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Synchronization for interacting clusters of generic linear agents and nonlinear oscillators: a unified analysis ⋆

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Abstract: This paper aims to exploit a unified analysis that is feasible in tackling the synchronization for multiple interacting clusters of generic linear agents and nonlinear oscillators, which are usually considered separately using different methodologies. With this newly proposed analysis, we are able to not only provide the sufficient conditions in terms of coupling strength and structure of the coupling topology which are easy to verify but also explicitly specify the lower bound for the coupling strength as well as the convergence rate. All the results and methodologies are applicable to the classic complete consensus/synchronization problem whereas there is only one cluster of agents.

Keywords: Group synchronization; unified analysis; coupling strength; coupling topology; convergence rate

1. INTRODUCTION

In networks of agents, the aim of "consensus" is to reach an agreement regarding the state of all agents by sharing information between them according to a prescribed structure Jadbabaie & Lin [2003], Ren *et al.* [2007], Ren & Beard [2005]. However, a real-world complex network may be composed of multiple smaller subnetworks, e.g., communities of natural oscillators are usually composed of interacting sub-populations Winfree [1980], and such a network in general exhibits richer scenarios than just consensus or synchronization. Very recently increasing attention has been paid to cluster/group synchronization [1] Montbrió *et al.* [2004], Kori *et al.* [2009], Belykh *et al.* [2001], Wu *et al.* [2009], Liu & Chen [2011], Yu & Wang [2010], Xia *et al.* [2011], Qin & Yu [2013a]. This phenomenon is observed when agents in a network fall into several subgroups, called clusters throughout this work, for which agents from the same cluster asymptotically reach state agreement in the presence of both intra- and inter-cluster couplings among agents. It frequently arises when agents within a cluster are cooperative, but are competitive with those in another cluster, as those considered in Liu & Chen [2011], Wu *et al.* [2009], Qin & Yu [2013a], Yu & Wang [2010], Qin *et al.* [2013b]. A basic question is to state under which conditions with respect to the coupling strengths and the coupling topology of the network each cluster of agents can converge to or maintain their synchronization behavior in the presence of interactions among different clusters.

With the cooperative and competitive inter-cluster coupling scheme, synchronization problem for interacting cluster of nonlinear oscillators is considered Wu *et al.* [2009], where pinning control technique is used to help secure there is no consensus between any two different clusters and moreover, coupling topologies are assumed to be undirected and connected. This undirected framework is later relaxed to be directed one in Liu & Chen [2011] via also pinning control technique, but the coupling topology for each cluster is assumed to be strongly connected. On the other hand, this scheme is developed in multi-agent consensus community to consider the group/cluster consensus problem. Specifically, Yu & Wang [2010] investigates group consensus for agents with single-integrator dynamics, whereas some sufficient conditions in terms of linear matrix inequalities (LMIs) are presented for guaranteeing the group consensus. It is not clear under what kind of coupling topologies such LMIs are feasible. This model

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[1] To clarify the difference between group and cluster synchronization, throughout the paper, by group synchronization is meant that for any initial states of the agents, all the agents within the same cluster finally reach complete synchronization, while there may or may not be consensus between different clusters, depending on the initial values of the agents Yu & Wang [2010], Xia *et al.* [2011]. If for some given initial states, not only all the agents within the same cluster reach complete synchronization, but also there is no consensus between any two different clusters, then cluster synchronization is said to be achieved for such initial states. Obviously, cluster synchronization implies group synchronization if it can be achieved for any initial states.

is later revisited in Xia *et al.* [2011], where a different algebraic condition is proposed. Although this condition has simpler form, it still cannot specify the relations between the coupling topology and the group synchronization behavior, which is a problem of focal interest in multi-agent consensus and synchronization community. Some efforts have been made towards this problem in Qin *et al.* [2013b], which investigates the group consensus for double-integrator agents and further provide some sufficient conditions in terms of the strength and structure of the coupling topology, but conditions on the coupling topology are still very restrictive. Very recently, Qin & Yu [2013a] considers the cluster synchronization for agents with generic linear agents and exploit, with the help of pinning control techniques, under what kind of conditions cluster synchronization can be achieved regardless of the strength of the coupling topology. Although the sufficient conditions provided in Qin & Yu [2013a] are easy to verify, it is still unclear what will happen if there is no pinning controller for each cluster of agents and what if there are collectively cyclic inter-cluster couplings.

