\dot{x_1} &= x_2 + x_2x_1 \\ \dot{x_2} &= \sin x_3 \\ \dot{x_3} &= x_3 + e^{x_2}u_1 \\ \dot{x_4} &= -(1+x_5)u_2\dot{x_5} = x_5 + (1+x_4)u_2\end{aligned} \tag{29}$$

where

$$f(x) = \begin{pmatrix} x_2+x_2x_1 \\ \sin x_3 \\ x_3 \\ 0 \\ x_5 \end{pmatrix} \quad g_1(x) = \begin{pmatrix} 0 \\ 0 \\ e^{x_2} \\ 0 \\ 0 \end{pmatrix}$$
$$g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -(1+x_5) \\ (1+x_4) \end{pmatrix} \tag{30}$$

The form representation of above system is given as

$$\begin{aligned}\omega_1 &= dx_1 - [x_2 + x_2x_1]dt \\ \omega_2 &= dx_2 - [\sin x_3]dt \\ \omega_3 &= dx_3 - [x_3 + e^{x_2}u_1]dt \\ \omega_4 &= dx_4 + [(1+x_5)u_2]dt \\ \omega_5 &= dx_5 - [(1+x_4)u_2]dt\end{aligned} \tag{31}$$

The annihilator of distribution is given by

$$(\Delta_0)^{\perp} = [\omega_1, \omega_2, \omega_3, \omega_4, \omega_5] \tag{32}$$

Division of towers based on the number of inputs (u_1, u_2) :

- $(\omega_1, \omega_4, \omega_5)$ belong to tower-1 based on input u_2
- (ω_2, ω_3) belong to tower-2 based on input u_1

$$I^{(0)} = \omega_1^1 \quad \omega_4^1 \quad \omega_5^1 \qquad \omega_2^2 \quad \omega_3^2$$

$$I^{(1)} = \overline{\omega_1^1} \quad \overline{\omega_4^1} \qquad \overline{\omega_2^2}$$

$$I^{(2)} = \overline{\overline{\omega_1^1}}$$

The annihilators of g_2 are given by,
$\overline{\omega_1^1} = \omega_1^1, \qquad \overline{\omega_4^1} = \omega_4^1 + \frac{1+x_5}{1+x_4}\omega_5^1$

The annihilators of g_1 are given by, $\quad \overline{\omega_2^2} = \omega_2^2$

The annihilators of $[f, g_2]$ is given by, $\quad \overline{\overline{\omega_1^1}} = \overline{\omega_1^1}$

$$\overline{\overline{\omega_1^1}} = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \frac{\partial h}{\partial x_4} & \frac{\partial h}{\partial x_5} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix} \tag{33}$$

$$\overline{\overline{\omega_1^1}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix} \tag{34}$$

Solving equation (34) for h, the linearizing output is given as $h(x) = x_1$. The linearizing coordinates for complete system is given by,

$$\phi(x) = \begin{pmatrix} \overline{\overline{\omega_1^1}} \\ \overline{\omega_4^1} \\ \omega_5^1 \\ \overline{\omega_2^2} \\ \omega_3^2 \end{pmatrix} \tag{35}$$

3.4 Advantages of the Blended Algorithm

The Blended Algorithm uses the tower structure for reduction purpose and the main advantages are as follows:

- The proposed method is computationally easy as there is no need to obtain linear transformation at every step as in the case of Quotient Submanifold method. The final step of obtaining linearizing output is also simple to calculate.
- In comparision to GS algorithm the proposed method of finding the linearizing coordinates is computationally easy and can be applied for all subclasses of feedback linearizable control affine systems.

4. CONCLUSION

A new iterative algorithm has been proposed for computing locally the linearizing output of multi-input nonlinear affine system. The idea of extended Goursat normal form is modified to obtain the successive integrations of one dimensional distributions of control system. The algorithm provides a simple way of finding the linearizing change of coordinates by computing the output function. The future work is aimed at extending the proposed algorithm to partial feedback linearization problem.

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