With mainly the above inspirations, this paper aims to develop a methodology that is feasible to perform a unified analysis to the group synchronization problem for generic linear agents and nonlinear oscillators under the general setting regarding the coupling topology. Our contributions are three-fold. (1) This unified analysis allows each cluster to have relaxed topological structure which is necessary to guarantee the complete synchronization of agents within the same cluster, i.e., coupling topology of each cluster is only required to have a directed spanning tree. (2) Differently from the algebraic conditions derived in most of the existing literature, we exploit sufficient conditions that guarantee group synchronization in terms of the structure and strength of the coupling topology and they are very easy to verify. More importantly, such conditions explicitly specify the relation between the group synchronization behavior and the coupling topology. (3) Thanks to the newly proposed Lyapunov method, which allows us to explicitly specify both the lower bound for the coupling strength of each cluster as well as the convergence rate under the general setting.

Notations: Let $\|x\|$ denote the Euclidean norm of a finite dimensional vector x.Let $diag\{\Xi_1,\dots,\Xi_p\}$ denote the block diagonal matrix with the i-th main diagonal block being square matrix Ξ_i. $\lambda_{min}(M)$ and $\lambda_{max}(M)$ denote respectively the smallest and largest eigenvalues of symmetric matrix M. For any $m\times 1$ vector α, denote by $\mathrm{diag}(\alpha)\in\mathbb{R}^{m\times m}$ the diagonal matrix with the ith $(i=1,2,\dots,m)$ diagonal element being the ith element of α.

2. PRELIMINARIES

2.1 Problem formulation

Consider a group of N identical agents taking the following generic linear system dynamics:

$$\dot{x}_i = Ax_i + BK\sum_{j=1,j\neq i}^{N} c_{ij}a_{ij}(x_j(t)-x_i(t)), \qquad (1)$$

where $x_i=[x_i^1,\dots,x_i^n]^T\in\mathbb{R}^n$ is the state of node i, $i=1,\dots,N$, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, and $K\in\mathbb{R}^{m\times n}$ is the feedback matrix to be determined. The complex network of N coupled nonlinear oscillators evolves according to the following dynamics Liu & Chen [2011], Wu *et al.* [2009], Qin *et al.* [2011], Wu [2005]:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1,j\neq i}^{n} c_{ij}a_{ij}\Gamma(x_j(t)-x_i(t)), \qquad (2)$$

where f is a continuous vector function which satisfies the following Lipschitz condition.

Assumption 1. There exists a constant $\rho>0$ such that

$$\|f(x_1)-f(x_2)\|^2 \le \rho\|x_1-x_2\|^2, \forall x_1,x_2\in\mathbb{R}^n;\ \forall t\ge 0.$$

and Γ is a positive-definite inner coupling matrix. For both the two systems, a_{ij} and c_{ij}, which specify the couplings among the nodes (refer to either the linear agent or the nonlinear oscillator), are respectively defined as follows: $a_{ij}\neq 0$ if there is a coupling (i.e., direct edge) from node j (which refers to either the linear agent or the nonlinear oscillator) to node i and otherwise $a_{ij}=0$; $c_{ij}=c_\ell$, if $\bar{i}=\bar{j}=\ell$, while $c_{ij}=1$ if $\bar{i}\neq\bar{j}$. Note that c_ℓ, $\ell=1,\dots,p$ measures the coupling strength for agents within cluster $\mathcal{V}_\ell$.

Group synchronization: The group synchronization is said to be achieved for system (2) if, for any initial states of the agents, there holds $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$, $\forall\bar{i}=\bar{j}$, $i,j=1,\dots,N$. The group synchronization is said to be achieved for system (1) if, there exists a feedback matrix K such that for any initial states of the agents, there holds $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$, $\forall\bar{i}=\bar{j}$, $i,j=1,\dots,N$.

Remark 2. In general, there is no consensus between any two different clusters of agents, as observed in Xia *et al.* [2011] and Qin & Yu [2013a], for systems (1) and (2), though complete synchronization does happen for only a thin set of initial states. One approach to avoid the complete synchronization is adding external controller to a small fraction of the nodes in each cluster via pinning control technique, as that considered in Qin & Yu [2013a], Liu & Chen [2011], Wu *et al.* [2009]. As mentioned earlier, this is equivalent to add a virtual leader to each cluster. Then no consensus between any two different clusters can be achieved by choosing leaders with differer trajectories. Since such leaders have the same dynamics as the follower agents, thus the pinning control framework is in fact a special case of the general leaderless framework as that considered in this paper.

2.2 Graph and matrix theory notions

Let $G=(\mathcal{V},\varepsilon,\mathcal{A})$ be a weighted digraph of order N with a finite nonempty set of nodes $\mathcal{V}=\{1,2,\dots,N\}$, a set of edges $\varepsilon\subset\mathcal{V}\times\mathcal{V}$, and a weighted *adjacency matrix* $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$, where a_{ij} is the weight, also called coupling strength in this work, of the directed edge (j,i) satisfying $a_{ij}\neq 0$ if (j,i) is an edge of G and $a_{ij}=0$ otherwise. Moreover, we assume $a_{ii}=0$ for all $i\in\mathcal{V}$. The Laplacian matrix L of $G=(\mathcal{V},\varepsilon,\mathcal{A})$ is defined as $L=diag\{\Delta_1,\dots,\Delta_N\}-\mathcal{A}$, where $\Delta_i=\sum_{j=1}^{N}a_{ij}$, $i=1,\dots,N$. Godsil & Doyle [2001]. An important fact of L

is that $\mathbf{1}_n = [1\ 1\ \cdots\ 1]^T \in \mathbb{R}^N$ is a right eigenvector of L associated with the eigenvalue $\lambda = 0$ Godsil & Doyle [2001].

A *directed path* is a sequence of edges in a directed graph of the form (i_1,i_2), $(i_2,i_3), \ldots, (i_{q-1}, i_q)$. A digraph G is called *strongly connected* if between any pair of distinct nodes i, j in G, there is a directed path from node i to node j. A *strongly connected component* (termed also *strong component* for short) of G is a maximal subgraph H of G such that H is strongly connected; and a *closed strong component* of G is a strong component of G which has no incoming edges from any nodes outside. A digraph *has a directed spanning tree* if there exists at least one node, called the root, having a directed path to all of the other nodes.

Consider p $(p > 1)$ disjoint clusters of nodes with respectively node set $\mathcal{V}_1, \ldots, \mathcal{V}_p$ $(\cup_{\ell=1}^{p} \mathcal{V}_\ell = \mathcal{V})$. For $i \in \mathcal{V}$, let $\bar{i}$ denote the subscript of the subset to which the integer i belongs, i.e. $i \in \mathcal{V}_{\bar{i}}$. Let G_ℓ denote the underlying topology of cluster $\mathcal{V}_\ell$, $\ell = 1, \ldots, p$, i.e., $\mathcal{V}(G_\ell) = \mathcal{V}_\ell$. Without loss of generality, assume the number of agents in a cluster, say $\mathcal{V}_\ell$, is N_ℓ, $1 \leq \ell \leq p$, and the N_ℓ agents in $\mathcal{V}_\ell$ are respectively indexed as $\sum_{j=0}^{\ell-1} N_j + 1, \ldots, \sum_{j=0}^{\ell} N_j$, where $N_0 = 0$, i.e., $\mathcal{V}_\ell = \left\{\sum_{j=0}^{\ell-1} N_j + 1, \ldots, \sum_{j=0}^{\ell} N_j\right\}$. Obviously, $N = N_1 + \cdots + N_p$.

As those imposed in Liu & Chen [2011], Yu & Wang [2010], Wu *et al.* [2009], the inter-cluster couplings are assumed to satisfy the following in-degree balanced condition to guarantee the group/cluster synchronization.

Assumption 3.

$$\sum_{j \in \mathcal{V}(G_\ell)} a_{ij} = 0, \ \ \forall i = 1, \ldots, N, \ i \in \mathcal{V} \backslash \mathcal{V}(G_\ell), \ \ell = 1, \ldots, p.$$

Remark 4. Inter-cluster couplings which are negatively weighted can be considered as the inhibitory mechanism to desynchronize the coupled nodes. However, the intra-cluster couplings are all positively weighted to serve as the synchronizing scheme, which is consistent with the cooperative scheme in multi-agent complete consensus problem Ren & Beard [2005], Jadbabaie & Lin [2003]. Note that in terms of the Laplacian matrix, in-degree balanced condition is equivalent to the condition that each row sum of $\mathcal{L}_{\ell k}$, $\ell \neq k$, which specifies the inter-cluster couplings from cluster $\mathcal{V}_k$ to cluster $\mathcal{V}_\ell$ in the Laplacian matrix $\mathcal{L}_s$ of G which takes the following form

$$\mathcal{L}_s = \begin{bmatrix} c_1\mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1p} \\ \mathcal{L}_{21} & c_2\mathcal{L}_{22} & \cdots & \mathcal{L}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{p1} & \mathcal{L}_{p2} & \cdots & c_p\mathcal{L}_{pp} \end{bmatrix}, \tag{3}$$

is zero. As a result, each $L_{\ell\ell}$ is the Laplacian matrix of digraph G_ℓ, $\ell = 1, \ldots, p$.

One may think that under the in-degree balanced condition, the inter-cluster couplings from the other clusters can be viewed as vanishing disturbance. This is not the case due to the existence of cyclic inter-cluster couplings.

3. MAIN RESULTS

3.1 Some notions for non-negatively weighted digraph

We need to introduce some results concerning non-negatively weighted graph. Given any non-negatively weighted digraph G of order m, without loss of generality, assume that G has q $(1 \leq q \leq m)$ strong components, say $\mathcal{G}_1, \ldots, \mathcal{G}_q$, with, respectively, the node sets $\mathcal{V}(\mathcal{G}_\ell) = \{\sum_{j=0}^{\ell-1} m_j + 1, \ldots, \sum_{j=0}^{\ell} m_j\}$, $1 \leq \ell \leq q$, where $m_0 = 0$; and the Laplacian matrix L associated with G takes in the following Frobenius normal form Wu [2005]:

$$L = \begin{bmatrix} L_{11} & & 0 \\ \vdots & \ddots & \\ L_{q1} & \cdots & L_{qq} \end{bmatrix}, \tag{4}$$

where $L_{ii} \in \mathbb{R}^{m_i \times m_i}$, $i = 1, \ldots, q$. Apparently, L_{11} is the Laplacian matrix of graph $\mathcal{G}_1$ and $\check{L}_{ii} = L_{ii} + \sum_{\ell=1}^{i-1} \mathcal{R}(L_{\ell i})$ is the Laplacian matrix of graph $\mathcal{G}_i$, $i = 2, \ldots, q$, where $\mathcal{R}(L_{\ell i})$ denotes the diagonal matrix with the kth diagonal element being the kth row sum of $L_{\ell i}$.

According to Lemma 4 in Qin *et al.* [2011] there exists positive-definite diagonal matrices Ξ_i, $i = 1, \ldots, q$, such that $\Xi_i L_{\ell\ell} + L_{\ell\ell}^T \Xi_i > 0$, $i = 2, \ldots, q$, while $\Xi_1 L_{11} + L_{11}^T \Xi_1 \geq 0$. In fact, one can choose $\Xi_i = diag(\alpha_i)$, where α_i is the unique column vector satisfying $\alpha_i^T \check{L}_{ii} = 0$ $(\check{L}_{11} = L_{11})$ and $\mathbf{1}^T \alpha_i = 1$, $i = 1, \ldots, q$.

Lemma 5. Given any non-negatively weighted digraph G, let L be the associated Laplacian matrix,

$$\Xi = diag\{\Xi_1, \delta_2\Xi_2, \ldots, \delta_q\Xi_q\},$$

and $\beta = [\alpha_1^T, 0, \ldots, 0]^T \in \mathbb{R}^m$, where $\delta_2, \ldots, \delta_q$ are any scalars. For any symmetric positive semi-definite matrix $S \in \mathbb{R}^{n\times n}$ and any column vector $Z \in \mathbb{R}^{mn}$ satisfying $(\beta \otimes I_n)^T Z = 0$, there holds the following inequality

$$Z^T \left[\left(\Xi L + L^T \Xi\right) \otimes S\right] \geq Z^T \left(\hat{L} \otimes S\right) Z,$$

where

$$\hat{L} = \begin{bmatrix} 2a(L_{11})\Xi_1 & \delta_2 L_{21}^T \Xi_2 & \cdots & \delta_q L_{q1}^T \Xi_q \\ \delta_2 \Xi_2 L_{21} & \delta_2(\Xi_2 L_{22} + L_{22}^T \Xi_2) & \cdots & \delta_q L_{q2}^T \Xi_q \\ \vdots & \vdots & \ddots & \vdots \\ \delta_q \Xi_q L_{q1} & \delta_q \Xi_q L_{q2} & \cdots & \delta_q(\Xi_q L_{qq} + L_{qq}^T \Xi_q) \end{bmatrix}$$

and $a(L_{11})$ is the algebraic connectivity of digraph $\mathcal{G}_1$ Wu [2005], Yu *et al.* [2010]

$$a(L_{11}) = \min_{\alpha_1^T x = 0, x \neq 0} \frac{x^T[(\Xi_1 L_{11} + L_{11}^T \Xi_1)/2]x}{x^T \Xi_1 x} > 0.$$

Moreover, there exist appropriate $\delta_i > 0$, $i = 2, \ldots, q$, such that $\hat{L} > 0$; in particular, one can choose any δ_i satisfying the condition that δ_{k+1} is sufficiently smaller than δ_j for any $j \leq k$.

3.2 Interacting clusters of generic linear agents

The compact system dynamics is as follows:

$$\dot{x}(t) = [I_N \otimes A - \mathcal{L}_s \otimes BK]\, x(t),$$

where $\mathcal{L}_s = [l_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$ is as that in (3). Recall that $\mathcal{L}_{\ell\ell}$ is the Laplacian matrix of non-negatively weighted digraph G_ℓ and thus all the results concerning Laplacian matrix of non-negatively weighted digraph are applicable to $\mathcal{L}_{\ell\ell}$, $\ell = 1,\ldots,p$.

Assuming, without loss of generality, that $\mathcal{L}_{\ell\ell}$ takes also in the Frobenius normal form as in (4), and the closed strong component in cluster $\mathcal{V}_\ell$ is of order $n_\ell (1 \le n_\ell \le N_\ell)$, $\ell = 1,\ldots,p$. Let $\beta_\ell = [\beta_\ell^1,\cdots,\beta_\ell^{n_\ell},0,\cdots,0]^T \in \mathbb{R}^{N_\ell}$, $\ell = 1,\ldots,p$, where $[\beta_\ell^1,\cdots,\beta_\ell^{n_\ell}]^T$ is the unique positive vector associated with the closed strong component in G_ℓ, and E_ℓ is the positive-definite diagonal matrix as that defined in Lemma 5 such that $\hat{\mathcal{L}}_{\ell\ell} > 0$ ($[\beta_\ell^1,\cdots,\beta_\ell^{n_\ell}]$, E_ℓ, and $\hat{\mathcal{L}}_{\ell\ell}$ correspond respectively to α_1^T, Ξ, and $\hat{L}$ in Lemma 5). Obviously, $\beta_\ell^T L_{\ell\ell} = 0$.

Let $s_0 = 0$, $s_\ell = \sum_{j=1}^{\ell} N_j$, and $\tilde{x}_\ell(t)$ be the stack of the state variables of agents in cluster $\mathcal{V}_\ell$, i.e., $\tilde{x}_\ell(t) = [x_{s_{\ell-1}+1},\ldots,x_{s_\ell}]^T$, $\ell = 1,2,\ldots,p$. Let

$$x_\ell^*(t) = \left(\beta_\ell^T \otimes I_n\right)\tilde{x}_\ell(t) \in \mathbb{R}^n,\quad \ell = 1,\ldots,p,$$
$$e_i(t) = x_i(t) - x_i^*(t) \in \mathbb{R}^n,\quad i = 1,\ldots,N,$$

and

$$\begin{aligned}\tilde{e}_\ell(t) &= \tilde{x}_\ell(t) - (\mathbf{1}_{N_\ell} \otimes I_n)\, x_\ell^*(t)\\ &= \tilde{x}_\ell(t) - \left(\mathbf{1}_{N_\ell}\beta_\ell^T \otimes I_n\right)\tilde{x}_\ell(t) \in \mathbb{R}^{nN_\ell}.\end{aligned}$$

Further, let $e(t)$ be the stack of the state error variables for all the agents, i.e.,

$$\begin{aligned}e(t) &= [\tilde{e}_1^T(t),\ldots,\tilde{e}_p^T(t)]^T\\ &= x(t) - diag\{\mathbf{1}_{N_1}\beta_1^T,\ldots,\mathbf{1}_{N_p}\beta_p^T\}x(t).\end{aligned}$$

Evidently, if one proves that $e(t) \to 0$ as t approaches infinity, then group synchronization is obtained. Denote by $\mathcal{L}_s = [l_{ij}]_{N\times N} \in \mathbb{R}^N$, then one obtains that

$$\sum_{j=1}^{N} l_{ij}x_j(t) = \sum_{j=1}^{N} l_{ij}e_j(t),$$

which yields that

$$(\mathcal{L}_s \otimes BK)\, x(t) = (\mathcal{L}_s \otimes BK)\, e(t).$$

This, in turn, gives the fact that

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - diag\{\mathbf{1}_{N_1}\beta_1^T,\ldots,\mathbf{1}_{N_p}\beta_p^T\}\dot{x}(t)\\ &= (I_N \otimes A - M \otimes BK)\, e(t),\end{aligned} \tag{5}$$

where $M = \left(I_N - diag\{\mathbf{1}_{N_1}\beta_1^T,\ldots,\mathbf{1}_{N_p}\beta_p^T\}\right)\mathcal{L}_s$.

Let $E = diag\{E_1,\cdots,E_p\}$ and consider the following Lyapunov function candidate

$$V(t) = e^T(t)\,(E \otimes P)\, e(t),$$

where $P > 0$ is a symmetric positive-definite matrix to be determined. Further, Let $K = B^TP$, then one obtains

$$\begin{aligned}&\dot{V}(t)\\ &= e^T(t)\left[E \otimes (A^TP + PA)\right.\\ &\quad \left.-(EM + M^TE) \otimes PBB^TP\right] e(t).\end{aligned} \tag{6}$$

Before proceeding further, we consider the second term in (6):

$$w(t) = -e^T(t)\left[(EM + M^TE) \otimes PBB^TP\right] e(t).$$

Since $(\beta_\ell \otimes I_n)^T \tilde{e}_\ell(t) \equiv 0$, $\ell = 1,\ldots,p$, it follows from Lemma 5 that

$$\begin{aligned}&\tilde{e}_\ell^T(t)\left[(E_\ell\mathcal{L}_{\ell\ell} + \mathcal{L}_{\ell\ell}^TE_\ell) \otimes PBB^TP\right]\tilde{e}_\ell(t)\\ &\ge \tilde{e}_\ell^T(t)\left(\hat{\mathcal{L}}_{\ell\ell} \otimes PBB^TP\right)\tilde{e}_\ell(t).\end{aligned}$$

Let

$$\begin{aligned}&\bar{M}\\ &= M - diag\{c_1\mathcal{L}_{11},\ldots,c_p\mathcal{L}_{pp}\}\\ &= \begin{bmatrix} 0 & \left(I_{N_1} - M_1\right)\mathcal{L}_{12} & \cdots & \left(I_{N_1} - M_1\right)\mathcal{L}_{1p}\\ \left(I_{N_2} - M_2\right)\mathcal{L}_{21} & 0 & \cdots & \left(I_{N_2} - M_2\right)\mathcal{L}_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \left(I_{N_p} - M_p\right)\mathcal{L}_{p1} & \left(I_{N_p} - M_p\right)\mathcal{L}_{p2} & \cdots & 0\end{bmatrix},\end{aligned}$$

where $M_\ell = \mathbf{1}_{N_\ell}\beta_\ell^T$, $\ell = 1,\ldots,p$. It follows that

$$w(t) = w_1(t) + w_2(t),$$

where

$$\begin{aligned}&w_1(t)\\ &= -e^T(t)\left(diag\left\{c_1(E_1\mathcal{L}_{11} + \mathcal{L}_{11}^TE_1),\cdots,\right.\right.\\ &\quad \left.\left.c_p(E_p\mathcal{L}_{pp} + \mathcal{L}_{pp}^TE_p)\right\} \otimes PBB^TP\right) e(t)\\ &= -\sum_{\ell=1}^{p} c_\ell\tilde{e}_\ell^T(t)\left[(E_\ell\mathcal{L}_{\ell\ell} + \mathcal{L}_{\ell\ell}^TE_\ell) \otimes PBB^TP\right]\tilde{e}_\ell(t)\\ &\le -\sum_{\ell=1}^{p} c_\ell\tilde{e}_\ell^T(t)\left(\hat{\mathcal{L}}_{\ell\ell} \otimes PBB^TP\right)\tilde{e}_\ell(t)\end{aligned}$$

and

$$w_2(t) = -e(t)^T\left(\bar{\mathcal{M}} \otimes PBB^TP\right) e(t)$$

with $\bar{\mathcal{M}} = E\bar{M} + \bar{M}^TE$. Note, similarly to $\bar{M}$, that matrix $\bar{\mathcal{M}}$ is also a block matrix with all diagonal blocks being zero matrices and thus it is irrelevant to coupling strengths c_ℓ, $\ell = 1,\ldots,p$. Let

$$\mathcal{M} = \bar{\mathcal{M}} + diag\{c_1\hat{\mathcal{L}}_{11},\cdots,c_p\hat{\mathcal{L}}_{pp}\},$$

then one obtains from $w_1(t)$ and $w_2(t)$ that

$$w(t) \le -e(t)^T\left(\mathcal{M} \otimes PBB^TP\right) e(t). \tag{7}$$

Next, we prove that $\mathcal{M} > 0$ can be guaranteed if each c_ℓ is larger than a threshold. In fact,one can choose such $c_\ell' s$ that satisfy

$$\lambda_{min}\left(diag\{c_1\hat{\mathcal{L}}_{11},\cdots,c_p\hat{\mathcal{L}}_{pp}\}\right) + \lambda_{min}\left(\bar{\mathcal{M}}\right) > 0. \tag{8}$$

Inequality (8) holds for any $c_\ell > 0$ if $\bar{\mathcal{M}} \ge 0$; while if $\lambda_{min}(\bar{\mathcal{M}}) < 0$, inequality (8) holds for any c_ℓ satisfying that $c_\ell > \frac{-\lambda_{min}(\bar{\mathcal{M}})}{\lambda_{min}(\hat{\mathcal{L}}_{\ell\ell})}$, $\ell = 1,\ldots,p$. Now fix any c_ℓ's that guarantees $\mathcal{M} > 0$ and choose any positive number, say $\eta > 0$, such that $\mathcal{M} \ge \eta E$. On the other hand, since

(A, B) is stabilizable, one can choose a solution $P > 0$ to the following Riccati inequality

$$PA + A^T P - \eta PBB^T P + \eta I_n < 0. \quad (9)$$

Then from from (6), (7), (9), and the property of Kronecker product, one obtains that

$$\begin{aligned} &\dot{V}(t) \\ &\le e^T(t)\left[E \otimes (A^T P + P^T A - \eta PBB^T P)\right] e(t) \\ &\le -\eta e^T(t)(E \otimes I_n) e(t) \le -\frac{\eta}{\lambda_{max}(P)} V(t), \end{aligned}$$

and thus $e(t) \to 0$ exponentially fast with the least rate of $\frac{\eta \lambda_{min}(E)}{2\lambda_{max}(E)\lambda_{max}(P)}$.

Summarizing the above analysis and notions gives the main result of this paper:

Theorem 6. Under Assumption 3, group synchronization can be achieved exponentially fast and further,

$$x_i(t) \to \left(\beta_\ell^T \otimes I_n\right) \tilde{x}_\ell(t) = \sum_{k=1}^{n_\ell} \beta_\ell^k x_{s_{\ell-1}+k}(t) \quad (10)$$

(assuming $\bar{i} = \ell$), if the following two conditions hold:

1) underlying topology of each cluster has a directed spanning tree;
2) coupling strength c_ℓ for cluster $\mathcal{V}_\ell$ satisfies

$$c_\ell > \max\left\{0, \frac{-\lambda_{min}(\bar{\mathcal{M}})}{\lambda_{min}(\hat{\mathcal{L}}_{\ell\ell})}\right\}.$$

Remark 7. For the case that the underlying topology of each cluster, say, G_ℓ, is undirected and connected, the case would be much simpler since $\beta_\ell = \frac{1}{N_\ell}\mathbf{1}_{N_\ell}$. Moreover, one has $x_i(t) \to \frac{1}{N_\ell}\sum_{k=1}^{N_\ell} x_{s_{\ell-1}+k}(t)$ (with $\bar{i} = \ell$), which implies that all agents within the same cluster contribute equally to the consensus value of such a cluster, thus leading to the average consensus.

Theorem 6 shows that the states of all the agents within each cluster converge asymptotically to the weighted sum of the states of agents in the closed strong component within the same cluster. However, differently from those results for complete consensus, such states cannot be expressed in terms of initial states of those agents in the closely strong component due to the presence of inter-cluster couplings from the other clusters.

3.3 Interacting clusters of nonlinear oscillators

The technique and analysis proposed in proof of Theorem 6 can be developed and extended to deal with the synchronization of coupled nonlinear oscillators Wu *et al.* [2009], Liu & Chen [2011]. Unless otherwise explicitly specified, all the notations used in the proof of Theorem 6 still work here.

Theorem 8. Under Assumption 3, group synchronization can be achieved exponentially fast for interacting clusters of nonlinear oscillators (2) and further, $x_i(t) \to \sum_{k=1}^{n_\ell} \beta_\ell^k x_{s_{\ell-1}+k}(t)$ (assuming $\bar{i} = \ell$), if the following two conditions hold:

1) underlying topology of each cluster has a directed spanning tree;
2) coupling strength c_ℓ, $\ell = 1, \ldots, p$, for cluster $\mathcal{V}_\ell$ satisfies

$$c_\ell > max\left\{\frac{\delta + \rho}{\lambda_{\min}(\Gamma)\lambda_{\min}(\hat{\mathcal{L}}_{\ell\ell})} - \frac{\lambda_{min}(\bar{\mathcal{M}})}{\lambda_{min}\left(\hat{\mathcal{L}}_{\ell\ell}\right)}, 0\right\}$$

where

$$\delta = \lambda_{\max}\Big(\left(E - Ediag\{\mathbf{1}_{N_1}\beta_1^T, \ldots, \mathbf{1}_{N_p}\beta_p^T\}\right) \times \left(E - Ediag\{\mathbf{1}_{N_1}\beta_1^T, \ldots, \mathbf{1}_{N_p}\beta_p^T\}\right)^T \Big).$$

Remark 9. It is worth mentioning here that one of the key points in the proof of the main results is to find such c_ℓ, $\ell = 1, \ldots, p$, that guarantees $\mathcal{M} > 0$. The lower bounds obtained in Theorem 6 and Theorem 8 may still be conservative, however, they show that increasing the coupling strength of intra-cluster couplings will not destroy the synchronization property of the whole system.

4. SIMULATION EXAMPLE

In this we present an illustrative example for interacting clusters of generic linear agents.

Example 2: Consider interacting two clusters of harmonic oscillators ($A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$) in $\mathcal{R}^2$ with coupling topology as shown in Figure 1, where $\mathcal{V}_1 = \{1, 2, 3, 4\}$ and $\mathcal{V}_2 = \{5, 6, 7, 8\}$. Assume that the initial values of all the 8 agents are randomly chosen from $[-50, 50] \times [-50, 50] \subset \mathbb{R}^2$. Let $\Delta_1(t) = \sum_{i=1}^4 \|x_i(t) - \frac{1}{3}\sum_{k=1}^3 x_k(t)\|$ and $\Delta_2(t) = \sum_{i=5}^8 \|x_i(t) - \frac{1}{4}\sum_{k=5}^8 x_k(t)\|$. Noting from (10) and the fact that $\beta_1 = \{1/3, 1/3, 1/3, 0\}$ and $\beta_2 = \{1/4, 1/4, 1/4, 1/4\}$, we know that $\Delta_1(t)$ and $\Delta_2(t)$ are the quantities describing respectively the process of agents in clusters $\mathcal{V}_1$ and $\mathcal{V}_2$ to the converged trajectories.

Further, it is easy to compute that

$$\hat{\mathcal{L}}_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & -\delta \\ 0 & -\delta & -\delta & 2\delta \end{bmatrix}, \quad \hat{\mathcal{L}}_{22} = diag\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}.$$

One chooses $\delta = \frac{4}{5}$ such that $\hat{\mathcal{L}}_{11} > 0$. As such, $E_1 = diag\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{4}{5}\}$ and $E_2 = diag\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$. Through computation, one can choose $c_1 = 4$ and $c_2 = 2$ such that $\mathcal{M} > 0$, and η can be chosen as $\eta = 0.1$ such that inequality $\mathcal{M} \ge \eta E$ holds. Finally, computing (9) gives $P = \begin{bmatrix} 1.7434 & 0.0747 \\ 0.0747 & 1.7304 \end{bmatrix}$ and $K = B^T P = [0.0747 \;\; 1.7304]$.

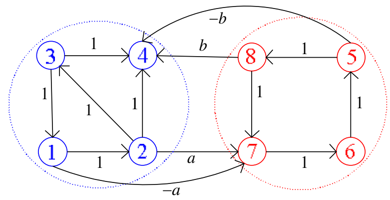

Fig. 1. Coupling topology

Figure 2, which plots the trajectories of the agents, shows that group synchronization is achieved. Figure 3 shows

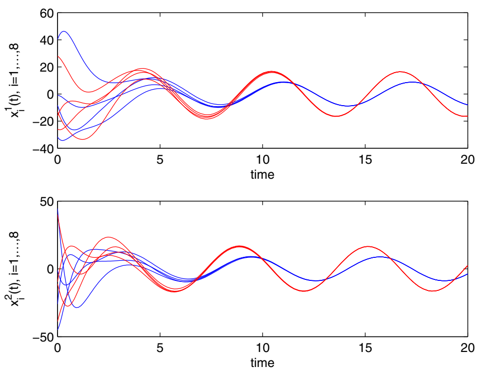

Fig. 2. State trajectories of the agents with $a = b = 1$

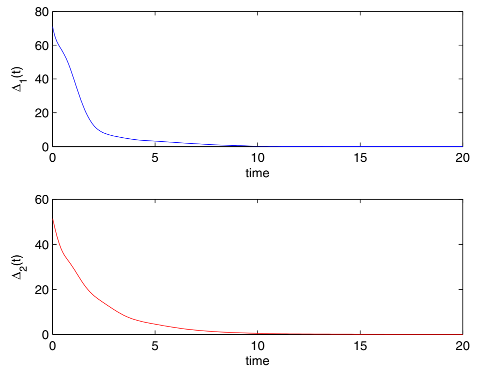

Fig. 3. Evolution trajectories of $\Delta_1(t)$ and $\Delta_2(t)$

that $x_i(t) \to \frac{1}{3}\sum_{k=1}^{3} x_k(t)$, $i = 1, \ldots, 4$, and $x_i(t) \to \frac{1}{4}\sum_{k=5}^{8} x_k(t)$, $i = 5, \ldots, 8$, which is consistent with the theoretical findings in Theorem 6.

5. CONCLUSION

In this paper, we have provided a unified analysis that is feasible to deal with the synchronization for multiple clusters of both generic linear agents and nonlinear oscillators in a general setting regarding the coupling topology. With this unified analysis, we have addressed a general concern that whether complete synchronization for agents/oscillators within the same cluster can be achieved if the coupling topology for each cluster has a directed spanning tree, and further, compared to the inter-cluster couplings, the intra-cluster couplings are sufficiently strong. Furthermore, both the lower bounds for such coupling strengths as well as the convergence rate have been explicitly specified.

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